

Shijun Liao

Homotopy Analysis Method in Nonlinear Differential Equations



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With 127 figures



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To my mother, wife and daughter

Preface

It is well-known that perturbation and asymptotic approximations of nonlinear problems often break down as nonlinearity becomes strong. Therefore, they are only valid for weakly nonlinear ordinary differential equations (ODEs) and partial differential equations (PDEs) in general.

The homotopy analysis method (HAM) is an analytic approximation method for highly nonlinear problems, proposed by the author in 1992. Unlike perturbation techniques, the HAM is independent of any small/large physical parameters at all: one can always transfer a nonlinear problem into an infinite number of linear sub-problems by means of the HAM. Secondly, different from all of other analytic techniques, the HAM provides us a convenient way to guarantee the convergence of solution series so that it is valid even if nonlinearity becomes rather strong. Besides, based on the homotopy in topology, it provides us extremely large freedom to choose equation type of linear sub-problems, base function of solution, initial guess and so on, so that complicated nonlinear ODEs and PDEs can often be solved in a simple way. Finally, the HAM logically contains some traditional methods such as Lyapunov's small artificial method, Adomian decomposition method, the δ -expansion method, and even the Euler transform, so that it has the great generality. Therefore, the HAM provides us a useful tool to solve highly nonlinear problems in science, finance and engineering.

This book consists of three parts. In Part I, the basic ideas of the HAM, especially its theoretical modifications and developments, are described, including the optimal HAM approaches, the theorems about the so-called homotopy-derivative operator and the different types of deformation equations, the methods to control and accelerate convergence, the relationship to Euler transform, and so on.

In Part II, inspired by so many successful applications of the HAM in different fields and also by the ability of "computing with functions instead of numbers" of computer algebra system like Mathematica and Maple, a Mathematica package BVPh (version 1.0) is developed by the author in the frame of the HAM for nonlinear boundary-value problems. A dozen of examples are used to illustrate its validity for highly nonlinear ODEs with singularity, multiple solutions and multipoint boundary conditions in either a finite or an infinite interval, and even for some types of non-

linear PDEs. As an open resource, the BVPh 1.0 is given in this book with a simple users guide and is free available online.

In Part III, we illustrate that the HAM can be used to solve some complicated highly nonlinear PDEs so as to enrich and deepen our understandings about these interesting nonlinear problems. For example, By means of the HAM, an explicit analytic approximation of the optimal exercise boundary of American put option was gained, which is often valid for a couple of decades prior to expiry, whereas the asymptotic and perturbation formulas are valid only for a couple of days or weeks in general. A Mathematica code based on such kind of explicit formula is given in this book for businessmen to gain accurate results in a few seconds. In addition, by means of the HAM, the wave-resonance criterion of arbitrary number of traveling gravity waves was found, for the first time, which logically contains the famous Phillips' criterion for four waves with small amplitude.

All of these show the originality, validity and generality of the HAM for highly nonlinear problems in science, finance and engineering.

All Mathematica codes and their input data files are given in the appendixes of this book and available (Accessed 25 Nov 2011, will be updated in the future) either at

<http://numericaltank.sjtu.edu.cn/HAM.htm>

or at

<http://numericaltank.sjtu.edu.cn/BVPh.htm>

This book is suitable for researchers and postgraduates in applied mathematics, physics, finance and engineering, who are interested in highly nonlinear ODEs and PDEs.

I would like to express my gratitude to my collaborators for their valuable discussions and communications, and to my postgraduates for their hard working. Thanks to Natural Science Foundation of China for the financial support.

I would like to express my sincere thanks to my parents, wife and daughter for their love, encouragement and support in the past 20 years.

Shanghai, China
March 2011

Shijun Liao

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Acronyms

2D	Two Dimensional
3D	Three Dimensional
APO	American Put Option
BEM	Boundary Element Method
BVP	Boundary Value Problem
BVPs	Boundary Value Problems
CPU	Central Processing Unit
DNS	Direct Numerical Simulation
FDM	Finite Difference Method
FEM	Finite Element Method
GBEM	Generalized Boundary Element Method
HAM	Homotopy Analysis Method
IVP	Initial Value Problem
IVPs	Initial Value Problems
ODE	Ordinary Differential Equation
ODEs	Ordinary Differential Equations
OHAM	Optimal Homotopy Analysis Method
PDE	Partial Differential Equation
PDEs	Partial Differential Equations

Part I

Basic Ideas and Theorems

“ The essence of mathematics lies entirely in its freedom. ”

by Georg Cantor (1845 — 1918)

Chapter 1

Introduction

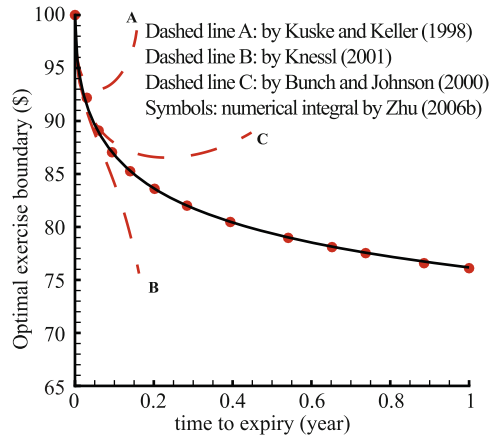
1.1 Motivation and purpose

It is well-known that nonlinear ordinary differential equations (ODEs) and partial differential equations (PDEs) for boundary-value problems are much more difficult to solve than linear ODEs and PDEs, especially by means of analytic methods. Traditionally, perturbation (Van del Pol, 1926; Von Dyke, 1975; Nayfeh, 2000) and asymptotic techniques are widely applied to obtain analytic approximations of nonlinear problems in science, finance and engineering. Unfortunately, perturbation and asymptotic techniques are too strongly dependent upon small/large physical parameters in general, and thus are often valid only for weakly nonlinear problems. For example, the asymptotic/perturbation approximations of the optimal exercise boundary of American put option are valid only for *a couple of days or weeks* prior to expiry, as shown in Fig. 1.1. Another famous example is the viscous flow past a sphere in fluid mechanics: the perturbation formulas of the drag coefficient are valid only for rather small Reynolds number $Re \ll 1$. Thus, it is necessary to develop some analytic approximation methods, which are independent of any small/large physical parameters at all and besides valid for strongly nonlinear problems.

In 1992, one of such kind of analytic approximation methods was proposed by the author (Liao, 1992), namely the homotopy analysis method (HAM) (Liao, 1997, 1999a,b; Liao and Campo, 2002; Liao, 2003a,b, 2004, 2005, 2006, 2009a, 2010a,b, 2011; Liao and Magyari, 2006; Liao and Tan, 2007; Li et al., 2010; Xu et al., 2010). Based on homotopy (Hilton, 1953) in topology (Sen, 1983), the HAM is independent of any small/large physical parameters. More importantly, unlike all other analytic techniques, it provides us a convenient way to guarantee the convergence of series solution of nonlinear problems by means of introducing an auxiliary parameter c_0 , called the convergence-control parameter. In 2003, the basic ideas of the HAM and some applications mostly related to nonlinear ODEs were described systematically by the author in the book “Beyond Perturbation” (Liao, 2003b).

Thereafter, the HAM attracts attention of many researchers in about a dozen of countries, and has been successfully applied to solve a lot of nonlinear problems

Fig. 1.1 Asymptotic/perturbation approximations of the optimal exercise boundary of American put option in the case of Example 13.1: $X = \$100$, $r = 0.1$, $\sigma = 0.3$ and $T = 1$ (year). Dashed line A: by Kuske and Keller (1998); Dashed line B: by Knessl (2001); Dashed line C: by Bunch and Johnson (2000); Symbols: numerical integral by Zhu (2006b) .



in science, finance and engineering. For example, Some new solutions (Liao, 2005; Liao and Magyari, 2006) have been found by means of the HAM, which had been neglected even by numerical techniques and never reported. Some analytic approximations (Zhu, 2006a; Cheng, 2008; Cheng et al., 2010) for the optimal exercise boundary of American put option were given, which are often valid for *a couple of years* prior to expiry and thus much better than the asymptotic/perturbation approximations (Bunch and Johnson, 2000; Knessl, 2001; Kuske and Keller, 1998) that are often valid only for *a couple of days or weeks*. Besides, the HAM has been successfully employed to solve some complicated nonlinear PDEs so as to enrich and deepen our physical understandings: the wave-resonance criterion for *arbitrary* number of traveling waves with *large* amplitudes was founded (Liao, 2011), *for the first time*, by means of the HAM, which logically contains the famous Phillips' criterion for *four* waves with *small* amplitudes. All of these applications show the originality, validity and generality of the HAM for nonlinear problems.

In addition, some theoretical studies and modifications have been done. For example, some mathematical theorems related to the equations for high-order approximations and the so-called homotopy-derivatives were proved (Liao, 2009a; Molabrahimi and Khani, 2009; Turkyilmazoglu, 2010a, 2011d). Some optimal HAM approaches (Yabushita et al., 2007; Marinca and Herişanu, 2008, 2009; Niu and Wang, 2010; Liao, 2010b) were developed, which greatly accelerate the convergence of solution series. Besides, it was proved (Liao, 2010a) that the HAM logically contains the famous Euler transform (Agnew, 1944), which reveals the reason why the HAM can guarantee the convergence of solution series. Furthermore, some HAM-based approaches (Liao, 2005; Abbasbandy et al., 2009; Xu et al., 2010; Marinca and Herişanu, 2008; Motsa et al., 2010b) were developed to search for multiple solutions of nonlinear problems, and/or to increase the computational efficiency. All of these greatly developed and modified the HAM in theory and provided the HAM a sound base.

Therefore, it is valuable to systematically describe the theoretical modifications and new applications of the HAM as a whole, and besides to discuss some open questions and its possible development in future.

It should be emphasized that our aim is to develop an analytic approximation method valid for *as many* nonlinear problems *as possible*, since it seems impossible to develop an analytic approach valid for *all* nonlinear problems, especially those related to chaos (Lorenz, 1963; Liao, 2009b) and turbulence.

1.2 Characteristic of homotopy analysis method

The HAM has the following characteristics which differ it from other traditional analytic techniques.

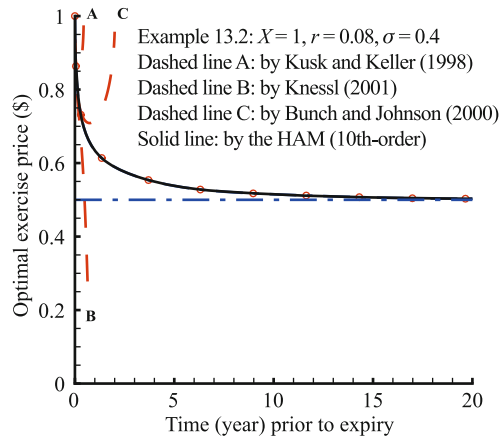
First of all, based on the homotopy of topology, the HAM is *independent* of any small/large physical parameters at all. So, unlike asymptotic/perturbation techniques, the HAM can be applied to solve many nonlinear problems in science, finance and engineering, especially those without small/large physical parameters. For example, an analytic approximation of the optimal exercise boundary $B(\tau)$ of American put option in polynomials of $\sqrt{\tau}$ to order $o(\tau^{48})$ was obtained by means of the HAM, which is often valid for *a couple of decades* or even for *half a century* prior to expiry, and thus is much better than the asymptotic/perturbation approximations (Bunch and Johnson, 2000; Knessl, 2001; Kuske and Keller, 1998) that are often valid for *a couple of days or weeks*, as shown in Fig. 1.2. For details, please refer to Chapter 13.

Secondly, unlike all other analytic techniques, the HAM provides us a convenient way to guarantee the convergence of solution series so that it is valid for *highly* nonlinear problems. For example, when perturbation results are valid only for a *small* physical parameter $0 \leq \varepsilon \leq 1$, one can gain accurate approximations valid in the *whole* interval $0 \leq \varepsilon < +\infty$ by means of the HAM, as shown by Abbasbandy (Abbasbandy, 2006). Besides, when results given by other methods are *divergent*, one can gain *convergent* series solution by means of the HAM, as shown by Liang and Jeffrey (2010).

Thirdly, the HAM provides us extremely large *freedom* to choose equation type of linear sub-problems and base functions of solution. Using such kind of freedom, some complicated nonlinear problems can be solved in a much easier way. For example, the two-dimensional *2nd-order* Gelfand equation was solved by means of the HAM in a rather easy way by transferring it into an infinite number of linear PDEs governed by a very simple *4th-order* linear operator, as shown by Liao and Tan (2007). Such kind of transformation has *never* been used by other analytic and numerical techniques. This suggests that we human beings might have much larger freedom to solve nonlinear problems than we traditionally thought so that we should always keep an open mind.

Finally, it has been proved (Liao, 2003b, 2010a) that the HAM logically contains the Lyapunov's artificial small parameter method (Lyapunov, 1992), Adomian de-

Fig. 1.2 The optimal exercise boundary of American put option in the case of Example 13.2: $X = \$1$, $r = 0.08$ and $\sigma = 0.4$. Solid line: 10th-order approximation by HAM in polynomial of $\sqrt{\tau}$ to $o(\tau^{48})$; Symbols: approximation by Padé method; Dashed lines: asymptotic/perturbation results; Dash-dotted line: perpetual optimal exercise price \$0.5.



composition method (Adomian, 1976, 1994), the δ -expansion method (Karmishin et al., 1990), and the Euler transform (Agnew, 1944). Thus, it has the great *generality*.

In summary, the HAM has the following advantages:

- Independent of small/large physical parameters;
- Guarantee of convergence;
- Flexibility on choice of base function and initial guess of solution;
- Great generality.

The HAM provides a useful analytic tool to investigate highly nonlinear problems with multiple solutions and singularity in science, finance and engineering.

1.3 Outline

This book consists of three parts. In Part I, the basic ideas of the HAM, its modifications and developments in theory are briefly described. In Chapter 2, two simple examples are used to describe the basic ideas of the HAM, all related concepts and approaches in details. For beginners of the HAM, it is strongly suggested to read Chapter 2 first. In Chapter 3, some optimal HAM approaches are described. In Chapter 4, the basic ideas of the HAM is systematically described, mathematical theorems about the so-called homotopy-derivative and equations for high-order approximations are proved, and some open questions are discussed. The relationship between the HAM with the famous Euler transform is revealed in Chapter 5. Some other HAM-based methods, together with the history of the HAM, are briefly described in Chapter 6, respectively.

In Part II, inspired by so many successful applications of the HAM in so many different fields of researches (Abbas et al., 2008; Abbasbandy, 2006, 2007; Abbasbandy and Parkes, 2008; Abbasbandy, 2008; Abbasbandy et al., 2009; Abbasbandy

and Parkes, 2010; Abbasbandy and Shivanian, 2011; Akyildiz, 2008; Akyildiz et al., 2009; Alizadeh-Pahlavan and Borjian-Boroujeni, 2008; Alizadeh-Pahlavan et al., 2009; Allan and Syam, 2005; Allan, 2007, 2009; Cai, 2006; Cheng, 2008; Gao, 2007; Hayat et al., 2004, 2005; Hayat and Sajid, 2007; Hayat et al., 2011; Jiao, 2009; Jiao et al., 2009; Kumari and Nath, 2010; Kumari et al., 2010; Liang, 2010; Liang and Jeffrey, 2009a,b, 2010; Liu, 2008; Liu and Li, 2009; Mahapatra et al., 2009; Marinca and Herişanu, 2008, 2009; Molabahrami and Khani, 2009; Motsa et al., 2010a,b; Niu and Wang, 2010; Pandey et al., 2011; Pirbodaghi et al., 2009; Sajid, 2006; Sajid and Hayat, 2008; Shidfar et al., 2010; Shidfar and Molabahrami, 2010; Siddheshwar, 2010; Singh et al., 2009; Song and Tao, 2007; Tao and Song, 2007; Turkyilmazoglu, 2009, 2010a,b, 2011a,b,c,d; Van Gorder and Vajravelu, 2008; Van Gorder et al., 2010a,b; Van Gorder and Vajravelu, 2011; Wu, 2009; Wu and Cheung, 2008, 2009; Yabushita et al., 2007; Zand et al., 2009; Zand and Ahmadian, 2009; Zhao and Wong, 2008; Zhu, 2008, 2006a,b; Zou, 2008; Zou et al., 2007) and the ability of “computing with functions instead of numbers” (Trefethen, 2007) provided by computer algebra system such as Mathematica (Abell and Braselton, 2004) and Maple, a Mathematica package BVPh (version 1.0) is developed by the author in the frame of the HAM, which is mainly for highly nonlinear ODEs in a finite or an infinite interval with multiple solutions, singularity and multipoint boundary conditions. In Chapter 7, the Mathematica package BVPh 1.0 is briefly described with related mathematical formulas and a simple users guide. Then, we illustrate its validity and generality for nonlinear boundary-value problems with multiple solutions in a finite interval (Chapter 8), nonlinear eigenvalue problems in a finite interval with multipoint boundary conditions, singularity and high nonlinearity (Chapter 9), nonlinear boundary-value problems governed by an ODE in an infinite interval with exponentially or algebraically decaying solutions (Chapter 10), and even some nonlinear PDEs related to non-similarity boundary layer flows (Chapter 11) and unsteady boundary-layer flows (Chapter 12), respectively. The BVPh 1.0 provides us a tool to solve some nonlinear boundary-value problems governed by a nonlinear ODE or PDE. As an open resource, the BVPh 1.0 is given in Appendix 7.1 and free available (Accessed 25 Nov 2011, will be updated in the future) at

<http://numericaltank.sjtu.edu.cn/BVPh.htm>

Besides, the input data files for all examples in Part II are free available at the same website.

In Part III, we illustrate that the HAM can be applied to investigate some rather complicated nonlinear PDEs with high nonlinearity so as to deepen and enrich our physical understandings about these nonlinear phenomena. In Chapter 13, the HAM is used to give an analytic approximation of the optimal exercise boundary of American put option, which is often valid for *a couple of decades* prior to expiry and sometimes may be valid even for *half a century*, and thus is much better than the perturbation/asymptotic results that are often valid only for *a couple of days or weeks*. Based on this kind of analytic results, a short Mathematica code APOh for businessmen is given in Appendix 13.3, which gives accurate results in a few seconds only! In Chapter 14, the two (three) dimensional *2nd-order* Gelfand equation is solved by

means of the HAM in a rather easier way by transferring it into an infinite number of *4th-order* (6th-order) linear PDEs. Such kind of transformation has *never* been used by other analytic and numerical methods, which suggests that we might have much larger freedom to solve nonlinear problems than we thought traditionally and thus we *must* always keep an open mind. In Chapter 15, the HAM is applied to solve a complicated nonlinear PDE describing the nonlinear interaction between gravity waves and exponential shear currents. It is found *for the first time* that the criterion for wave breaking is the same for waves on uniform and non-uniform currents. In Chapter 16, the HAM is successfully applied to investigate the nonlinear interaction of arbitrary number of traveling gravity waves. And it is found, *for the first time*, the wave-resonance criterion for *arbitrary* number of traveling waves with *large* amplitudes, which logically contains the famous Phillips' criterion for *four* waves with *small* amplitudes.

All of these show the *originality*, *validity* and *generality* of the HAM for some highly nonlinear problems with multiple solutions, singularity and multipoint boundary conditions.

For readers' convenience, the related Mathematica codes and their input data files of nearly all examples are given in the appendixes of this book and available either at

<http://numericaltank.sjtu.edu.cn/HAM.htm>

or at

<http://numericaltank.sjtu.edu.cn/BVPh.htm>

respectively.

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Chapter 2

Basic Ideas of the Homotopy Analysis Method

Abstract The basic ideas and all fundamental concepts of the homotopy analysis method (HAM) are described in details by means of two simple examples, including the concept of the homotopy, the flexibility of constructing equations for continuous variations, the way to guarantee convergence of solution series, the essence of the convergence-control parameter c_0 , the methods to accelerate convergence, and so on. The corresponding Mathematica codes are given in appendixes and free available online. Beginners of the HAM are strongly suggested to read it first.

2.1 Concept of homotopy

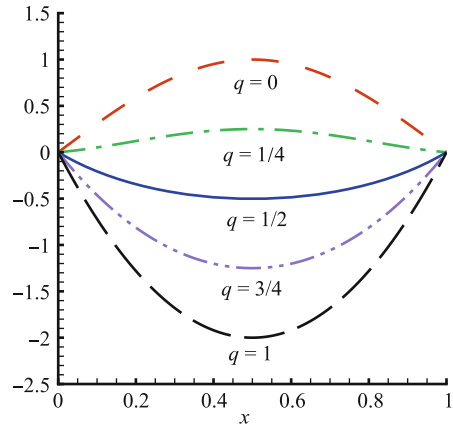
The homotopy analysis method (HAM) (Liao, 1992, 1997, 1999a,b; Liao and Campo, 2002; Liao, 2003a,b, 2004, 2005, 2006, 2009, 2010a,b; Liao and Tan, 2007; Xu et al., 2010) proposed by Shijun Liao (1992) is based on the concept of the homotopy, a fundamental concept in topology and differential geometry (Armstrong, 1983; Sen, 1983). The concept of the homotopy can be traced back to Jules Henri Poincaré (1854 — 1912), a French mathematician. Shortly speaking, a homotopy describes a kind of *continuous* variation or deformation in mathematics. For example, a circle can be *continuously* deformed into a square or an ellipse, the shape of a coffee cup can deform continuously into the shape of a doughnut. However, the shape of a coffee cup can not be distorted continuously into the shape of a football. Essentially, a homotopy defines a connection between different things in mathematics, which contain same characteristics in some aspects.

For example, the two different real functions $\sin(\pi x)$ and $8x(x-1)$ in the interval $x \in [0, 1]$ can be connected by constructing such a family of functions

$$\mathcal{H}(x; q) = (1 - q) \sin(\pi x) + q [8x(x - 1)], \quad (2.1)$$

where $q \in [0, 1]$ is called *the embedding parameter*. Note that $\mathcal{H}(x; q)$ depends on not only the independent variable $x \in [0, 1]$ but also the embedding parameter

Fig. 2.1 Continuous deformation of the homotopy $\mathcal{H}(x; q) : \sin(\pi x) \sim 8x(x-1)$. Dashed line: $q = 0$; Dash-dotted line: $q = 1/4$; Solid line: $q = 1/2$; Dash-double-dotted line: $q = 3/4$; Long-dashed line: $q = 1$.



$q \in [0, 1]$. Especially, when $q = 0$, we have

$$\mathcal{H}(x; 0) = \sin(\pi x), \quad x \in [0, 1],$$

and when $q = 1$, it holds

$$\mathcal{H}(x; 1) = 8x(x-1), \quad x \in [0, 1],$$

respectively. So, as the embedding parameter $q \in [0, 1]$ increases from 0 to 1, the real function $\mathcal{H}(x; q)$ varies *continuously* from a trigonometric function $\sin(\pi x)$ to a polynomial $8x(x-1)$, as shown in Fig. 2.1. In topology, $\mathcal{H}(x; q)$ is called a *homotopy*, $\sin(\pi x)$ and $8x(x-1)$ are called *homotopic*, denoted by

$$\mathcal{H} : \sin(\pi x) \sim 8x(x-1).$$

Let $C[a, b]$ denote a set of all continuous real functions in the interval $a \leq x \leq b$. In general, if a continuous function $f \in C[a, b]$ can be deformed continuously into another continuous function $g \in C[a, b]$, one can construct a homotopy

$$\mathcal{H} : f(x) \sim g(x)$$

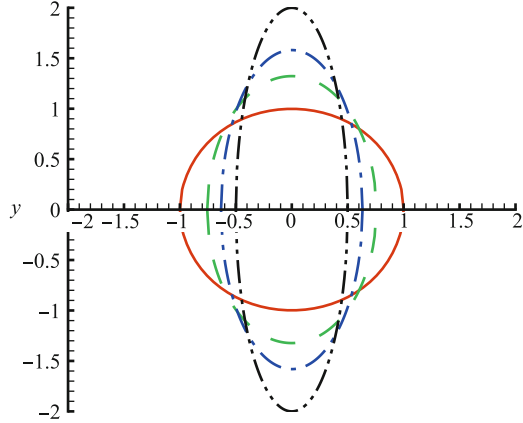
in the way

$$\mathcal{H}(x; q) = (1 - q)f(x) + qg(x), \quad x \in [a, b]. \quad (2.2)$$

However, a continuous real function can *not* be deformed continuously into a discontinuous function. For example, $\sin(x)$ can not be deformed continuously into the step function

$$s(x) = \begin{cases} 1, & \text{when } x < 0, \\ 0, & \text{when } x = 0, \\ -1, & \text{when } x > 0. \end{cases}$$

Fig. 2.2 Continuous deformation of the solution $y(x; q)$ of the homotopy equation (2.3). Solid line: $q = 0$; Dashed line: $q = 1/4$; Dash-dotted line: $q = 1/2$; Dash-double-dotted line: $q = 1$.



Definition 2.1. A *homotopy* between two continuous functions $f(\mathbf{x})$ and $g(\mathbf{x})$ from a topological space $\mathbf{X}$ to a topological space $\mathbf{Y}$ is formally defined to be a continuous function $\mathcal{H} : \mathbf{X} \times [0, 1] \rightarrow \mathbf{Y}$ from the product of the space $\mathbf{X}$ with the unit interval $[0, 1]$ to $\mathbf{Y}$ such that, if $\mathbf{x} \in \mathbf{X}$ then $\mathcal{H}(\mathbf{x}; 0) = f(\mathbf{x})$ and $\mathcal{H}(\mathbf{x}; 1) = g(\mathbf{x})$.

Since curves can be defined by algebraic or differential equations, the concept of homotopy defined above for functions can be easily expanded to equations. For example, let us consider such a family of algebraic equations

$$\mathcal{E}(q) : (1 + 3q)x^2 + \frac{y^2}{(1 + 3q)} = 1, \quad q \in [0, 1], \quad (2.3)$$

where $q \in [0, 1]$ is the embedding parameter. When $q = 0$, we have the circle equation

$$\mathcal{E}_0 : x^2 + y^2 = 1, \quad (2.4)$$

whose solution is a circle $y = \pm\sqrt{1 - x^2}$. When $q = 1$, we have the ellipse equation

$$\mathcal{E}_1 : 4x^2 + y^2/4 = 1, \quad (2.5)$$

whose solution is an ellipse $y = \pm 2\sqrt{1 - 4x^2}$. Thus, as the embedding parameter q increases from 0 to 1, Equation (2.3) varies continuously from the circle equation $\mathcal{E}_0$ into the ellipse equation $\mathcal{E}_1$, while its solution y deforms continuously from the circle $y = \pm\sqrt{1 - x^2}$ to the ellipse $y = \pm 2\sqrt{1 - 4x^2}$, as shown in Fig. 2.2. So, more precisely speaking, the solution y of (2.3) is dependent not only on x but also on $q \in [0, 1]$, and thus (2.3) should be expressed more precisely in the form

$$\mathcal{E}(q) : (1 + 3q)x^2 + \frac{y^2(x; q)}{(1 + 3q)} = 1, \quad q \in [0, 1], \quad (2.6)$$

which defines *two* homotopies: one is the *homotopy of equation*

$$\mathcal{E}(q) : \mathcal{E}_0 \sim \mathcal{E}_1,$$

where $\mathcal{E}_0$ and $\mathcal{E}_1$ denote (2.4) and (2.5), respectively, the other is the *homotopy of function*

$$y(x; q) : \pm \sqrt{1 - x^2} \sim \pm 2\sqrt{1 - 4x^2}.$$

In other words, the solution $y(x; q)$ of (2.6) is also a homotopy. Notice that such kind of continuous deformation is completely defined by (2.6). For simplicity, we call (2.6) *the zeroth-order deformation equation*. The same idea can be easily extended to other types of equations, such as differential equations, integral equations and so on, as shown later in this book.

Definition 2.2. The embedding parameter $q \in [0, 1]$ in a homotopy of functions or equations is called *homotopy-parameter*.

Definition 2.3. Given an equation denoted by $\mathcal{E}_1$, which has at least one solution u . Let $\mathcal{E}_0$ denote a proper, simpler equation, called *the initial equation*, whose solution u_0 is known. If one can construct a homotopy of equation $\tilde{\mathcal{E}}(q) : \mathcal{E}_0 \sim \mathcal{E}_1$ such that, as the homotopy-parameter $q \in [0, 1]$ increases from 0 to 1, $\tilde{\mathcal{E}}(q)$ deforms (or varies) *continuously* from the the initial equation $\mathcal{E}_0$ to the the original equation $\mathcal{E}_1$, while its solution varies *continuously* from the known solution u_0 of $\mathcal{E}_0$ to the unknown solution u of $\mathcal{E}_1$, then this kind of homotopy of equations is called *the zeroth-order deformation equation*.

Note that we can construct many different homotopies which connect the circle equation (2.4) and the ellipse equation (2.5). For example, the following zeroth-order deformation equation

$$\mathcal{E}(q, \mu) : (1 + 3q^\mu)x^2 + \frac{y^2(x; q)}{(1 + 3q^\mu)} = 1, \quad q \in [0, 1], \quad (2.7)$$

where $\mu > 0$ is a constant, defines a two-parameter (q, μ) family of homotopy

$$\mathcal{E}(q, \mu) : \mathcal{E}_0 \sim \mathcal{E}_1,$$

where $\mathcal{E}_0$ and $\mathcal{E}_1$ denote (2.4) and (2.5), respectively. For different values of μ , it defines a different homotopy. Since $\mu \in (0, +\infty)$, there exists an infinite number of different homotopies of equations which connect the circle equation (2.4) and the ellipse equation (2.5), and correspondingly, an infinite number of homotopies of functions which connect the circle $y = \pm \sqrt{1 - x^2}$ and the ellipse $y = \pm 2\sqrt{1 - 4x^2}$. This illustrates the great flexibility of constructing a homotopy for given two homotopic functions or equations. All of these belong to the basic concepts in topology and differential geometry (Armstrong, 1983; Sen, 1983).

Based on the homotopy mentioned above, some new concepts can be derived. Note that the homotopy

$$\mathcal{H}(x; q) = (1 - q) \sin(\pi x) + q [8x(x - 1)]$$

can be rewritten in the form

$$\mathcal{H}(x; q) = \sin(\pi x) + [8x(x - 1) - \sin(\pi x)] q,$$

and thus we have

$$\frac{\partial \mathcal{H}(x; q)}{\partial q} = 8x(x - 1) - \sin(\pi x), \quad q \in [0, 1], \quad (2.8)$$

which describes the *ratio* (or the *speed*) of the continuous deformation from $\sin(\pi x)$ to $8x(x - 1)$, called the *1st-order homotopy-derivative*. In general, the homotopy

$$\mathcal{H}(x; q) = (1 - q) f(x) + q g(x), \quad x \in [a, b]$$

completely defines the corresponding 1st-order homotopy-derivative

$$\frac{\partial \mathcal{H}(x; q)}{\partial q} = g(x) - f(x), \quad q \in [0, 1]. \quad (2.9)$$

Generalizing this concept, we can further define the high-order homotopy-derivatives, as shown later.

Since topology is not a common course for undergraduate students, most readers are unfamiliar with the concept of homotopy. Fortunately, the HAM is based on the simple fundamental concept of homotopy only, and other knowledge in topology are almost unnecessary, as shown later in this book. Thus, it is easy to understand and apply the HAM to gain analytic approximations of nonlinear differential equations in practice. In the following part of this chapter, we will use two simple examples to describe the basic ideas of the HAM in details. One is a nonlinear algebraic equation, the other is a 2nd-order differential equation for periodic oscillations.

2.2 Example 2.1: generalized Newtonian iteration formula

Let us first consider a nonlinear algebraic equation

$$f(x) = 0,$$

where $f(x) \in C^\infty[a, b]$ is a continuous real function. Assume that the above equation has at least one solution in the region $x \in [a, b]$.

Let $x_0 \in [a, b]$ denote an initial guess of the unknown solution x . Obviously, $f(x) - f(x_0) \in C^\infty[a, b]$ can be deformed continuously to $f(x) \in C^\infty[a, b]$, i.e. they are homotopic. Thus, we can construct such a homotopy of function

$$\mathcal{H}(x; q) = (1 - q) [f(x) - f(x_0)] + q f(x),$$

where $q \in [0, 1]$ is the homotopy-parameter. When $q = 0$ and $q = 1$, we have

$$\mathcal{H}(x; 0) = f(x) - f(x_0), \quad \mathcal{H}(x; 1) = f(x),$$

respectively. Thus, as q increases from 0 to 1, $\mathcal{H}(x; q)$ varies continuously from $f(x) - f(x_0)$ to $f(x)$. Such kind of continuous variation is called *deformation* in topology (Sen, 1983).

Now, enforcing $\mathcal{H}(x; q) = 0$, i.e.

$$(1 - q)[f(x) - f(x_0)] + q f(x) = 0, \quad q \in [0, 1],$$

we have now a parameter-family of algebraic equations. The solution of the above parameter-family of algebraic equations is dependent upon the homotopy-parameter q . Replacing x by $\tilde{x}(q)$, this family of equations can be rewritten more precisely in the form

$$(1 - q)\{f[\tilde{x}(q)] - f(x_0)\} + q f[\tilde{x}(q)] = 0. \quad (2.10)$$

When $q = 0$, we have

$$f[\tilde{x}(0)] - f(x_0) = 0,$$

whose solution is

$$\tilde{x}(0) = x_0.$$

When $q = 1$, we have

$$f[\tilde{x}(1)] = 0,$$

which is exactly the same as the original algebraic equation $f(x) = 0$. So, it holds

$$\tilde{x}(1) = x.$$

Therefore, as the homotopy-parameter q increases from 0 to 1, $\tilde{x}(q)$ deforms from the initial guess x_0 to the solution x of the original equation $f(x) = 0$. So, Equation (2.10) defines a homotopy of function $\tilde{x}(q) : x_0 \sim x$. For simplicity, the parameter-family of equations (2.10) is called *the zeroth-order deformation equation*, because it defines a continuous deformation from the initial guess x_0 to the solution x of the original equation $f(x) = 0$.

Note that $\tilde{x}(q)$ is a function of the homotopy-parameter q . Assume that $\tilde{x}(q)$ is *analytic* at $q = 0$ so that it can be expanded into a Maclaurin series with respect to the homotopy-parameter q , i.e.

$$\tilde{x}(q) \sim x_0 + \sum_{k=1}^{+\infty} x_k q^k, \quad (2.11)$$

where $\tilde{x}(0) = x_0$ is employed, and

$$x_k = \frac{1}{k!} \left. \frac{d^k \tilde{x}(q)}{dq^k} \right|_{q=0} = \mathcal{D}_k [\tilde{x}(q)]. \quad (2.12)$$

Here, the series (2.11) is called *homotopy-Maclaurin series*, $\mathcal{D}_k$ is called *the homotopy-derivative operator*, and $\mathcal{D}_k[\tilde{x}(q)]$ is called *the k th-order homotopy-derivative* of $\tilde{x}(q)$. We will give more rigorous definitions and some related theorems about the operator $\mathcal{D}_k$ in Chapter 3. Note that $\mathcal{D}_1[\tilde{x}(q)] = x_1$ is exactly the above-mentioned deformation-ratio (2.9) of $\tilde{x}(q)$ at $q = 0$. So, $\mathcal{D}_k[\tilde{x}(q)]$, the k th-order homotopy-derivative ($k \geq 1$) of $\tilde{x}(q)$, can be regarded as the high-order deformation-ratio of $\tilde{x}(q)$ at $q = 0$.

Assuming that $\tilde{x}(q)$ is *analytic* in $q \in [0, 1]$, then the homotopy–Maclaurin series (2.11) is convergent at $q = 1$ to $\tilde{x}(1)$. Thus, using the relationship $\tilde{x}(1) = x$, we have from (2.11) the so-called *homotopy-series solution*

$$x = x_0 + \sum_{k=1}^{+\infty} x_k. \quad (2.13)$$

It should be emphasized that we had to make such an assumption here, because a Maclaurin series of a function $f(x)$ may not converge to $f(x)$. The assumption may come true by means of properly constructing the zeroth-order deformation equation, as shown later. In practice, only finite terms can be obtained, and the M th-order approximation of x is given by

$$x \approx x_0 + \sum_{k=1}^M x_k. \quad (2.14)$$

So, as long as $x_1, x_2, \dots, x_M$ become known, we obtain the M th-order approximation of x by means of the above formula.

Definition 2.4. Given a nonlinear equation denoted by $\mathcal{E}_1$, which has at least one solution $u(\mathbf{z}, t)$, where $\mathbf{z}$ and t denote the spatial and temporal independent variables, respectively. Let $q \in [0, 1]$ denote a homotopy-parameter and $\tilde{\mathcal{E}}(q)$ the zeroth-order deformation equation, which connects the original equation $\mathcal{E}_1$ and an initial equation $\mathcal{E}_0$ with the known initial approximation $u_0(\mathbf{z}, t)$. Assuming that the zeroth-order deformation equation $\tilde{\mathcal{E}}(q)$ is so properly constructed that its solution $\phi(\mathbf{z}, t; q)$ exists and is analytic at $q = 0$, we have the *homotopy-Maclaurin series*:

$$\phi(\mathbf{z}, t; q) \sim u_0(\mathbf{z}, t) + \sum_{n=1}^{+\infty} u_n(\mathbf{z}, t) q^n, \quad q \in [0, 1]$$

and the *homotopy-series*:

$$\phi(\mathbf{z}, t; 1) \sim u_0(\mathbf{z}, t) + \sum_{n=1}^{+\infty} u_n(\mathbf{z}, t).$$

The equations related to the unknown $u_n(\mathbf{z}, t)$ are called *the n th-order deformation equations*.

Definition 2.5. If the solution $\phi(\mathbf{z}, t; q)$ of the zeroth-order deformation equation $\tilde{\mathcal{E}}(q) : \mathcal{E}_0 \sim \mathcal{E}_1$ exists and is analytic about q in $q \in [0, 1]$, then we have the *homotopy-series solution* of the original equation $\mathcal{E}_1$:

$$u(\mathbf{z}, t) = u_0(\mathbf{z}, t) + \sum_{n=1}^{+\infty} u_n(\mathbf{z}, t),$$

and the *Mth-order homotopy-approximation*

$$u(\mathbf{z}, t) \approx u_0(\mathbf{z}, t) + \sum_{n=1}^M u_n(\mathbf{z}, t).$$

According to the fundamental theorem in calculus about Taylor series, the coefficient x_k of the homotopy-Maclaurin series (2.11) is unique. Therefore, the governing equation of x_k is unique, too, and can be deduced directly from the zeroth-order deformation equation (2.10). First, differentiating the zeroth-order deformation equation (2.10) with respect to the homotopy-parameter q , we have

$$f'[\tilde{x}(q)] \frac{d\tilde{x}(q)}{dq} + f(x_0) = 0. \quad (2.15)$$

Then, setting $q = 0$ in the above equation and using the relationship $\tilde{x}(0) = x_0$, we have

$$f'(x_0) \left. \frac{d\tilde{x}(q)}{dq} \right|_{q=0} + f(x_0) = 0.$$

According to the definition (2.12), we have

$$\left. \frac{d\tilde{x}(q)}{dq} \right|_{q=0} = x_1.$$

Thus, we obtain the so-called *1st-order deformation equation*

$$x_1 f'(x_0) + f(x_0) = 0,$$

whose solution is

$$x_1 = -\frac{f(x_0)}{f'(x_0)}.$$

Similarly, differentiating the zeroth-order deformation equation (2.10) twice with respect to the homotopy-parameter q and dividing it by $2!$, we have

$$\frac{1}{2} f''(\tilde{x}) \left(\frac{d\tilde{x}}{dq} \right)^2 + f'(\tilde{x}) \left(\frac{1}{2!} \frac{d^2 \tilde{x}}{dq^2} \right) = 0. \quad (2.16)$$

Then, setting $q = 0$ and using $\tilde{x}(0) = x_0$, the above equation reads

$$\frac{1}{2}f''(x_0)\left(\left.\frac{d\tilde{x}}{dq}\right|_{q=0}\right)^2 + f'(x_0)\left(\left.\frac{1}{2!}\frac{d^2\tilde{x}}{dq^2}\right|_{q=0}\right) = 0.$$

Using the definition (2.12), we have the *2nd-order deformation equation*

$$\frac{1}{2}x_1^2 f''(x_0) + x_2 f'(x_0) = 0,$$

whose solution is

$$x_2 = -\frac{x_1^2 f''(x_0)}{2f'(x_0)} = -\frac{f^2(x_0)f''(x_0)}{2[f'(x_0)]^3}.$$

Alternatively, using the linear operator $\mathcal{D}_k$ defined by (2.12) and directly taking the 2nd-order homotopy-derivative on both sides of the zeroth-order deformation equation (2.10), we obtain exactly the same 2nd-order deformation equation as mentioned above. In this way, we obtain x_k one by one in the order $k = 1, 2, 3, \dots$. In general, given a zeroth-order deformation equation, it is straightforward to give the corresponding high-order deformation equations, as described in Chapter 3. Here, we would like to emphasize that all of above high-order deformation equations are *linear*, and therefore are easy to solve.

Using (2.14), we have the *1st-order homotopy-approximation*

$$x \approx x_0 + x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, \quad (2.17)$$

and the *2nd-order homotopy-approximation*

$$x \approx x_0 + x_1 + x_2 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{f^2(x_0)f''(x_0)}{2[f'(x_0)]^3}. \quad (2.18)$$

Note that (2.17) is exactly the famous Newton's iteration formula given by Sir Isaac Newton (1643 — 1727), and (2.18) is the Olver's iteration formula. In fact, one can give a family of iteration formulas in a similar way. This shows the potential of the homotopy approach.

The above approach has some interesting characteristics. First of all, it is *independent* of any physical parameters at all: no matter whether a nonlinear problem contains small/large physical parameters or not, one can always introduce the homotopy-parameter $q \in [0, 1]$ to construct such a zeroth-order deformation equation and then to obtain the homotopy-series solution or a homotopy approximation. Secondly, as mentioned above, all high-order deformation equations are linear and thus are easy to solve. In this way, this approach transforms a nonlinear equation into an infinite number of linear sub-problems, but does not need any small/large physical parameters. Giving up the dependence on small/large physical parameters, the HAM can be applied to solve more complicated nonlinear problems, as shown later in this book.

However, the above approach has a limitation: the convergence of homotopy-Maclaurin series (2.11) at $q = 1$ is not guaranteed, therefore the homotopy-series

solution (2.13) might be divergent. This is mainly because the above approach is based on such an assumption that $\tilde{x}(q)$ is *analytic* in $q \in [0, 1]$ that the homotopy-Maclaurin series (2.11) is convergent at $q = 1$, but it does not provide a way to guarantee that such an assumption indeed holds, especially for nonlinear problems with strong nonlinearity. This explains why the famous Newton's iteration formula (2.17) and Olver's iteration formula (2.18) often fail.

To overcome this limitation of the early HAM mentioned above, Liao (1997) greatly modified the approach by introducing a non-zero auxiliary parameter c_0 , called now *the convergence-control parameter*¹, to construct such a more generalized zeroth-order deformation equation

$$(1 - q) \{f[\tilde{x}(q)] - f(x_0)\} = q c_0 f[\tilde{x}(q)]. \quad (2.19)$$

Since $c_0 \neq 0$, when $q = 1$, the above equation becomes

$$c_0 f[\tilde{x}(1)] = 0,$$

which is equivalent to the original equation $f(x) = 0$, provided $x = \tilde{x}(1)$. All other formulas like (2.11) and (2.13) are the same, except the high-order deformation equation, which can be derived as follows.

Taking the 1st-order homotopy-derivative on both sides of the zeroth-order deformation equation (2.19), i.e. differentiating (2.19) with respect to q and then setting $q = 0$, we have the corresponding 1st-order deformation equation

$$x_1 f'(x_0) - c_0 f(x_0) = 0,$$

whose solution is

$$x_1 = c_0 \frac{f(x_0)}{f'(x_0)}.$$

Taking the 2nd-order homotopy-derivative on both sides of the zeroth-order deformation equation (2.19), i.e. differentiating (2.19) twice with respect to q and then setting $q = 0$ and dividing by $2!$, we have the 2nd-order deformation equation

$$x_2 f'(x_0) - (1 + c_0)x_1 f'(x_0) + \frac{1}{2}x_1^2 f''(x_0) = 0,$$

whose solution reads

$$x_2 = (1 + c_0)x_1 - \frac{x_1^2 f''(x_0)}{2f'(x_0)} = c_0(1 + c_0) \frac{f(x_0)}{f'(x_0)} - \frac{c_0^2 f^2(x_0) f''(x_0)}{2[f'(x_0)]^3}.$$

Similarly, we have

¹ The same non-zero auxiliary parameter was denoted by $\hbar$ when Liao (1997) first introduced it into the frame of the HAM. However, since $\hbar$ has a special meaning in quantum mechanics, we replace $\hbar$ by c_0 in the whole book, which means the "basic" convergence-control parameter.

$$x_3 = c_0(1+c_0)^2 \frac{f(x_0)}{f'(x_0)} - c_0^2(1+c_0) \frac{f^2(x_0)f''(x_0)}{[f'(x_0)]^3} \\ + \frac{c_0^3 f^3(x_0) \left\{ 3 [f''(x_0)]^2 - f'(x_0)f'''(x_0) \right\}}{6[f'(x_0)]^5},$$

and so on.

Then, according to (2.14), we have the corresponding first-order homotopy-approximation

$$x \approx x_0 + x_1 = x_0 + c_0 \frac{f(x_0)}{f'(x_0)}, \quad (2.20)$$

the 2nd-order homotopy-approximation

$$x \approx x_0 + x_1 + x_2 = x_0 + c_0(c_0 + 2) \frac{f(x_0)}{f'(x_0)} - \frac{c_0^2 f^2(x_0)f''(x_0)}{2[f'(x_0)]^3}, \quad (2.21)$$

and the 3rd-order homotopy-approximation

$$x \approx x_0 + x_1 + x_2 + x_3 \\ = x_0 + c_0(c_0^2 + 3c_0 + 3) \frac{f(x_0)}{f'(x_0)} - c_0^2(2c_0 + 3) \frac{f^2(x_0)f''(x_0)}{2[f'(x_0)]^3} \\ + \frac{c_0^3 f^3(x_0) \left\{ 3 [f''(x_0)]^2 - f'(x_0)f'''(x_0) \right\}}{6[f'(x_0)]^5}. \quad (2.22)$$

By means of computer algebra system, it is easy to derive an iteration formula at any given order of approximation.

It is interesting that Newton's iteration formula (2.17) and Olver's iteration formula (2.18) are only special cases of (2.20) and (2.21) in case of $c_0 = -1$, respectively. In practice, the convergence-control parameter c_0 in (2.20) and (2.22) can be regarded as an iteration factor, which is widely used in numerical computations to modify Newton's iteration formula (2.17) and Olver's iteration formula (2.18). It is well-known that a properly chosen iteration factor can greatly modify the convergence of many numerical iteration approaches. Similarly, the convergence-control parameter c_0 can greatly modify the convergence of the homotopy-series solution. This is indeed true. We will show this point in details later in this book: it is the convergence-control parameter c_0 that provides us a simple way to ensure the convergence of homotopy-control series solution. That is the reason why we call c_0 the convergence-control parameter. In this way, the above mentioned limitation of the early HAM is overcome. Indeed, the convergence-control parameter c_0 in the zeroth-order deformation equation introduced by Liao (1997) greatly modifies the early HAM. We will show this point in details by means of Example 2.2.

The Padé technique can be applied to give a new iteration formula. Regarding q as an independent variable, then using $[1,1]$ Padé approximant about q to the homotopy-series

$$\tilde{x}(q) \approx x_0 + x_1 q + x_2 q^2 + x_3 q^3 + \cdots,$$

and finally setting $q = 1$, we have the iteration formula

$$x \approx x_0 + \frac{2f(x_0)f'(x_0)}{f(x_0)f''(x_0) - 2[f'(x_0)]^2}. \quad (2.23)$$

It is very interesting that the above formula is *independent* of the convergence-control parameter c_0 . The so-called homotopy-Padé technique mentioned above provides us an alternative way to modify the convergence of homotopy series solution, as shown later in details in Sect. 2.3.5.

2.3 Example 2.2: nonlinear oscillation

The approach described above also works for other types of nonlinear equations. For example, let us consider here a body moving on a smooth horizontal plane acted by a horizontal force f . Let m denote the mass of the body, t the time, and $x(t)$ the horizontal co-ordinate of the body, as shown in Fig. 2.3. Assume that there is no friction force between the body and the plane. According to Newton's second law, the motion of the body is described by

$$m\ddot{x}(t) = f,$$

where the dot denotes the differentiation with respect to the time t . For simplicity, we consider here such a case

$$x(0) = x^*, \quad \dot{x}(0) = 0, \quad m = 1, \quad f = -(\lambda x + \varepsilon x^3)$$

that the motion of the body is described by

$$\ddot{x}(t) + \lambda x(t) + \varepsilon x^3(t) = 0, \quad x(0) = x^*, \quad \dot{x}(0) = 0, \quad (2.24)$$

where $\lambda \in (-\infty, +\infty)$ and $\varepsilon \in (-\infty, +\infty)$ are constant physical parameters.

2.3.1 Analysis of the solution characteristic

Let us first consider the case of $\varepsilon = 0$, i.e.

$$\ddot{x}(t) + \lambda x(t) = 0, \quad x(0) = x^*, \quad \dot{x}(0) = 0, \quad (2.25)$$

whose solution is

Fig. 2.3 A body moving on a smooth horizontal plane acted by a horizontal force f .

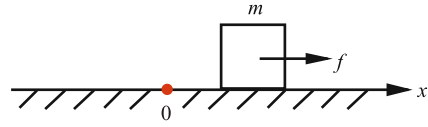
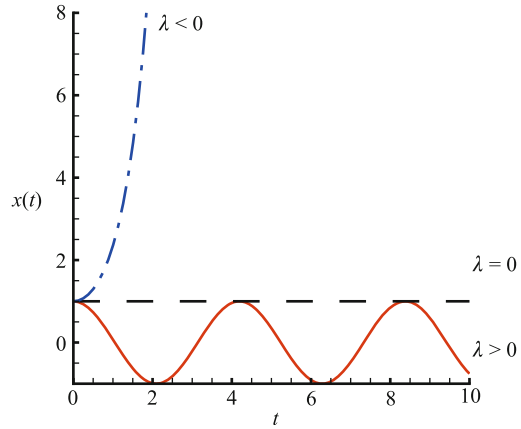


Fig. 2.4 Solutions of equation $\ddot{x} + \lambda x = 0$ with $x(0) = 1$ and $\dot{x}(0) = 0$. Solid line: when $\lambda = 9/4$; Dashed line: when $\lambda = 0$; Dash-dotted line: when $\lambda = -9/4$.



$$x(t) = \begin{cases} x^* \cos(\sqrt{\lambda} t), & \text{when } \lambda > 0, \\ x^*, & \text{when } \lambda = 0, \\ x^* \cosh(\sqrt{|\lambda|} t), & \text{when } \lambda < 0. \end{cases} \quad (2.26)$$

As shown in Fig. 2.4, the solution $x(t)$ has quite different characteristic for different values of λ : it is a periodic function when $\lambda > 0$ but quickly tends to infinity when $\lambda < 0$.

We can explain this from the physical points of view. In general, the equilibrium point of a dynamic system is defined by $f = 0$. Enforcing $f = -\lambda x = 0$, we have the equilibrium point $x = 0$ for both $\lambda > 0$ and $\lambda < 0$. When $\lambda > 0$, the force $f = -\lambda x = -|\lambda|x$ always points to $x = 0$ so that $x = 0$ is a stable equilibrium point, thus the body oscillates around $x = 0$ and is expressed by a periodic function. However, when $\lambda < 0$, the force $f = -\lambda x = |\lambda|x$ never points to the equilibrium point $x = 0$ so that $x = 0$ is an unstable equilibrium point, and thus the body departs from the starting point $x = x^*$ farther and farther, and will never return, i.e. $x(t)$ is expressed by an exponential function. When $\lambda = 0$, the body is still, because there is no force acted on the body. These are the physical reasons why the solution $x(t)$ has quite different characteristic for $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$. Note that, $\sqrt{\lambda}$ denotes the frequency of oscillation in case of $\lambda > 0$, and $\sqrt{|\lambda|}$ denotes the escaping-speed of the body from the starting-point in case of $\lambda < 0$.

In summary, the characteristic of solution and the equilibrium point of linear equation $\ddot{x} + \lambda x = 0$ are listed in the Table 2.1.

In case of $\varepsilon \neq 0$, the equilibrium point is determined by

$$f = -\lambda x - \varepsilon x^3 = -\varepsilon x \left(\frac{\lambda}{\varepsilon} + x^2 \right) = 0. \quad (2.27)$$

Table 2.1 The characteristic of the solution and the equilibrium point of $\ddot{x} + \lambda x = 0$ with $x(0) = x^*, \dot{x}(0) = 0$.

	$\lambda > 0$	$\lambda = 0$	$\lambda < 0$
Solution expression	periodic	constant	exponential
Equilibrium point	$x = 0$, stable	$x = x^*$, neutral	$x = 0$, unstable

When $\lambda/\varepsilon \geq 0$, Equation (2.27) has only one real solution $x = 0$. When $\lambda/\varepsilon < 0$, it has three real solutions $x = 0$ and $x = \pm \sqrt{|\lambda/\varepsilon|}$. Physically, we have one equilibrium point $x = 0$ in case of $\lambda/\varepsilon \geq 0$, but have three equilibrium points $x = 0$ and $x = \pm \sqrt{|\lambda/\varepsilon|}$ in case of $\lambda/\varepsilon < 0$. The characteristics of the equilibrium point and solution for different values of ε and λ are listed below:

- In case of $\varepsilon > 0$ and $\lambda \geq 0$, the force $f = -(\lambda + \varepsilon x^2)x$ always points to the unique equilibrium point $x = 0$, so that $x = 0$ is a stable equilibrium point and thus the body always oscillates about $x = 0$, i.e. $x(t)$ is a periodic function.
- In case of $\varepsilon > 0$ but $\lambda < 0$, the force

$$f = -\lambda x - \varepsilon x^3 = |\lambda| x - \varepsilon x^3 = -(\varepsilon x^2 - |\lambda|) x$$

points to either the equilibrium point $x = +\sqrt{|\lambda|/\varepsilon}$ or the equilibrium point $x = -\sqrt{|\lambda|/\varepsilon}$. Thus, we have two stable equilibrium points $x = \pm \sqrt{|\lambda|/\varepsilon}$ but one unstable equilibrium point $x = 0$. So, the body oscillates about one of the stable equilibrium points $x = \pm \sqrt{|\lambda|/\varepsilon}$, and the solution is a periodic function.

- In case of $\varepsilon < 0$ and $\lambda > 0$, the force

$$f = -\lambda x - \varepsilon x^3 = |\varepsilon| x \left(x^2 - \frac{|\lambda|}{\varepsilon} \right)$$

points to the equilibrium point $x = 0$ when $|x^*| < \sqrt{|\lambda/\varepsilon|}$, but never points to any equilibrium points when $|x^*| > \sqrt{|\lambda/\varepsilon|}$. So, $x = 0$ is a stable equilibrium point, and $x = \pm \sqrt{|\lambda/\varepsilon|}$ are two unstable equilibrium points. Thus, the body oscillates about the stable equilibrium point $x = 0$ when $|x^*| < \sqrt{|\lambda/\varepsilon|}$, but departs from the starting point to infinity when $|x^*| > \sqrt{|\lambda/\varepsilon|}$.

- In case of $\varepsilon < 0$ and $\lambda \leq 0$, the force $f = -(\lambda + \varepsilon x^2)x = (|\lambda| + |\varepsilon|x^2)x$ never points to the unique equilibrium point $x = 0$, so that $x = 0$ is a unstable equilibrium point and thus the body departs from the starting point to infinity.

The characteristic of the equilibrium point and solution expression of the nonlinear dynamic system (2.24) for different values of ε and λ are listed in Tables 2.2 and 2.3, respectively. Note that, in case of $\varepsilon > 0$, the body always oscillates around

Table 2.2 Characteristic of equilibrium point of the nonlinear dynamic system (2.24).

	$\varepsilon > 0$	$\varepsilon < 0$
$\lambda \geq 0$	one equilibrium point: $x = 0$, stable	three equilibrium points: $x = 0$, stable; $x = \pm \sqrt{ \lambda/\varepsilon }$, unstable
$\lambda < 0$	three equilibrium point: $x = 0$, unstable; $x = \pm \sqrt{ \lambda/\varepsilon }$, stable	one equilibrium point: $x = 0$, unstable

Table 2.3 Characteristic of the solution expression of the nonlinear dynamic system (2.24).

	$\varepsilon > 0$	$\varepsilon < 0$
$\lambda \geq 0$	periodic	periodic: when $ x^* < \sqrt{ \lambda/\varepsilon }$ exponential: when $ x^* > \sqrt{ \lambda/\varepsilon }$
$\lambda < 0$	periodic	exponential

a stable equilibrium point, for *arbitrary* value of λ , so that $x(t)$ is always a periodic function. Besides, in case of $\varepsilon < 0$, there exists a periodic solution when $\lambda > 0$ and $|x^*| < \sqrt{|\lambda/\varepsilon|}$. These periodic solutions describe the periodic oscillations around a stable equilibrium point. In other cases, the solution is an exponential function, describing the quick escaping of the body from the starting point to infinity.

Therefore, dependent on the physical parameters λ , ε and the starting position x^* , there are two different types of solutions for the nonlinear dynamic system (2.24). One is a periodic function, describing the periodic oscillation around a stable equilibrium point. In this case, the important physical quantity is the frequency of the motion, denoted by ω . The other is a non-periodic function, which tends to infinity exponentially as $t \rightarrow +\infty$, describing the quick escaping of the body from the starting point x^* to infinity. In this case, there exists an important physical quantity, denoted by μ , which describes the escaping speed of the body from the starting point x^* to infinity. Each of ω and μ corresponds to a kind of time scale, so that we can define a dimensionless time variable $\tau = \omega t$ for the periodic function, or $\tau = \mu t$ for the non-periodic function, respectively. Although we now do not know the details of the solution of the nonlinear dynamic system (2.24), we are quite sure that the periodic solution can be expressed by

$$x = a_0 + \sum_{k=1}^{+\infty} a_k \cos(k\tau), \quad \tau = \omega t, \quad (2.28)$$

and the non-periodic solution can be expressed by

$$x = b_0 + \sum_{k=-\infty}^{+\infty} b_k \exp(k\tau), \quad \tau = \mu t, \quad (2.29)$$

respectively, where a_k, b_k are constant to be determined. Our aim is to find convergent series in above forms for all possible physical parameters $-\infty < \lambda < +\infty$ and $-\infty < \varepsilon < +\infty$. For simplicity, we call (2.28) *the solution expression* for periodic oscillations of the nonlinear dynamic system (2.24), and (2.29) *the solution expression* for non-periodic ones, respectively.

Definition 2.6. Given a differential equation $\mathcal{E}$ that has at least one solution u . If there exists such a complete set of base functions $\mathbf{e}_k$ ($0 \leq k < +\infty$) that the series $\sum_{k=0}^{+\infty} a_k \mathbf{e}_k$ converges to the solution u , then $u = \sum_{k=0}^{+\infty} a_k \mathbf{e}_k$ is called the *solution expression* of the given equation $\mathcal{E}$.

Based on the above analysis, we define two different sets of functions

$$\mathbf{S}_p := \left\{ x(\tau) \mid x(\tau) = a_0 + \sum_{k=1}^{+\infty} a_k \cos(k\tau), \quad a_k \text{ is constant} \right\} \quad (2.30)$$

and

$$\mathbf{S}_e := \left\{ x(\tau) \mid x(\tau) = b_0 + \sum_{k=-\infty}^{+\infty} b_k \exp(k\tau), \quad b_k \text{ is constant} \right\}. \quad (2.31)$$

Obviously, if $f_1 \in \mathbf{S}_p$ and $f_2 \in \mathbf{S}_p$ are two periodic functions, then f_1 can be continuously deformed into f_2 . So, $f_1 \in \mathbf{S}_p$ and $f_2 \in \mathbf{S}_p$ are homotopic. Similarly, two exponential functions $f_1 \in \mathbf{S}_e$ and $f_2 \in \mathbf{S}_e$ are homotopic, too. However, it is impossible for a periodic function $f_1 \in \mathbf{S}_p$ to be deformed into an exponential function $f_2 \in \mathbf{S}_e$.

The above analysis of the characteristic of the solution and the equilibrium point of the nonlinear dynamic system (2.24) is rather helpful for the choice of the initial guess of $x(t)$ and the auxiliary linear operator $\mathcal{L}$ in the frame of the HAM, as mentioned below.

When the starting position $x(0) = x^*$ is at the equilibrium point, the solution is $x(t) = x^*$. So, $x = x^*$ is assumed to be not the equilibrium point. In addition, only the periodic solutions are considered for the sake of simplicity. As mentioned above, the periodic solutions exist when $\varepsilon > 0$, or $\varepsilon < 0, \lambda > 0$ and $|x^*| < \sqrt{|\lambda/\varepsilon|}$.

Let ω and $T = 2\pi/\omega$ denote the frequency and the period of the solution $x(t)$, respectively. Using the transformation $\tau = \omega t$, Equation (2.24) becomes

$$\gamma x''(\tau) + \lambda x(\tau) + \varepsilon x^3(\tau) = 0, \quad x(0) = x^*, x'(\tau) = 0, \quad (2.32)$$

where the prime denotes differentiation with respect to τ , and $\gamma = \omega^2$ is an unknown constant, which depends on ε, λ and x^* . As mentioned above, although the frequency square γ is unknown, we are quite sure that $x(\tau)$ is a function with the known period 2π , expressed by (2.28), i.e. $x(\tau) \in \mathbf{S}_p$. Our aim is to give such kind of

a series solution convergent for *all* possible physical parameters λ, ε and the starting point x^* .

2.3.2 Mathematical formulations

Let $x_0(\tau)$ denote the initial approximation of $x(\tau)$. According to the solution expression (2.28) and considering the initial conditions in (2.24), we choose

$$x_0(\tau) = \beta + (x^* - \beta) \cos \tau, \quad (2.33)$$

where $x = \beta$ corresponds to the stable equilibrium point near the starting position $x = x^*$. According to Tables 2.2 and 2.3, we have

$$\beta = \begin{cases} 0, & \text{when } \varepsilon > 0 \text{ and } \lambda \geq 0, \\ \sqrt{|\lambda/\varepsilon|}, & \text{when } \varepsilon > 0, \lambda < 0 \text{ and } x^* > 0, \\ -\sqrt{|\lambda/\varepsilon|}, & \text{when } \varepsilon > 0, \lambda < 0 \text{ and } x^* < 0, \\ 0, & \text{when } \varepsilon < 0, \lambda \geq 0 \text{ and } |x^*| < \sqrt{|\lambda/\varepsilon|}. \end{cases} \quad (2.34)$$

Note that the above defined $x_0(\tau)$ satisfies the initial conditions $x(0) = x^*$ and $x'(0) = 0$ of the nonlinear dynamic system (2.32).

Let $\mathcal{L}$ denote an auxiliary linear operator with the property $\mathcal{L}(0) = 0$. We will show how to choose $\mathcal{L}$ later. Here, we just emphasize that we have great freedom to choose the so-called auxiliary linear operator $\mathcal{L}$.

Let c_0 denote the convergence-control parameter, $q \in [0, 1]$ the homotopy-parameter, respectively. For the sake of simplicity, we define such a nonlinear operator

$$\mathcal{N}(x) = \gamma x''(\tau) + \lambda x(\tau) + \varepsilon x^3(\tau). \quad (2.35)$$

Note that $x_0 \in \mathbf{S}_p$ is a periodic function, where $\mathbf{S}_p$ is defined by (2.30). Obviously, if $x(\tau) \in \mathbf{S}_p$, then $c_0 \mathcal{N}[x(\tau)] \in \mathbf{S}_p$. Assume that $\mathcal{L}$ is properly chosen so that

$$\mathcal{L}[x(\tau) - x_0(\tau)] \in \mathbf{S}_p, \quad \text{if } x(\tau) \in \mathbf{S}_p. \quad (2.36)$$

Thus, $\mathcal{L}[x(\tau) - x_0(\tau)]$ and $c_0 \mathcal{N}[x(\tau)]$ are periodic functions with the same period and therefore can be deformed into each other. So, we can construct such a homotopy of functions

$$\mathcal{H}(x; q) := (1 - q)\mathcal{L}[x(\tau) - x_0(\tau)] - c_0 q \mathcal{N}[x(\tau)]. \quad (2.37)$$

When $q = 0$ and $q = 1$, we have respectively

$$\mathcal{H}(x; 0) := \mathcal{L}[x(\tau) - x_0(\tau)], \quad \text{when } q = 0, \quad (2.38)$$

$$\mathcal{H}(x; 1) := -c_0 \mathcal{N}[x(\tau)], \quad \text{when } q = 1. \quad (2.39)$$

Thus, as q increases from 0 to 1, the homotopy $\mathcal{H}(x, q)$ continuously changes (or deforms) from the periodic function

$$\mathcal{L}[x(\tau) - x_0(\tau)] \in \mathbf{S}_p$$

to the periodic function

$$-c_0 \mathcal{N}[x(\tau)] \in \mathbf{S}_e.$$

Then, enforcing

$$\mathcal{H}(x; q) = 0,$$

we have a two-parameter (q, c_0) family of differential equations

$$(1 - q)\mathcal{L}[x - x_0(\tau)] = c_0 q \mathcal{N}(x),$$

i.e.

$$(1 - q)\mathcal{L}[x - x_0(\tau)] = c_0 q (\gamma x'' + \lambda x + \varepsilon x^3), \quad (2.40)$$

subject to the initial conditions

$$x = x^*, \quad x' = 0, \quad \text{when } \tau = 0, \quad (2.41)$$

where the dot denotes the differentiation with respect to τ . Obviously, the solution x of the above dynamic system is dependent not only on the dimensionless time $\tau \geq 0$ but also on the homotopy-parameter $q \in [0, 1]$ that has no physical meaning. So, x should be expressed more precisely by $\tilde{x}(\tau; q)$. Note that $\gamma = \omega^2$ is an *unknown* constant in the original equation (2.32), so γ in (2.40) can be regarded as a constant, too. However, because we have great freedom to construct the family of differential equation (2.40), we can also regard γ as a function of q , denoted by $\tilde{\gamma}(q)$, which varies continuously from $\tilde{\gamma}(0) = \gamma_0 = \omega_0^2$ to $\tilde{\gamma}(1) = \gamma = \omega^2$, where ω_0 is the initial guess of the unknown frequency ω . In other words, we regard $\tilde{\gamma}(q)$ as a kind of homotopy, denoted by $\tilde{\gamma}(q) : \gamma_0 \sim \gamma$, which connects the initial guess $\gamma_0 = \omega_0^2$ and the unknown quantity $\gamma = \omega^2$. We will explain later why we should regard γ as such a kind of continuous function of q . Here, we just point out that it is necessary in order to avoid the so-called secular terms such as $\tau \cos(\tau)$.

Then, replacing x and γ in (2.40) by $\tilde{x}(\tau; q)$ and $\tilde{\gamma}(q)$, respectively, we have a two-parameter (q, c_0) family of differential equation $\mathcal{E}(q)$:

$$(1 - q)\mathcal{L}[\tilde{x}(\tau; q) - x_0(\tau)] = c_0 q [\tilde{\gamma}(q) \tilde{x}''(\tau; q) + \lambda \tilde{x}(\tau; q) + \varepsilon \tilde{x}^3(\tau; q)], \quad (2.42)$$

subject to the initial conditions

$$\tilde{x}(0; q) = x^*, \quad \tilde{x}'(0; q) = 0, \quad (2.43)$$

where the prime denotes the differentiation with respect to τ . When $q = 0$, we have the equation

$$\mathcal{E}_0 : \mathcal{L}[\tilde{x}(\tau; 0) - x_0(\tau)] = 0, \quad \tilde{x}(0; 0) = x^*, \quad \tilde{x}'(0; 0) = 0. \quad (2.44)$$

According to the linear property of $\mathcal{L}$, i.e. $\mathcal{L}(0) = 0$, and using the fact that $x_0(\tau)$ satisfies the initial conditions, it is obvious that

$$\tilde{x}(\tau; 0) = x_0(\tau). \quad (2.45)$$

When $q = 1$, since $c_0 \neq 0$, Equations (2.42) and (2.43) are equivalent to the equation

$$\mathcal{E}_1 : \tilde{\gamma}(1) \tilde{x}''(\tau; 1) + \lambda \tilde{x}(\tau; 1) + \varepsilon \tilde{x}^3(\tau; 1) = 0, \quad \tilde{x}(0; 1) = x^*, \quad \tilde{x}'(0; 1) = 0, \quad (2.46)$$

which is exactly the same as the original equation (2.32), provided

$$\tilde{x}(\tau; 1) = x(\tau), \quad \tilde{\gamma}(1) = \gamma. \quad (2.47)$$

So, as the homotopy-parameter $q \in [0, 1]$ increases from 0 to 1, the equation-family $\mathcal{E}(q)$ varies from the equation $\mathcal{E}_0$ into the equation $\mathcal{E}_1$, while $\tilde{x}(\tau; q)$ deforms *continuously* from the initial guess $x_0(\tau)$ to the unknown solution $x(\tau)$ of (2.32), so does $\tilde{\gamma}(q)$ from its initial guess $\gamma_0 = \omega_0^2$ to the unknown quantity $\gamma = \omega^2$. In other words, $\mathcal{E}(q)$ is a homotopy of equations, denoted by $\mathcal{E}(q) : \mathcal{E}_0 \sim \mathcal{E}_1$, and $\tilde{x}(\tau; q)$ is a homotopy of functions, denoted by $\tilde{x}(\tau; q) : x_0(\tau) \sim x(\tau)$. Note that, $\tilde{\gamma}(q) : \gamma_0 \sim \gamma$ has been defined as a homotopy, as mentioned above. Note also that both of γ_0 and γ are unknown now, which will be determined later. This kind of continuous variations is called deformation in topology. So, Equations (2.42) and (2.43) are called *the zeroth-order deformation equations*.

Since $\tilde{x}(\tau; q)$ and $\tilde{\gamma}(q)$ depend on the embedding parameter $q \in [0, 1]$, they can be expanded in a power series of q as follows

$$\tilde{x}(\tau; q) \sim x_0(\tau) + \sum_{n=1}^{+\infty} x_n(\tau) q^n, \quad (2.48)$$

$$\tilde{\gamma}(q) \sim \gamma_0 + \sum_{n=1}^{+\infty} \gamma_n q^n, \quad (2.49)$$

where

$$x_n(\tau) = \frac{1}{n!} \left. \frac{\partial^n \tilde{x}(\tau; q)}{\partial q^n} \right|_{q=0} = \mathcal{D}_n [\tilde{x}(\tau; q)], \quad \gamma_n = \frac{1}{n!} \left. \frac{d^n \tilde{\gamma}(q)}{dq^n} \right|_{q=0} = \mathcal{D}_n [\tilde{\gamma}(q)]. \quad (2.50)$$

Here, $\tilde{x}(\tau; 0) = x_0(\tau)$ and $\tilde{\gamma}(0) = \gamma_0$ are used. We call (2.48) and (2.49) the homotopy-Maclaurin series of $\tilde{x}(\tau; q)$ and $\tilde{\gamma}(q)$, respectively. According to (2.47), we obtain the solution of (2.32), if the homotopy-Maclaurin series (2.48) and (2.49) are convergent at $q = 1$ to $\tilde{x}(\tau; 1)$ and $\tilde{\gamma}(1)$, respectively. So, it is very important to guarantee the convergence of the homotopy-Maclaurin series (2.48) and (2.49) at $q = 1$. However, a power series has in general a bounded convergence radius. Fortunately, both $\tilde{x}(\tau; q)$ and $\tilde{\gamma}(q)$ are dependent upon *not only* the homotopy-parameter $q \in [0, 1]$ *but also* the convergence-control parameter c_0 and the auxiliary linear operator $\mathcal{L}$, because both of c_0 and $\mathcal{L}$ appear in the zeroth-order deformation equation (2.42). More importantly, we have great *freedom* to choose the convergence-

control parameter c_0 and the auxiliary linear operator $\mathcal{L}$. Thus, assuming that the convergence-control parameter c_0 and the auxiliary linear operator $\mathcal{L}$ are properly chosen so that the homotopy-Maclaurin series (2.48) and (2.49) are convergent to $\tilde{x}(\tau; q)$ and $\tilde{\gamma}(q)$ at $q = 1$, we have, according to the expressions (2.47) to (2.49), the *homotopy-series solution*

$$x(\tau) = x_0(\tau) + \sum_{n=1}^{+\infty} x_n(\tau), \quad (2.51)$$

$$\gamma = \gamma_0 + \sum_{n=1}^{+\infty} \gamma_n, \quad (2.52)$$

and the *mth-order homotopy-approximation*

$$x(\tau) \approx x_0(\tau) + \sum_{n=1}^m x_n(\tau), \quad (2.53)$$

$$\gamma \approx \gamma_0 + \sum_{n=1}^m \gamma_n. \quad (2.54)$$

For simplicity, define the sets

$$X_m = \{x_0(\tau), x_1(\tau), x_2(\tau), \dots, x_m(\tau)\}, \quad \Gamma_m = \{\gamma_0, \gamma_1, \gamma_2, \dots, \gamma_m\}.$$

According to the fundamental theorems in calculus (Fitzpatrick, 1996), $x_n(\tau)$ and γ_n are *unique*, and are completely determined by $\tilde{x}(\tau; q)$ and $\tilde{\gamma}(q)$, respectively, which are governed by the zeroth-order deformation equations (2.42) and (2.43). There are two different ways to derive the corresponding equations of $x_n(\tau)$ and γ_n . But, each of them gives the same results, as shown below. First, differentiating n times (2.42) and (2.43) with respect to q , then dividing by $n!$, and finally setting $q = 0$, we have, according to the definition (2.50) of $x_n(\tau)$ and γ_n , the so-called *nth-order deformation equation*

$$\mathcal{L}[x_n(\tau) - \chi_n x_{n-1}(\tau)] = c_0 \delta_{n-1}(X_{n-1}, \Gamma_{n-1}), \quad (2.55)$$

subject to the initial conditions

$$x_n(0) = 0, \quad x'_n(0) = 0, \quad (2.56)$$

where

$$\begin{aligned} & \delta_k(X_k, \Gamma_k) \\ &= \mathcal{D}_k [\tilde{\gamma}(q) \tilde{x}''(\tau; q) + \lambda \tilde{x}(\tau; q) + \varepsilon \tilde{x}^3(\tau; q)] \\ &= \sum_{i=0}^k \gamma_i x''_{k-i}(\tau) + \lambda x_k(\tau) + \varepsilon \sum_{i=0}^k x_{k-i}(\tau) \sum_{j=0}^i x_{i-j}(\tau) x_j(\tau), \end{aligned} \quad (2.57)$$

and

$$\chi_n = \begin{cases} 0, & n \leq 1, \\ 1, & n > 1. \end{cases} \quad (2.58)$$

The detailed derivation of (2.55) and (2.56) is given in Appendix 2.2. Alternatively, one can directly substitute the homotopy-Maclaurin series (2.48) and (2.49) into the zeroth-order deformation equations (2.42) and (2.43), then equates the coefficients of the like power of q . As proved by Hayat and Sajid (2007) in general, the 2nd approach gives exactly the *same* equations as (2.55) and (2.56) by the 1st approach, as shown in Appendix 2.2. This is mainly because, due to the fundamental theorems in calculus (Fitzpatrick, 1996), $x_n(\tau)$ in the homotopy-Maclaurin series (2.48) is *unique*, and thus must be governed by the *unique* equation. So does γ_n . In general, given a zeroth-order deformation equation, it is straight-forward to obtain the corresponding high-order deformation equations. We will show this point in Chapter 3 in details.

It should be emphasized here that the high-order deformation equations (2.55) and (2.56) are *linear*. Thus, according to (2.51) and (2.52), the original nonlinear differential equation (2.32) has been transformed into an infinite number of linear differential equations (2.55) and (2.56). Note that, different from perturbation techniques (Cole, 1992; Hilton, 1953; Hinch, 1991; Murdock, 1991; Nayfeh, 1973, 2000), such kind of transformation does *not* need any small/large physical parameters. Thus, giving up the dependence of perturbation techniques on small/large physical parameters, the HAM is valid for more equations, especially for those with strong nonlinearity. More importantly, without the restriction of small/large physical parameters, the HAM provides us great freedom to choose the auxiliary linear operator $\mathcal{L}$, as shown later. So, it is an obvious advantage of the HAM to be independent of any small/large physical parameters.

Note that the auxiliary linear operator $\mathcal{L}$ in the zeroth-order deformation equation (2.42) is not determined up to now. As mentioned before, it has the property $\mathcal{L}(0) = 0$. Besides, if $x(\tau)$ is a periodic function, then $\mathcal{L}[x(\tau)]$ should be also a periodic function, too. Even so, $\mathcal{L}$ is still rather general. Since the HAM is based on the concept of homotopy in topology, we have great freedom to choose the auxiliary linear operator $\mathcal{L}$, as shown later in this book.

Note that the original equation (2.32) contains the linear part $\gamma x'' + \lambda x$. Regarding ε as a small parameter (i.e. perturbation quantity) and substituting

$$x = \bar{x}_0 + \sum_{n=1}^{+\infty} \bar{x}_n \varepsilon^n$$

into (2.32) and then equating the like-power of ε , one has the perturbed equations

$$\begin{aligned} \gamma \bar{x}_0'' + \lambda \bar{x}_0 &= 0, & \bar{x}_0(0) &= x^*, & \bar{x}_0'(0) &= 0, \\ \gamma \bar{x}_1'' + \lambda \bar{x}_1 &= -\varepsilon \bar{x}_0^3, & \bar{x}_1(0) &= 0, & \bar{x}_1'(0) &= 0, \\ &\vdots \end{aligned}$$

The solution $\bar{x}_0$ of the above perturbed equations is periodic when $\lambda > 0$ but is non-periodic when $\lambda < 0$. However, as listed in Table 2.3, when $\varepsilon > 0$, the solution of (2.32) is periodic even in case of $\lambda < 0$. So, the above perturbation method fails to get good approximations in case of $\varepsilon > 0$ and $\lambda < 0$. Therefore, $\mathcal{L}x = \gamma x'' + \lambda x$ is not a proper linear operator to get periodic approximations of (2.32).

Fortunately, the HAM is independent of any small/large physical parameters, as mentioned above. So, we have great freedom to choose a proper auxiliary linear operator, as shown below. Because the original nonlinear oscillation problem is governed by the 2nd-order differential equation (2.32), it seems natural for us to choose such a 2nd-order differential operator

$$\mathcal{L}[x(\tau)] = x''(\tau) + A_1(\tau) x'(\tau) + A_2(\tau) x(\tau), \quad (2.59)$$

where the prime denotes the differentiation with respect to τ , and $A_1(\tau), A_2(\tau)$ are periodic real functions so that $\mathcal{L}[x(\tau)] \in \mathbf{S}_p$ if $x(\tau) \in \mathbf{S}_p$. Let $y_1(\tau), y_2(\tau)$ denote the non-zero solutions of $\mathcal{L}[x(\tau)] = 0$, and $x_n^s(\tau)$ a special solution of (2.55). Then, the general solution of (2.55) reads

$$x_n(\tau) = \chi_n x_{n-1}(\tau) + x_n^s(\tau) + C_1 y_1(\tau) + C_2 y_2(\tau), \quad (2.60)$$

where C_1, C_2 are integral coefficients. Note that our aim is to obtain convergent series solution $x(\tau)$ with the known period 2π , i.e. $x(\tau) \in \mathbf{S}_p$, which implies, according to (2.51), that $x_n(\tau)$ must be a periodic function with the period 2π , i.e. $x_n(\tau) \in \mathbf{S}_p$. So, we choose the simplest periodic function with the period 2π :

$$y_1(\tau) = \cos \tau, \quad y_2(\tau) = \sin \tau.$$

Because $y_1(\tau)$ and $y_2(\tau)$ are non-zero solutions (i.e. kernel) of $\mathcal{L}(x) = 0$, it holds for any non-zero constant coefficients C_1 and C_2 that

$$\mathcal{L}(C_1 \cos \tau + C_2 \sin \tau) = 0.$$

Then, using the above equation and the definition (2.59) of $\mathcal{L}$, we have

$$\begin{aligned} & \{[A_2(\tau) - 1] \cos \tau - A_1(\tau) \sin \tau\} C_1 \\ & + \{[A_2(\tau) - 1] \sin \tau + A_1(\tau) \cos \tau\} C_2 = 0. \end{aligned} \quad (2.61)$$

The above equation holds for arbitrary coefficients C_1 and C_2 in the interval $\tau \in [0, +\infty)$, if and only if

$$A_1(\tau) = 0, \quad A_2(\tau) = 1. \quad (2.62)$$

Substituting them into the general definition (2.59) of $\mathcal{L}$, we obtain the auxiliary linear operator

$$\mathcal{L}(x) = x'' + x, \quad (2.63)$$

which has the property

$$\mathcal{L}(C_1 \cos \tau + C_2 \sin \tau) = 0 \quad (2.64)$$

for any constants C_1 and C_2 . In other words, $\cos \tau$ and $\sin \tau$ belong to the kernel of the auxiliary linear operator $\mathcal{L}$. Then, the general solution of (2.55) reads

$$x_n(\tau) = \chi_n x_{n-1}(\tau) + x_n^s(\tau) + C_1 \cos \tau + C_2 \sin \tau, \quad (2.65)$$

where the integral coefficients

$$C_1 = -\chi_n x_{n-1}(0) - x_n^s(0), \quad C_2 = 0$$

are determined by the initial condition (2.56). Thus, the solution of the n th-order deformation equations (2.55) and (2.56) reads

$$x_n(\tau) = \chi_n x_{n-1}(\tau) + x_n^s(\tau) - [\chi_n x_{n-1}(0) + x_n^s(0)] \cos \tau. \quad (2.66)$$

Since $x_n \in \mathbf{S}_p$, the above expression implies that $x_n^s \in \mathbf{S}_p$, i.e. the special solution $x_n^s(\tau)$ must be a function with the period 2π . Note that both of $x_n^s(\tau)$ and γ_{n-1} are unknown, but we have only one governing equation for $x_n^s(\tau)$. So, one additional algebraic equation is needed to determine γ_{n-1} . To show how to get $x_n^s(\tau)$ and γ_{n-1} , let us consider the first-order equation

$$\mathcal{L}[x_1(\tau)] = c_0 \delta_0(x_0, \gamma_0), \quad x_1(0) = 0, x_1'(0) = 0,$$

where we have, according to (2.33) and (2.57), that

$$\begin{aligned} \delta_0(x_0, \gamma_0) &= \gamma_0 x_0''(\tau) + \lambda x_0(\tau) + \varepsilon x_0^3(\tau) \\ &= A_{1,0} + A_{1,1} \cos \tau + A_{1,2} \cos 2\tau + A_{1,3} \cos 3\tau, \end{aligned} \quad (2.67)$$

in which $A_{1,0}, A_{1,1}, A_{1,2}$ and $A_{1,3}$ are constant coefficients, especially

$$A_{1,1} = (\beta - x^*) \left\{ \gamma_0 - \lambda - \frac{3}{4} \varepsilon [5\beta^2 - 2\beta x^* + (x^*)^2] \right\}.$$

According to the property (2.64), if $A_{1,1} \neq 0$, then $x_1(\tau)$ contains the so-called secular term $\tau \cos \tau$, which however is *not* periodic, i.e. $x_n^s \notin \mathbf{S}_p$. Such kind of non-periodic solution must be avoided, since our aim is to obtain periodic solution of (2.32). For this reason, we had to enforce $A_{1,1} = 0$, which provides us with one additional algebraic equation of γ_0 , i.e.

$$(\beta - x^*) \left\{ \gamma_0 - \lambda - \frac{3}{4} \varepsilon [4\beta^2 + (\beta - x^*)^2] \right\} = 0.$$

Since $\beta \neq x^*$, we have the initial guess

$$\gamma_0 = \lambda + \frac{3}{4} \varepsilon [4\beta^2 + (\beta - x^*)^2], \quad (2.68)$$

corresponding to the initial guess of the frequency

$$\omega_0 = \sqrt{\lambda + \frac{3}{4}\varepsilon[4\beta^2 + (\beta - x^*)^2]} . \quad (2.69)$$

Note that $\gamma_0 = \omega_0^2$ defined above is always positive for all possible values of λ, ε and x^* in case of periodic solution $x(\tau)$, therefore ω_0 is always a real positive number. The above expression indicates that, physically, the frequency of the periodic oscillation of the nonlinear dynamic system (2.32) depends not only on the physical parameters λ and ε , but also on x^* , the starting position, and β , the position (i.e. the coordinate) of the stable equilibrium point.

Thereafter, it is easy to get the special solution

$$x_1^s(\tau) = A_{1,0} - \frac{A_{1,2}}{3} \cos 2\tau - \frac{A_{1,3}}{8} \cos 3\tau.$$

Then, substituting it into (2.66), we have

$$x_1(\tau) = A_{1,0} - \left(A_{1,0} - \frac{A_{1,2}}{3} - \frac{A_{1,3}}{8} \right) \cos \tau - \frac{A_{1,2}}{3} \cos 2\tau - \frac{A_{1,3}}{8} \cos 3\tau.$$

Similarly, we can solve $\gamma_1, x_2(\tau), \gamma_2, x_3(\tau)$, and so on. In this way, it is easy to successively solve the linear deformation equations (2.55) and (2.56) by means of computer algebra system such as Mathematica, Maple and so on, and then obtain a high-order approximation of $x(\tau)$ and $\gamma = \omega^2$ in a few seconds using a laptop.

It should be emphasized that the HAM mentioned above provides us great freedom to construct the zeroth-order deformation equation, mainly because it is based on the concept of homotopy in topology and thus is independent of any small/large physical parameters. It is due to this kind of freedom that we can choose the auxiliary linear operator $\mathcal{L}(x) = x'' + x$ to obtain periodic solutions for *all* possible values of λ, ε and x^* , even including $\lambda < 0$. Besides, it is due to this freedom that we can regard γ as a function of q , i.e. a homotopy $\gamma: \gamma_0 \sim \gamma$, in the zeroth-order deformation equation (2.42), so that the secular term $\tau \cos \tau$ is avoided and the periodic solutions are obtained. Furthermore, it is due to this kind of freedom that we can introduce the convergence-control parameter c_0 into the zeroth-order deformation equation, which provides us a simple way to guarantee the convergence of homotopy-series solution. By means of this freedom, we can obtain much better approximations of many problems with strong nonlinearity. This is the 2nd advantage of the HAM.

2.3.3 Convergence of homotopy-series solution

First of all, let us prove the following two theorems about the homotopy-series

$$\sum_{n=0}^{+\infty} x_n(\tau) \text{ and } \sum_{n=0}^{+\infty} \gamma_n.$$

Theorem 2.1. *If the homotopy-series $\sum_{k=0}^{+\infty} x_k(\tau)$ and $\sum_{k=0}^{+\infty} x_k''(\tau)$ are convergent, then $\sum_{k=0}^{+\infty} \delta_k = 0$, where δ_k is defined by (2.57).*

Proof. According to (2.55) and (2.58), we have

$$\begin{aligned}\mathcal{L}(x_1) &= c_0 \delta_0, \\ \mathcal{L}(x_2 - x_1) &= c_0 \delta_1, \\ \mathcal{L}(x_3 - x_2) &= c_0 \delta_2, \\ &\vdots \\ \mathcal{L}(x_m - x_{m-1}) &= c_0 \delta_{m-1},\end{aligned}$$

where $\mathcal{L}(u) = u'' + u$ is the auxiliary linear operator. Since $\mathcal{L}$ is a linear operator, the sum of all above equations gives

$$\mathcal{L}(x_m) = c_0 \sum_{n=0}^{m-1} \delta_n.$$

Since the homotopy-series $\sum_{n=0}^{+\infty} x_n$ and $\sum_{n=0}^{+\infty} x_n''$ converge, it holds

$$\lim_{n \rightarrow +\infty} x_n = 0, \quad \lim_{n \rightarrow +\infty} x_n'' = 0.$$

Then, we have

$$c_0 \sum_{n=0}^{+\infty} \delta_n = \lim_{m \rightarrow +\infty} \mathcal{L}(x_m) = \lim_{m \rightarrow +\infty} (x_m'' + x_m) = \lim_{m \rightarrow +\infty} x_m'' + \lim_{m \rightarrow +\infty} x_m = 0,$$

which gives, since $c_0 \neq 0$, that $\sum_{n=0}^{+\infty} \delta_n = 0$. □

Theorem 2.2. *If the convergence-control parameter c_0 is so properly chosen that the homotopy-series $\sum_{n=0}^{+\infty} x_n(\tau)$ and $\sum_{n=0}^{+\infty} \gamma_n$ are absolutely convergent to $x(\tau)$ and γ , respectively, and besides $\sum_{n=0}^{+\infty} x_n''(\tau)$ is convergent to $x''(\tau)$, then the homotopy-series $\sum_{n=0}^{+\infty} x_n(\tau)$ and $\sum_{n=0}^{+\infty} \gamma_n$ satisfy the original nonlinear differential equation (2.32).*

Proof. Since the homotopy-series $\sum_{n=0}^{+\infty} x_n$ and $\sum_{n=0}^{+\infty} x_n''$ converge, we have according to

Theorem 2.1 that $\sum_{n=0}^{+\infty} \delta_n = 0$, i.e.

$$\sum_{n=0}^{+\infty} \left[\sum_{k=0}^n \gamma_k x''_{n-k}(\tau) + \lambda x_n(\tau) + \varepsilon \sum_{k=0}^n x_{n-k}(\tau) \sum_{j=0}^k x_{k-j}(\tau) x_j(\tau) \right] = 0.$$

Since $\sum_{n=0}^{+\infty} x_n$ and $\sum_{n=0}^{+\infty} \gamma_n$ are absolutely convergent to $x(\tau)$ and γ , respectively, and besides $\sum_{n=0}^{+\infty} x''_n$ is convergent to $x''(\tau)$, we have due to the theorems of Cauchy product that

$$\sum_{n=0}^{+\infty} \left(\sum_{k=0}^n \gamma_k x''_{n-k} \right) = \sum_{k=0}^{+\infty} \sum_{n=k}^{+\infty} \gamma_k x''_{n-k} = \sum_{k=0}^{+\infty} \sum_{m=0}^{+\infty} \gamma_k x''_m = \left(\sum_{j=0}^{+\infty} \gamma_j \right) \left(\sum_{m=0}^{+\infty} x''_m \right)$$

and

$$\sum_{n=0}^{+\infty} \left(\sum_{k=0}^n x_{n-k} \sum_{j=0}^k x_{k-j} x_j \right) = \left(\sum_{n=0}^{+\infty} x_n \right)^3.$$

Besides, it holds obviously

$$\sum_{n=0}^{+\infty} \lambda x_n = \lambda \left(\sum_{n=0}^{+\infty} x_n \right).$$

Substituting the above three expressions into the equation from $\sum_{n=0}^{+\infty} \delta_n = 0$ gives

$$\left(\sum_{j=0}^{+\infty} \gamma_j \right) \left(\sum_{n=0}^{+\infty} x_n \right)'' + \lambda \left(\sum_{n=0}^{+\infty} x_n \right) + \varepsilon \left(\sum_{n=0}^{+\infty} x_n \right)^3 = 0.$$

According to (2.33), it holds $x_0(0) = x^*$ and $x'_0(0) = 0$. Then, using (2.56), we have

$$\sum_{n=0}^{+\infty} x_n(0) = x_0(0) = x^*, \quad \sum_{n=0}^{+\infty} x'_n(0) = x'_0(0) = 0.$$

Thus, the absolutely convergent homotopy-series $x = \sum_{n=0}^{+\infty} x_n$ and $\gamma = \sum_{n=0}^{+\infty} \gamma_n$ satisfy the original equation (2.32). This ends the proof. $\square$

According to Theorem 2.2, it is important to guarantee the convergence of homotopy-series. Theorem 2.1 provides us a convenient way to check the convergence of homotopy-series, as shown below. The above two theorems are tenable in general, as proved in Chapter 3.

Obviously, it is very important to guarantee the convergence of an approximation series. Unfortunately, near all previous analytic approximation methods such as perturbation methods, Lyapunov's artificial small-parameter method, Adomian decomposition method, the δ -expansion method and so on, can not guarantee the convergence of approximation series, as mentioned in Chapter 1. This is the essen-

tial reason why these traditional analytic techniques are valid mainly for weakly nonlinear problems.

Fortunately, based on the homotopy in topology, the HAM provides us great freedom. Using this kind of freedom, we introduce a non-zero auxiliary parameter c_0 , namely the convergence-control parameter, into the so-called zeroth-order deformation equation (2.42). Note that the high-order deformation equation (2.55) also contains the convergence-control parameter c_0 . Therefore, the homotopy-series $\sum_{n=0}^{+\infty} x_n(\tau)$ and $\sum_{n=0}^{+\infty} \gamma_n$ contain the convergence-control parameter c_0 , too. More importantly, as mentioned above, we have great freedom to choose the value of c_0 so that we can find some proper values of c_0 to guarantee the convergence of the homotopy-series. Thus, the convergence-control parameter c_0 provides us a convenient way to guarantee the convergence of homotopy-series, as shown below.

According to Theorem 2.2, the residual² of the original governing equation (2.32) tends to zero in $\tau \in [0, 2\pi]$ if the homotopy-series $\sum_{n=0}^{+\infty} x_n(\tau)$ and $\sum_{n=0}^{+\infty} \gamma_n$ are absolutely convergent and besides $\sum_{n=0}^{+\infty} x_n''(\tau)$ is convergent. The averaged value of the squared residual of the governing equation clearly indicates the accuracy of an analytic approximation. So, in order to choose a proper value of c_0 , we use the squared residual

$$E_m(c_0) = \frac{1}{2\pi} \int_0^{2\pi} [\Delta_m(\tau; c_0)]^2 d\tau, \quad (2.70)$$

where

$$\Delta_m(\tau; c_0) = \check{\gamma} \check{x}''(\tau) + \lambda \check{x}(\tau) + \varepsilon \check{x}^3(\tau) \quad (2.71)$$

is the residual of the governing equation (2.32), and

$$\check{x} = \sum_{n=0}^m x_n(\tau), \quad \check{\gamma} = \sum_{n=0}^m \gamma_n$$

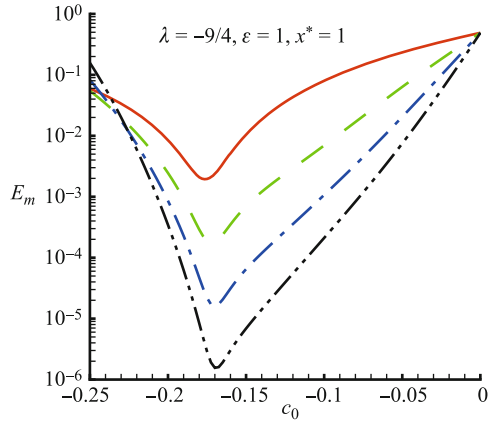
are the m th-order approximation of $x(\tau)$ and γ , respectively. For the sake of computational efficiency, we calculate the discrete squared residual $E_m(c_0)$ numerically, i.e.

$$E_m(c_0) \approx \frac{1}{(N+1)} \sum_{k=0}^N [\Delta_m(\tau_k; c_0)]^2, \quad \tau_k = \frac{2k\pi}{N}, \quad (2.72)$$

where N is an integer. Obviously, the above expression is a good approximation of $E_m(c_0)$ for large enough N . In this chapter, we use $N = 50$. Note that $E_m(c_0)$ depends on the convergence-control parameter c_0 . Obviously, the smaller the value of $E_m(c_0)$ for given m , the better the approximation. At the given order of approximation m ,

² For precise definition about residual of equations, please refer to the website at <http://en.wikipedia.org/wiki/Residual>. Accessed 15 April 2011.

Fig. 2.5 Discrete squared residual $E_m(c_0)$ in case of $\varepsilon = 1, \lambda = -9/4$ and $x^* = 1$. Solid line: 1st-order approx.; Dashed line: 3rd-order approx.; Dash-dotted line: 5th-order approx.; Dash-double-dotted line: 7th-order approx.



the “best” or “optimal” approximation is defined by the minimum of $E_m(c_0^*)$ with the corresponding optimal convergence-control parameter c_0^* .

Note that, it is convenient to solve the linear high-order deformation equations (2.55) and (2.56) by means of the computer algebra systems like Mathematica, Maple and so on. The corresponding Mathematica code (without iteration approach) is free available (Accessed 25 Nov 2011, will be updated in the future) at

<http://numericaltank.sjtu.edu.cn/HAM.htm>

Without loss of the generality, let us first consider the case $\varepsilon = 1, \lambda = -9/4$ and $x^* = 1$. As shown in Fig. 2.5, as the order of approximation increases, the discrete squared residual $E_m(c_0)$ decreases in the interval

$$\mathbf{R}_c = \{c_0 | -0.2 \leq c_0 \leq -0.05\}.$$

Therefore, if $c_0 \in \mathbf{R}_c$, then the corresponding homotopy-series solution converges. To confirm this, we investigate the convergence of the homotopy-series solutions by different values of c_0 , such as $c_0 = -1/5, -3/20, -1/10$ and $-1/20$, respectively. It is found that, as the order of approximation increases, the discrete squared residuals of *all* these homotopy-series decrease monotonously, as shown in Table 2.4, and besides *all* these homotopy-series give the *same* value $\gamma = 3.9278$, as shown in Table 2.5. Our calculations illustrate that there indeed exists such a set $\mathbf{R}_c$ that, if $c_0 \in \mathbf{R}_c$, then the corresponding homotopy-series converges to the *same* result, although with different convergence-rate. Mathematically, according to Theorem 2.2, all convergent homotopy-series of $x(\tau)$ and γ satisfy the original equation (2.32). Since (2.32) has unique solution, then all of these convergent homotopy-series of $x(\tau)$ and γ must be the same. Physically, the convergence-control parameter c_0 is an artificial parameter, which has no physical meanings at all. Thus, from physical points of view, all convergent homotopy-series must be independent of the convergence-control parameter c_0 (otherwise, it is physically wrong). In this way, we can explain all of our above results from both mathematical and physical view-points.

Table 2.4 Discrete squared residual $E_m(c_0)$ in case of $\varepsilon = 1, \lambda = -9/4, x^* = 1$ by means of different values of c_0 .

order of approx.	$c_0 = -1/5$	$c_0 = -3/20$	$c_0 = -1/10$	$c_0 = -1/20$
5	8.6×10^{-4}	6.5×10^{-5}	1.1×10^{-3}	2.0×10^{-2}
10	1.1×10^{-4}	2.8×10^{-7}	2.3×10^{-5}	2.0×10^{-3}
20	5.2×10^{-6}	9.4×10^{-12}	3.0×10^{-8}	4.7×10^{-5}
30	4.4×10^{-7}	4.4×10^{-16}	5.7×10^{-11}	1.8×10^{-6}
40	4.5×10^{-8}	2.3×10^{-20}	1.2×10^{-13}	8.7×10^{-8}
50	5.4×10^{-9}	1.3×10^{-24}	2.8×10^{-16}	4.5×10^{-9}
60	6.9×10^{-10}	8.3×10^{-29}	6.8×10^{-19}	2.5×10^{-10}
70	9.3×10^{-11}	5.3×10^{-33}	1.7×10^{-21}	1.4×10^{-11}
80	1.3×10^{-11}	3.5×10^{-37}	4.4×10^{-24}	8.0×10^{-13}

Table 2.5 Approximations of $\gamma = \omega^2$ in case of $\varepsilon = 1, \lambda = -9/4, x^* = 1$ by means of different values of c_0 .

order of approx.	$c_0 = -1/5$	$c_0 = -3/20$	$c_0 = -1/10$	$c_0 = -1/20$
5	3.8944	3.9322	3.9550	4.0801
10	3.9381	3.9280	3.9307	3.9687
20	3.9297	3.9278	3.9279	3.9325
30	3.9283	3.9278	3.9278	3.9285
40	3.9280	3.9278	3.9278	3.9279
50	3.9279	3.9278	3.9278	3.9278
60	3.9278	3.9278	3.9278	3.9278
70	3.9278	3.9278	3.9278	3.9278
80	3.9278	3.9278	3.9278	3.9278

Table 2.6 Approximations of $\gamma = \omega^2$ in case of $\varepsilon = 1, \lambda = -9/4$ and $x^* = 1$ by means of the optimal convergence-control parameter $c_0^* = -17/100$.

m , order of approx.	$\gamma = \omega^2$	$E_m(c_0^*)$
0	4.6875	0.49
1	3.9223	2.4×10^{-3}
5	3.9263	1.6×10^{-5}
10	3.9281	1.0×10^{-7}
15	3.9278	2.2×10^{-10}
20	3.9278	1.3×10^{-12}
25	3.9278	7.0×10^{-15}
30	3.9278	4.8×10^{-17}
40	3.9278	2.6×10^{-21}
50	3.9278	1.7×10^{-25}
60	3.9278	1.2×10^{-29}
70	3.9278	8.5×10^{-34}
80	3.9278	6.5×10^{-38}

Definition 2.7. Each unknown auxiliary parameter in the zeroth-order deformation equation, except the homotopy-parameter $q \in [0, 1]$, is called a *convergence-control parameter*, if it can influence the convergence of the homotopy-series. All of these convergence-control parameters construct the so-called *convergence-control vector*, denoted by $\mathbf{c} = (c_0, c_1, c_2, \dots)$, where $c_0, c_1, c_2, \dots$ are convergence-control parameters.

Definition 2.8. A set $\mathbf{R}_c$ of all possible values of a convergence-control parameter c_0 is called *the effective-region* of the convergence-control parameter c_0 , if the corresponding homotopy-series converges for each $c_0 \in \mathbf{R}_c$.

Definition 2.9. A set $\mathbf{R}_c$ of all possible values of the convergence-control vector $\mathbf{c}$ is called *the effective-region* of the convergence-control vector $\mathbf{c}$, if the corresponding homotopy-series converges for each $\mathbf{c} \in \mathbf{R}_c$.

According to Fig. 2.5 and Table 2.4, the homotopy-series given by different values of $c_0 \in \mathbf{R}_c$ converges in a quite different rate. For example, the homotopy-series given by $c_0 = -3/20$ converges much faster than the homotopy-series given by $c_0 = -1/5$ and $c_0 = -1/20$, as shown in Table 2.4. Besides, according to Fig. 2.5, the minimum of $E_m(c_0)$ exists near $c_0 = -0.17$. So, we have the optimal convergence-control parameter $c_0^* = -0.17$ in the case of $\varepsilon = 1$, $\lambda = -9/4$ and $x^* = 1$. This is indeed true. By means of the optimal convergence-control parameter $c_0^* = -17/100$, the discrete squared residual $E_m(c_0)$ quickly decreases and γ tends to a fixed value 3.9278 rather faster, as shown in Table 2.6. Thus, the convergence-control parameter c_0 indeed provides us a convenient way to guarantee the quick convergence of homotopy-series solution. Note that, although the initial guess of γ has 19.3% relative error, the 3rd and 5th-order approximation of γ have only 0.1% and 0.04% relative error, respectively. Thus, in this case, a few terms can give a rather accurate approximation of $x(\tau)$ by means of the optimal value of c_0 , as shown in Fig. 2.6. Note also that, the homotopy-series given by $c_0 = -3/20$ quickly converges, too. Thus, in practice, it is unnecessary to use the “exact” value of the optimal convergence-control parameter c_0 .

The above approach has general meaning. For example, we further consider the following three cases:

- $\varepsilon = 1, \quad \lambda = 9/4, \quad x^* = 1;$
- $\varepsilon = 1, \quad \lambda = 0, \quad x^* = 1;$
- $\varepsilon = -1, \quad \lambda = 4, \quad x^* = -1.$

For each case, we investigate the curves of the discrete squared residual $E_m(c_0)$ versus c_0 in a similar way so as to find out a region of c_0 for the convergence of homotopy-series and besides the optimal value of c_0 . According to Figs. 2.7 to 2.9, we have the optimal value $c_0 = -1/3$ in case of $\varepsilon = 1, \lambda = 9/4, x^* = 1$, the optimal

Fig. 2.6 Comparison of numerical result with analytic approximations of $x(\tau)$ in case of $\varepsilon = 1, \lambda = -9/4$ and $x^* = 1$ by means of $c_0 = -0.17$. Solid line: 5th-order approx.; Dashed line: initial approx.; Symbols: numerical result.

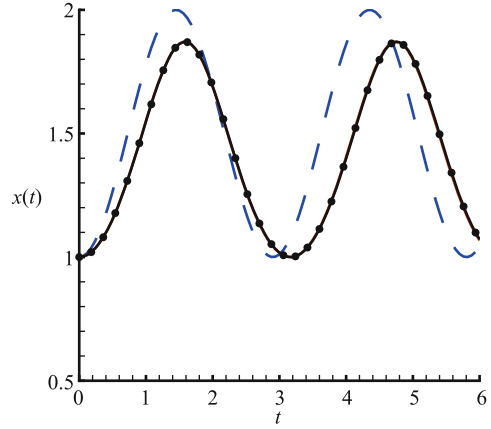


Fig. 2.7 Discrete squared residual $E_m(c_0)$ in case of $\varepsilon = 1, \lambda = 9/4$ and $x^* = 1$. Solid line: 1st-order approx.; Dashed line: 3rd-order approx.; Dash-dotted line: 5th-order approx.

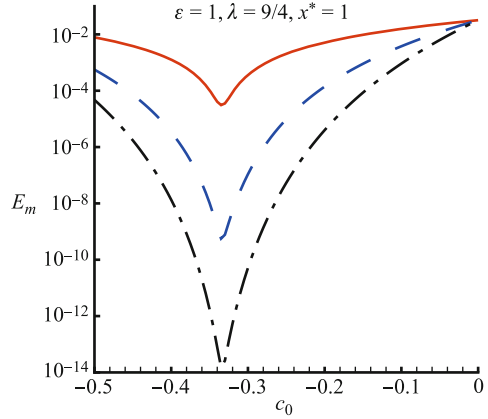


Fig. 2.8 Discrete squared residual $E_m(c_0)$ in case of $\varepsilon = 1, \lambda = 0$ and $x^* = 1$. Solid line: 1st-order approx.; Dashed line: 3rd-order approx.; Dash-dotted line: 5th-order approx.

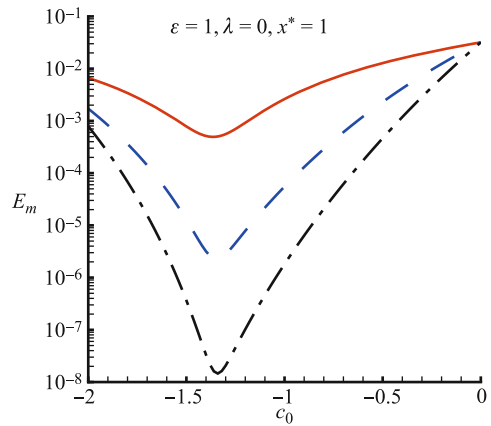


Fig. 2.9 Discrete squared residual $E_m(c_0)$ in case of $\varepsilon = -1, \lambda = 4$ and $x^* = -1$. Solid line: 1st-order approx.; Dashed line: 3rd-order approx.; Dash-dotted line: 5th-order approx.

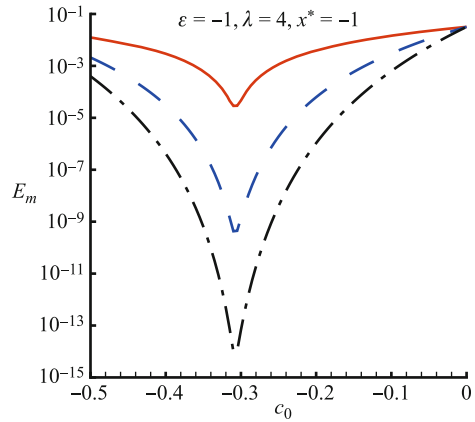
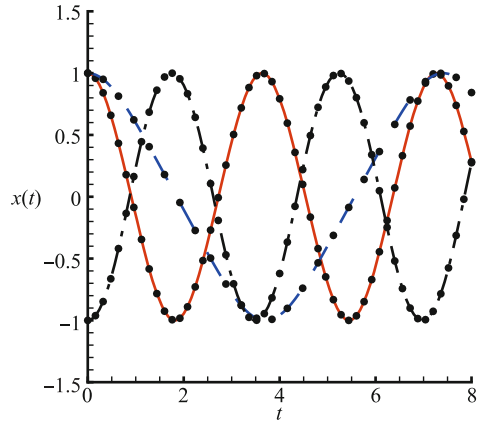


Fig. 2.10 Comparison of numerical results with analytic approximations of $x(\tau)$. Solid line: 1st-order of approx. in case of $\varepsilon = 1, \lambda = 9/4, x^* = 1$ by $c_0 = -1/3$; Dashed line: 2nd-order approx. in case of $\varepsilon = 1, \lambda = 0, x^* = 1$ by $c_0 = -4/3$; Dash-dotted line: 2nd-order approx. in case of $\varepsilon = -1, \lambda = 4, x^* = -1$ by $c_0 = -3/10$; Symbols: numerical results.



value $c_0 = -4/3$ in case of $\varepsilon = 1, \lambda = 0, x^* = 1$, and the optimal value $c_0 = -3/10$ in case of $\varepsilon = -1, \lambda = 4, x^* = -1$, respectively. Using the corresponding optimal value of c_0 , the discrete squared residual E_m decreases quickly for each case, as shown in Table 2.7, and besides $\gamma = \omega^2$ quickly tends to a fixed value, as shown in Table 2.8. Furthermore, even the corresponding 1st or 2nd-order approximations of $x(t)$ are rather accurate, as shown in Fig. 2.10. All of these indicate the convergence of the corresponding homotopy-series solution. Therefore, the convergence-control parameter c_0 indeed provides us a convenient way to guarantee the convergence of the homotopy-series solution. It should be emphasized that other analytic techniques have no ways to guarantee the convergence of series. So, this is an obvious advantage of the HAM.

Considering the theoretical rigorousness of the HAM, we often mention the homotopy-series and investigate the convergence of this kind of infinite series. However, this does not mean that we must use many terms to get an accurate enough approximation! In many cases, only a few terms of the homotopy-approximations

Table 2.7 Discrete squared residual $E_m(c_0)$ of m th-order approximation in different cases.

m , order of approx.	$\varepsilon = 1, \lambda = 9/4$ $x^* = 1, c_0 = -1/3$	$\varepsilon = 1, \lambda = 0$ $x^* = 1, c_0 = -4/3$	$\varepsilon = -1, \lambda = 4$ $x^* = -1, c_0 = -3/10$
0	0.032	0.032	0.032
1	3.1×10^{-5}	5.1×10^{-4}	4.6×10^{-5}
5	1.3×10^{-14}	1.5×10^{-8}	6.7×10^{-14}
10	5.8×10^{-26}	7.3×10^{-14}	2.4×10^{-24}
15	5.9×10^{-37}	7.7×10^{-19}	1.1×10^{-34}

Table 2.8 The m th-order approximation of $\gamma = \omega^2$ for different cases.

m , order of approx.	$\varepsilon = 1, \lambda = 9/4$ $x^* = 1, c_0 = -1/3$	$\varepsilon = 1, \lambda = 0$ $x^* = 1, c_0 = -4/3$	$\varepsilon = -1, \lambda = 4$ $x^* = -1, c_0 = -3/10$
0	3	0.75	3.25
1	2.9921875	0.71875	3.24296875
5	2.9921730367	0.7177741910	3.2427770978
10	2.9921730364	0.7177700399	3.2427770917
15	2.9921730364	0.7177700110	3.2427770917
20	2.9921730364	0.7177700110	3.2427770917
25	2.9921730364	0.7177700110	3.2427770917
30	2.9921730364	0.7177700110	3.2427770917

can give very accurate result. For example, we obtain rather accurate 1st-order homotopy-approximation

$$x \approx \frac{1}{96} [95 \cos(\tau) + \cos(3\tau)], \quad \tau = \sqrt{3} t, \quad (2.73)$$

in the case of $\varepsilon = 1$, $\lambda = 9/4$, $x^* = 1$ by means of the optimal value $c_0 = -1/3$, the rather accurate 2nd-order homotopy-approximation

$$x \approx \frac{1}{576} [551 \cos \tau + 24 \cos(3\tau) + \cos(5\tau)], \quad \tau = \sqrt{\frac{23}{32}} t, \quad (2.74)$$

in the case of $\varepsilon = 1$, $\lambda = 0$, $x^* = 1$ by means of the optimal value $c_0 = -4/3$, and the rather accurate 2nd-order homotopy-approximation

$$x \approx -1.00952 \cos \tau + 9.60938 \times 10^{-3} \cos(3\tau) - 8.78906 \times 10^{-5} \cos(5\tau), \quad \tau = \sqrt{\frac{4151}{1280}} t, \quad (2.75)$$

in the case of $\varepsilon = -1$, $\lambda = 4$, $x^* = -1$ by means of the optimal value $c_0 = -3/10$, respectively, as shown in Fig. 2.10. In practice, using the optimal convergence-control parameter, we often obtain accurate enough approximations in a few terms by means of the HAM. However, as the nonlinearity of equations becomes stronger,

more and more terms are needed, due to the complexity of strong nonlinear problems.

Without computer, it is hard to find the optimal value c_0 , corresponding to the minimum of the squared residual $E_m(c_0)$. However, in the times of computers, it is easy to do so by means of computer algebra system such as Mathematica, Maple, MathLab and so on. For example, using Mathematica, we obtain up-to 30th-order approximations just in a few seconds even by a laptop, although, as mentioned above, such high-order approximations are *not* necessary at all for the above cases. However, when necessary, we can obtain rather high order homotopy-approximations by computer algebra system, and more importantly, find out an optimal convergence-control parameter c_0 to guarantee the convergence of the corresponding homotopy-series so as to have an accurate enough result. Note also that, a laptop can save/read an enormous amount of data to/from a diskette in a few seconds. Besides, rather lengthy expressions can be calculated in a few seconds by a computer. Using a laptop with computer algebra system, we generally can not feel obvious difference between calculating a short formula and evaluating a rather complicated expression which might be one hundred pages long if printed out. So, the HAM is indeed for the times of computer: it combines the great flexibility of constructing a homotopy in theory and the increasing computing-power of an electronic computer in practice.

2.3.4 Essence of the convergence-control parameter c_0

To reveal the essence of the convergence-control parameter c_0 , let us further consider a little more general case $\lambda = 0, x^* = 1$ and $\varepsilon > 0$, which has the exact solution

$$\omega = \frac{\Gamma(3/4)}{\Gamma(5/4)} \sqrt{\frac{\pi \varepsilon}{8}},$$

where Γ denotes Gamma function. Thus, we have the exact solution

$$\gamma = \omega^2 \approx 0.7177700110 \varepsilon. \quad (2.76)$$

According to (2.34) and (2.68), we have the initial guess $x_0(\tau) = \cos \tau$ and

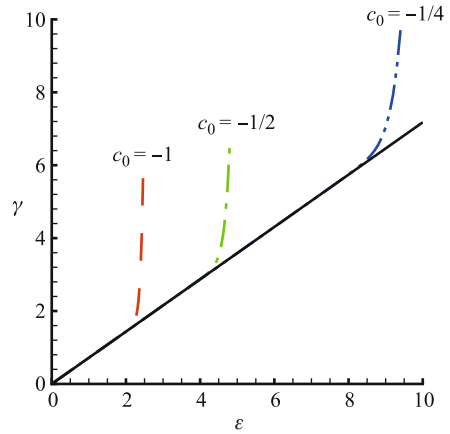
$$\gamma_0 = \frac{3}{4} \varepsilon.$$

Regarding c_0 as unknown and using the same approach mentioned above, we obtain the 1st-order homotopy-approximation

$$\gamma \approx \frac{3}{4} \varepsilon + \frac{3}{128} c_0 \varepsilon^2,$$

the 2nd-order homotopy-approximation

Fig. 2.11 Comparison of exact formula (2.76) with the 20th-order homotopy-approximations of γ in case of $\lambda = 0, x^* = 1$ and $\varepsilon > 0$. Solid line: the exact formula (2.76); Dashed line: $c_0 = -1$; Dash-dotted line: $c_0 = -1/2$; Dash-double-dotted line: $c_0 = -1/4$.



$$\gamma \approx \frac{3}{4} \varepsilon + \frac{3}{64} c_0 \varepsilon^2 + \frac{9}{512} c_0^2 \varepsilon^3,$$

the 3rd-order homotopy-approximation

$$\gamma \approx \frac{3}{4} \varepsilon + \frac{9}{128} c_0 \varepsilon^2 + \frac{27}{512} c_0^2 \varepsilon^3 + \frac{1779}{131072} c_0^3 \varepsilon^4,$$

and so on. It is found that γ is a kind of power series of ε with coefficients dependent upon the convergence-control parameter c_0 . So, mathematically speaking, we have different approximations of γ for different values of c_0 . The comparison of the exact formula (2.76) with the 20th-order approximations of γ by means of different values of c_0 is given in Fig. 2.11, which clearly indicates that the convergence radius of the homotopy-series of γ becomes larger when $c_0 < 0$ closer to zero is used. Therefore, different values of c_0 correspond to different convergence radius of the homotopy series of $\gamma = \omega^2$. In other words, we can adjust and control the convergence region of the homotopy-series by choosing different values of c_0 : this is exactly the reason why we call c_0 the convergence-control parameter.

According to Fig. 2.11, the smaller the absolute value of $c_0 < 0$, the larger the convergence radius of the power series of γ about ε . This suggests us to choose $c_0 = -1/(1 + \varepsilon)$, which gives the first-order approximation

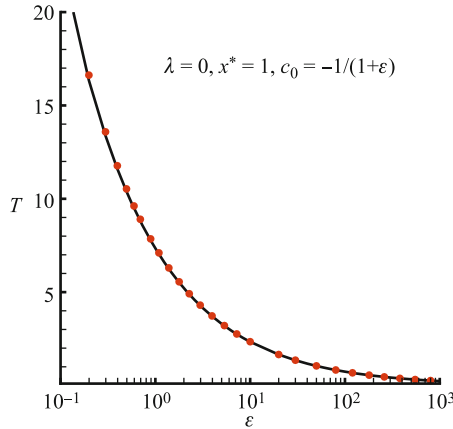
$$\gamma \approx \frac{3\varepsilon(32 + 31\varepsilon)}{128(1 + \varepsilon)} \quad (2.77)$$

and the second-order approximation

$$\gamma \approx \frac{3\varepsilon(128 + 248\varepsilon + 123\varepsilon^2)}{512(1 + \varepsilon)^2}, \quad (2.78)$$

respectively. Compared to the exact formula (2.76), the above expressions have 1.2% and 0.4% relative error, respectively, for *all* possible values of ε , i.e. $0 \leq$

Fig. 2.12 Comparison of the exact period $T = 7.4163/\sqrt{\varepsilon}$ with the 2nd-order homotopy-approximation (2.79) in case of $\lambda = 0, x^* = 1$ by means of $c_0 = -1/(1 + \varepsilon)$. Solid line: formula (2.79); Symbols: exact result.



$\varepsilon < +\infty$. Using the 2nd-order approximation (2.78) of $\gamma = \omega^2$, we have a simple approximate formula of the period

$$T \approx 32\pi (1 + \varepsilon) \sqrt{\frac{2}{3 \varepsilon (128 + 248 \varepsilon + 123 \varepsilon^2)}}, \tag{2.79}$$

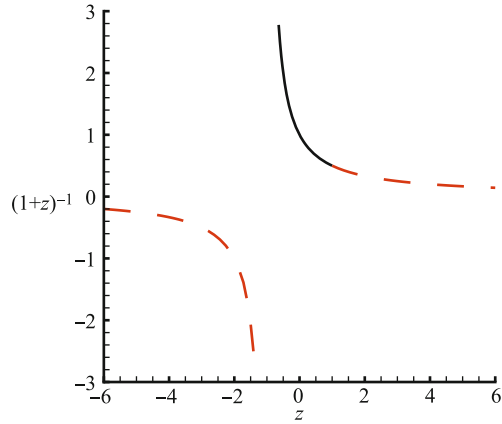
which agrees quite well with the exact period $T = 7.4163/\sqrt{\varepsilon}$ in the *whole* region $0 < \varepsilon < +\infty$, as shown in Fig. 2.12.

This is mainly because we have freedom to choose different values of the convergence-control parameter c_0 . Especially, when we choose $c_0 = -1/\gamma_0 = -4/(3\varepsilon)$, the homotopy-approximations of γ quickly converges even to the exact formula (2.76), as shown in Table 2.9. Note that the maximum relative errors of the 1st and 3rd-order approximation of γ given by $c_0 = -4/(3\varepsilon)$ are only 0.14% and 0.008%, respectively! It is very interesting that the corresponding 17th and 19th-order approximations of γ are the *same* as the exact formula (2.76) ! So, it seems

Table 2.9 Approximation of $\gamma = \omega^2$ in case of $\lambda = 0$ and $x^* = 1$ by means of $c_0 = -4/(3\varepsilon)$.

Order of approx.	γ
1	0.71875ε
3	0.7178276910ε
5	0.7177741910ε
7	0.7177703474ε
9	0.7177700399ε
11	0.7177700136ε
13	0.7177700113ε
15	0.7177700111ε
17	0.7177700110ε
19	0.7177700110ε

Fig. 2.13 Curve of the real function $(1+z)^{-1}$ defined in the interval $z \in (-\infty, +\infty)$. Solid line: the part which can be expressed by the series $\sum_{n=0}^{+\infty} (-z)^n$; Dashed line: the part which can not be expressed by the same series.



that $c_0 = -1/\gamma_0$ is the optimal convergence-control parameter in this special case. It illustrates the great potential of the HAM for strongly nonlinear problems.

Note that, the larger the value of ε , the stronger the nonlinearity of (2.32). However, in case of $\lambda = 0$ and $x^* = 1$, for *all* possible values of $\varepsilon > 0$, even including $\varepsilon \rightarrow +\infty$, we obtain simple but rather accurate approximations (2.77) and (2.78), and even the exact formula (2.76), by choosing a proper convergence-control parameter c_0 . All of these indicate that the convergence-control parameter c_0 indeed provides us a convenient way to guarantee the convergence of homotopy-series so that the HAM is valid for strongly nonlinear problems.

Finally, we use a simple example to explain why an auxiliary parameter can greatly influence the convergence of a series. The real function $(1+z)^{-1}$ is well defined in an infinite interval $-\infty < z < +\infty$, except the singular point $z = -1$. However, according to the so-called Newtonian binomial theorem, the series

$$\sum_{n=0}^{+\infty} (-z)^n = 1 - z + z^2 - z^3 + \dots \quad (2.80)$$

converges to $(1+z)^{-1}$ only in a rather small interval $|z| < 1$, as shown in Fig. 2.13. This is because $z = -1$ is a singular point of the function $(1+z)^{-1}$, and according to the traditional theorems, this kind of singular point determines the convergence radius of the power series $\sum_{n=0}^{+\infty} (-z)^n$, as shown in Fig. 2.13.

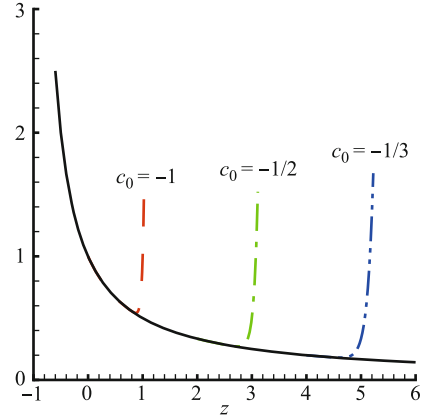
It is interesting that, using the Newtonian binomial theorem, we can rigorously prove the following theorem:

Theorem 2.3. *For a real number z and an auxiliary parameter $c_0 \neq 0$, it holds*

$$\frac{1}{1+z} = \lim_{m \rightarrow +\infty} \sum_{n=0}^m \mu_0^{m+1, n+1}(c_0) (-z)^n, \quad (2.81)$$

in the interval

Fig. 2.14 Comparison of $(1+z)^{-1}$ with the series (2.81) by different values of c_0 . Dashed line: $c_0 = -1$; Dash-dotted line: $c_0 = -1/2$; Dash-double-dotted line: $c_0 = -1/3$; Solid line: the exact function $(1+z)^{-1}$ when $z > -1$.



$$-1 < z < \frac{2}{|c_0|} - 1, \quad \text{when } c_0 < 0,$$

or

$$-\frac{2}{c_0} - 1 < z < -1, \quad \text{when } c_0 > 0,$$

where

$$\mu_0^{m,n}(c_0) = (-c_0)^n \sum_{i=0}^{m-n} \binom{n-1+i}{i} (1+c_0)^i. \quad (2.82)$$

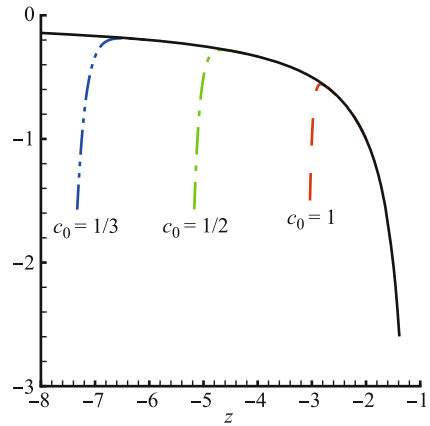
The detailed proof is given in Appendix 2.3.

According to the above theorem, the convergence region of the power series

$$\frac{1}{1+z} = \lim_{m \rightarrow +\infty} \sum_{n=0}^m \mu_0^{m+1,n+1}(c_0)(-z)^n$$

is determined by the auxiliary parameter c_0 . As shown in Fig. 2.14, when $c_0 = -1$, it is exactly the traditional Newtonian binomial and thus converges to $(1+z)^{-1}$ in the interval $-1 < z < 1$; when $c_0 = -1/2$, it converges to $(1+z)^{-1}$ in the interval $-1 < z < 3$; when $c_0 = -1/3$, it converges to $(1+z)^{-1}$ in the interval $-1 < z < 5$, respectively. Especially, according to Theorem 2.3, the above series converges to $(1+z)^{-1}$ in the infinite interval $-1 < z < +\infty$ when $c_0 < 0$ tends to zero! Similarly, as shown in Fig. 2.15, when $c_0 = 1$, it converges to $(1+z)^{-1}$ in the interval $-3 < z < -1$; when $c_0 = 1/2$, it converges to $(1+z)^{-1}$ in the interval $-5 < z < -1$; and when $c_0 = 1/3$, it converges to $(1+z)^{-1}$ in the interval $-7 < z < -1$. It is very interesting that, according to Theorem 2.3, the above series converges to $(1+z)^{-1}$ in the infinite interval $-\infty < z < -1$ when $c_0 > 0$ tends to zero! Thus, by simply introducing such a kind of auxiliary parameter c_0 , the convergence of the power series of $(1+z)^{-1}$ is modified fantastically: different from the traditional Newtonian binomial $(1+z)^{-1} = \sum_{n=0}^{+\infty} (-z)^n$ which is valid only in the small interval $|z| < 1$, the

Fig. 2.15 Comparison of $(1+z)^{-1}$ with the series (2.81) by different values of c_0 . Dashed line: $c_0 = 1$; Dash-dotted line: $c_0 = 1/2$; Dash-double-dotted line: $c_0 = 1/3$; Solid line: the exact function $(1+z)^{-1}$ when $z < -1$.



power series (2.81) can converge to $(1+z)^{-1}$ in the *infinite* interval $-\infty < z < +\infty$, except the singular point $z = -1$ only!

The above simple example illustrates that such an auxiliary parameter c_0 can indeed fantastically modify the convergence of a series. Note that Figs. 2.14 and 2.15 are in essence similar to Fig. 2.11. Therefore, in essence, the auxiliary parameter c_0 in the frame of the HAM provides us a convenient way to adjust and control the convergence of the homotopy-series. This is the reason why the auxiliary parameter c_0 is called the convergence-control parameter. There are some different ways to introduce such a kind of non-zero auxiliary parameter in approximation series. The way mentioned above about $(1+z)^{-1}$ is one of them. Another example is the Euler transform, which also uses an non-zero auxiliary parameter to enlarge the convergence region of a given series. Unfortunately, these two approaches can not be directly applied to nonlinear differential equations in general. It is the HAM which introduces such a kind of non-zero auxiliary parameter c_0 for nonlinear differential equations in general, which provides us a convenient way to guarantee the convergence of homotopy-series solution and to get optimal approximation by choosing an optimal value of c_0 . In fact, the HAM even logically contains the Euler transform: we can derive the famous Euler transform in the frame of the HAM, and give a similar but more general transform, called the generalized Euler transform, as shown in Chapter 5.

From the mathematical points of view, We can explain why the convergence of the power series (2.81) of $(1+z)^{-1}$ can be fantastically modified by the convergence-control parameter c_0 . Regard $\mathbf{S}_0 = (1, -1, 1, -1, \dots)$ as a point of an infinite-dimension space $\mathbf{R}^\infty$ (for example, a Hilbert space). Then, the traditional binomial $\sum_{n=0}^{+\infty} (-z)^n$ corresponds to an *unique* limit which tends to this point along such a traditional path:

$$\begin{aligned}
(1, 0, 0, 0, 0, \dots) &\in \mathbf{R}^\infty, \\
(1, -1, 0, 0, 0, \dots) &\in \mathbf{R}^\infty, \\
(1, -1, 1, 0, 0, \dots) &\in \mathbf{R}^\infty, \\
(1, -1, 1, -1, 0, \dots) &\in \mathbf{R}^\infty, \\
&\vdots
\end{aligned}$$

However, Liao (2003) proved in the book “Beyond Perturbation” that the function $\mu_0^{m,n}(c_0)$ has the property

$$\lim_{m \rightarrow +\infty} \mu_0^{m,n}(c_0) = \begin{cases} 1, & \text{when } -2 < c_0 < 0, \\ \infty, & \text{otherwise,} \end{cases}$$

for a given integer $n \geq 0$. Thus, when $-2 < c_0 < 0$, the series (2.81) corresponds to a *family* of limits which tend to the *same* point $\mathbf{S}_0 = (1, -1, 1, -1, \dots) \in \mathbf{R}^\infty$ along different approach paths dependent upon the value of c_0 :

$$\begin{aligned}
(\mu_0^{0,0}(c_0), 0, 0, 0, 0, \dots) &\in \mathbf{R}^\infty, \\
(\mu_0^{1,0}(c_0), -\mu_0^{1,1}(c_0), 0, 0, 0, \dots) &\in \mathbf{R}^\infty, \\
(\mu_0^{2,0}(c_0), -\mu_0^{2,1}(c_0), \mu_0^{2,2}(c_0), 0, 0, \dots) &\in \mathbf{R}^\infty, \\
(\mu_0^{3,0}(c_0), -\mu_0^{3,1}(c_0), \mu_0^{3,2}(c_0), -\mu_0^{3,3}(c_0), 0, \dots) &\in \mathbf{R}^\infty, \\
&\vdots
\end{aligned}$$

It is well-known that a limit of a function with multiple variables might be quite different for different approach paths. For example, the limit

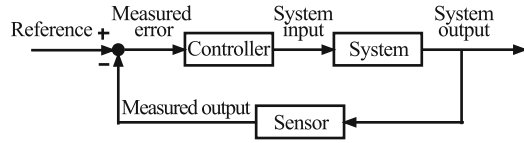
$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sqrt{x^2 + y^2}}{|x|} = \sqrt{1 + \alpha^2}$$

is dependent on an arbitrary real number α , if we gain the limit along the different approach paths $y = \alpha x$. This is the mathematical reason why the convergence region of the series (2.81) of $(1+z)^{-1}$ (when $z > 0$) can be greatly enlarged when $-2 < c_0 < 0$.

It is a pity that, this can not explain why the series (2.81) converges to $(1+z)^{-1}$ (when $z < -1$) by means of $c_0 > 0$. Note that, $\lim_{m \rightarrow +\infty} \mu_0^{m,n}(c_0) = \infty$ in case of $c_0 > 0$, as proved by Liao (2003) in his book. For example, when $c_0 = 1$, the polynomial $\sum_{n=0}^m \mu_0^{m+1,n+1}(c_0) (-z)^n$ reads

$$\begin{aligned}
-3 - z, & \quad \text{when } m = 1, \\
-15 - 17z - 7z^2 - z^3, & \quad \text{when } m = 3, \\
-63 - 129z - 111z^2 - 49z^3 - 11z^4 - z^5, & \quad \text{when } m = 5, \\
-255 - 769z - 1023z^2 - 769z^3 - 351z^4 - 97z^5 - 15z^6 - z^7, & \quad \text{when } m = 7, \\
& \vdots
\end{aligned}$$

Fig. 2.16 The feedback loop to control dynamic behavior of a system.



Note that, the constant term tends to infinity, so do the coefficients of *all* terms, conformed to the property $\lim_{m \rightarrow +\infty} \mu_0^{m,n}(c_0) = \infty$ for $c_0 > 0$. However, it is very interesting that, as $m \rightarrow +\infty$, the corresponding series (2.81) converges to $(1+z)^{-1}$ in $-3 < z < -1$, according to Theorem 2.3. So, in case of $c_0 > 0$, even if each term seems to be divergent, the series of (2.81) as a whole is convergent. Although this phenomena can not be clearly explained by the traditional method of determining the convergence radius of a power series, it shows the great power of such kind of auxiliary parameter c_0 .

Finally, from the view points of control theory, we explain why the convergence-control parameter c_0 can ensure the convergence of homotopy series. The feedback loop, as shown in Fig. 2.16, is a basic concept in control theory to control dynamic behaviors of a system. Let us regard the governing equation and the related boundary/initial conditions as a system. Then, the initial guess, the auxiliary linear operator and the convergence-control parameter c_0 can be regarded as “the system input”, and the m th-order homotopy-approximation as “the system output”, respectively. The squared residual of the governing equations of the m th-order homotopy-approximation can be regarded as the “measured error”. For different inputs, especially the different convergence-control parameter c_0 , the “measured error” is different. Our strategy is: keep c_0 as an unknown parameter in the “system input” and then choose its value in such a way that the “measured error”, i.e. the squared residual of the governing equations, is the minimum. In essence, this constructs a negative feedback loop to control the residual of governing equations! Note that, only by means of the computer algebra systems like Mathematica and Maple, the convergence-control parameter c_0 can be used as an unknown variable to appear in solution expressions. This is an essential difference between the symbolic computation and numerical techniques: all input of numerical methods must be assigned numerical values at the beginning of computation so that the input completely determine the convergence of iteration. For example, the so-called under-relaxation factor is widely used in most numerical techniques for strongly nonlinear equations to make an iteration approach convergent. However, at the beginning of iteration, one had to assign a value to the under-relaxation factor, and as long as its value is chosen, one can not “control” the convergence of the iteration approach any more: in principle, all numerical iterations can not construct a negative feedback loop from the viewpoints of control theory. However, using computer algebra systems like Mathematica and Maple, we need *not* assign a value to the convergence-control parameter c_0 at the beginning of computation, whose value is determined *after* the finish of the computation by means of the minimum of the squared residual of governing equations. So, from the viewpoint of the control theory, the convergence-control parameter c_0 provides us

in principle a *negative feedback loop* so that we can guarantee the convergence of the homotopy-series.

2.3.5 Convergence acceleration by homotopy-Padé technique

Let

$$x_0(\tau) + \sum_{n=1}^{+\infty} x_n(\tau) q^n$$

denote a homotopy-series gained by the HAM in general. Since the homotopy-series solution is given by

$$x(\tau) = x_0(\tau) + \sum_{n=1}^{+\infty} x_n(\tau),$$

it is very important to guarantee the convergence of the homotopy-series at $q = 1$. As shown above, the optimal convergence-control parameter c_0 of the HAM provides us a convenient way to guarantee the quick convergence of homotopy-series. In addition, there exist some other techniques to accelerate the convergence of a given series. Among them, the so-called Padé approximant developed by the French mathematician Henri Eugène Padé (1863 — 1953) is widely applied, which gives the “best” approximation of a given function by a rational function of given order.

For a power series

$$\sum_{n=0}^{+\infty} \alpha_n z^n,$$

the corresponding $[m, n]$ Padé approximant is expressed by

$$\frac{\sum_{k=0}^m a_{m,k} z^k}{\sum_{k=0}^n b_{m,k} z^k},$$

where $a_{m,k}, b_{m,k}$ are determined by the coefficients α_j ($j = 0, 1, 2, 3, \dots, m+n$). In many cases the traditional Padé technique can greatly increase the convergence region and rate of a given series. Note that such a traditional Padé approximant is a fraction, whose numerator and denominator are polynomials of z that is often a physical parameter.

The so-called homotopy-Padé technique (Liao, 2003b) is a combination of the traditional Padé technique with the homotopy analysis method. Regarding a homotopy-series

$$\tilde{x}(\tau; q) \sim x_0(\tau) + \sum_{n=1}^{+\infty} x_n(\tau) q^n$$

as a power series of q , we first employ the traditional $[m, n]$ Padé technique about the homotopy-parameter q to obtain the $[m, n]$ Padé approximant

$$\frac{\sum_{k=0}^m A_{m,k}(\tau) q^k}{\sum_{k=0}^n B_{m,k}(\tau) q^k}, \quad (2.83)$$

where the coefficients $A_{m,k}(\tau)$ and $B_{m,k}(\tau)$ are determined by the first $m+n$ terms

$$x_0(\tau), x_1(\tau), x_2(\tau), \dots, x_{m+n}(\tau)$$

of the homotopy-series. Then, setting $q = 1$ in (2.83), we have the so-called $[m, n]$ homotopy-Padé approximation

$$x(\tau) \approx \frac{\sum_{k=0}^m A_{m,k}(\tau)}{\sum_{k=0}^n B_{m,k}(\tau)}. \quad (2.84)$$

Note that the Padé approximant (2.83) is a fraction, whose numerator and denominator are polynomials of q , the homotopy-parameter without physical meanings. Therefore, both of the numerator and denominator of the $[m, n]$ homotopy-Padé approximation (2.84) are *unnecessary* to be polynomials: they can be any proper base functions. This provides us great freedom to choose different base functions. Thus, the so-called homotopy-Padé technique mentioned above is more general than the traditional ones.

For example, let us reconsider the nonlinear differential equation (2.32) in case of $\lambda = 0, x^* = 1$ and $\varepsilon > 0$. By means of the HAM mentioned before, we have

$$\begin{aligned} x_0 &= \cos \tau, \\ x_1 &= \frac{1}{32} c_0 \varepsilon (\cos \tau - \cos 3\tau), \\ x_2 &= \frac{c_0 \varepsilon}{1024} [(32 + 23c_0 \varepsilon) \cos \tau - (32 + 24c_0 \varepsilon) \cos 3\tau + c_0 \varepsilon \cos 5\tau], \\ &\vdots \end{aligned}$$

where c_0 is the convergence-control parameter. By means of the homotopy-Padé technique mentioned above, we have the $[1, 1]$ homotopy-Padé approximation

$$x(\tau) \approx \frac{21 \cos \tau}{(23 - 2 \cos 2\tau)}, \quad (2.85)$$

and the $[2, 2]$ homotopy-Padé approximation

$$x(\tau) \approx \frac{7723 \cos \tau - 513 \cos 3\tau - \cos 5\tau}{(9099 - 1940 \cos 2\tau + 50 \cos 4\tau)}, \quad (2.86)$$

respectively, where $\tau = \omega t$. Note that the numerator and denominator of these two homotopy-Padé approximations are not polynomials but trigonometric functions!

Similarly, based on the homotopy-series

$$\tilde{\gamma}(q) = \frac{3}{4}\varepsilon + \left(\frac{3}{128}c_0\varepsilon^2\right)q + \left[\frac{3}{512}c_0\varepsilon^2(4 + 3c_0\varepsilon)\right]q^2 + \cdots,$$

we obtain the corresponding $[m, m]$ homotopy-Padé approximations, as shown in Table 2.10. Note that the homotopy-Padé approximation of γ converges rather quickly to the exact result $\gamma = 0.7177700110\varepsilon$, even more quickly than the homotopy-series of γ given in Sect. 2.3.4 by $c_0 = -4/(3\varepsilon)$. Besides, it should be emphasized that all of these homotopy-Padé approximations of $x(\tau)$ and γ are *independent* of the *unknown* convergence-control parameter c_0 . Therefore, even if a bad value of the convergence-control parameter c_0 is chosen so that the corresponding homotopy-series is convergent slow or even divergent, we can obtain quickly convergent homotopy-series by means of the homotopy-Padé technique!

Table 2.10 $[m, m]$ homotopy-Padé approximations of $\gamma = \omega^2$ in case of $\lambda = 0, x^* = 1$ for arbitrary $\varepsilon > 0$ and the arbitrary unknown convergence-control parameter c_0 .

m	$\gamma = \omega^2$ given by the $[m, m]$ homotopy-Padé approx.
1	0.71875\varepsilon
2	0.7177996422\varepsilon
3	0.7177708977\varepsilon
4	0.7177700374\varepsilon
5	0.7177700118\varepsilon
6	0.7177700111\varepsilon
7	0.7177700110\varepsilon
8	0.7177700110\varepsilon
9	0.7177700110\varepsilon
10	0.7177700110\varepsilon

Besides, the homotopy-Padé approximations of $x(\tau)$ reveals that $x(\tau)$ is independent of the physical parameter ε . This is indeed true. As shown in Sect. 2.3.4, rather accurate approximations are obtained by means of the convergence-control parameter $c_0 = -1/\gamma_0 = -4/(3\varepsilon)$. Substituting $c_0\varepsilon = -4/3$ into x_1, x_2 and so on, it is found that the corresponding homotopy-series of $x(\tau)$ is indeed independent of ε . Note that the $[1, 1]$ homotopy-Padé approximation (2.85) is very simple, but rather accurate, as shown in Fig. 2.17. Therefore,

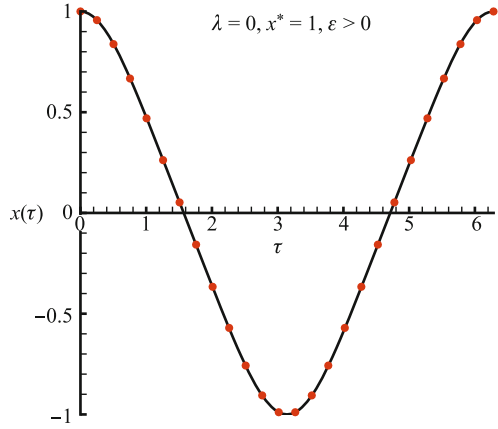
$$x \approx \frac{21 \cos \tau}{(23 - 2 \cos 2\tau)}, \quad \tau = 0.7177700110 \varepsilon t \quad (2.87)$$

is a simple but accurate approximation of $x(t)$ for *all* possible values of $\varepsilon \in (0, +\infty)$. All of these illustrate the great potential of the so-called homotopy-Padé technique.

Furthermore, for arbitrary λ, ε and x^* , replacing $x(\tau) = x^* y(\tau)$, we can rewrite

$$\gamma x''(\tau) + \lambda x(\tau) + \varepsilon x^3(\tau) = 0, \quad x(0) = x^*, \quad x'(0) = 0$$

Fig. 2.17 Comparison of the $[1,1]$ homotopy-Padé approximation (2.85) of $x(\tau)$ with the numerical ones in case of $\lambda = 0$, $x^* = 1$ and $\varepsilon > 0$. Solid line: approximation formula (2.85); Symbols: numerical result.



by

$$\gamma y''(\tau) + \lambda y(\tau) + \bar{\varepsilon} y^3(\tau) = 0, \quad y(0) = 1, \quad y'(0) = 0,$$

where $\bar{\varepsilon} = \varepsilon (x^*)^2$. In fact, this is the reason why we mainly consider $x^* = 1$ in Sect. 2.3. So, in case of $\lambda = 0$, using (2.87), we have a simple but accurate homotopy-Padé approximation

$$y \approx \frac{21 \cos \tau}{(23 - 2 \cos 2\tau)}, \quad \tau = 0.7177700110 \bar{\varepsilon} t,$$

which gives a rather simple but accurate approximation of the periodic oscillation

$$x(t) \approx \frac{21 x^* \cos \tau}{(23 - 2 \cos 2\tau)}, \quad \tau = 0.7177700110 \varepsilon (x^*)^2 t \quad (2.88)$$

with the corresponding period

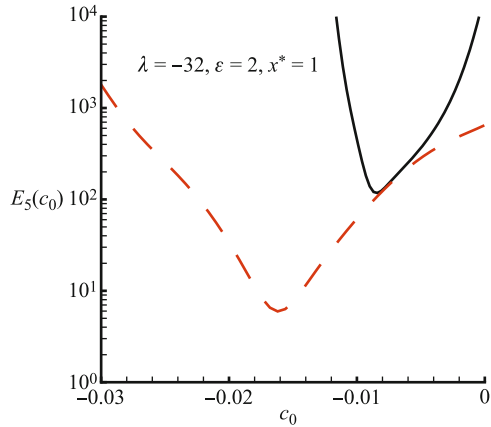
$$T \approx \frac{7.4163}{\sqrt{\bar{\varepsilon}} |x^*|} \quad (2.89)$$

for *arbitrary* values of $x^* \in (-\infty, +\infty)$ and $\varepsilon \in (0, +\infty)$. Note that the above accurate approximation of T clearly reveals the relationship between the period and the physical parameters and initial conditions. This illustrates that, using the HAM, we can indeed obtain rather simple but accurate approximations of some problems with very strong nonlinearity!

2.3.6 Convergence acceleration by optimal initial approximation

In the frame of the HAM, we have great freedom to choose the initial approximation $x_0(x)$ in the so-called zeroth-order deformation equation (2.42): any a periodic real function satisfying $x_0(0) = x^*, x'_0(0) = 0$ can be used. In Sect. 2.3.2, we choose the

Fig. 2.18 Discrete squared residual $E_5(c_0)$ versus c_0 in case of $\lambda = -32$, $\varepsilon = 2$ and $x^* = 1$ by means of different initial approximation $x_0(\tau)$. Solid line: $x_0 = 4 - 3 \cos \tau$; Dashed line: $x_0 = 2.9116 - 1.9116 \cos \tau$.



initial approximation

$$x_0(\tau) = \beta + (x^* - \beta) \cos \tau,$$

where β is defined by (2.34), although it can be an arbitrary real number in theory. According to the analysis of solution characteristic in Sect. 2.3.1, when $x = 0$ is the stable equilibrium point, the body oscillates about $x = 0$ with the amplitude x^* , so that the centre of the motion $x(\tau)$ is exactly the stable equilibrium point $x = 0$. In this case, we have $\beta = 0$ by the definition (2.34), which is physically correct and thus can give accurate homotopy-approximations, as mentioned before. However, when $x = \pm \sqrt{|\lambda/\varepsilon|}$ is the stable equilibrium point, the body oscillates about the stable equilibrium point, but due to the non-asymmetry of the force f near the stable equilibrium point, the centre of the motion $x(\tau)$ is different from the stable equilibrium point $x = \pm \sqrt{|\lambda/\varepsilon|}$. In this case, β given by (2.34) is only an approximation of the center of motion $x(\tau)$: the more asymmetric the force f near the stable equilibrium point $x = \pm \sqrt{|\lambda/\varepsilon|}$, the worse the approximation. Therefore, in some cases of $\beta \neq 0$, the homotopy-series obtained by the “normal” approach of the HAM mentioned in Sect. 2.3.2 converge slowly.

For example, let us consider the case

$$\lambda = -32, \quad \varepsilon = 2, \quad x^* = 1.$$

According to the analysis of solution characteristic in Sect. 2.3.1, $x = 4$ is a stable equilibrium point, and the body oscillates near $x = 4$ but not exactly around it. According to (2.34), we have $\beta = 4$, which gives the initial approximation

$$x_0 = 4 - 3 \cos \tau.$$

As mentioned in Sect. 2.3.2 and Sect. 2.3.3, we can obtain the curves of the discrete squared residual $E_m(c_0)$ versus c_0 , as shown in Fig. 2.18, which suggests an optimal convergence-control parameter $c_0^* = -0.008$. However, even by means of this optimal convergence-control parameter, the corresponding homotopy-series

converges rather slowly, as shown in Table 2.11. Note that the discrete squared residual $E_0(c_0^*)$ is rather large, which indicates that the initial approximation $x_0(\tau) = \beta + (x^* - \beta) \cos \tau$ with the definition (2.34), i.e.

$$x_0(\tau) = 4 - 3 \cos \tau,$$

is a bad initial approximation.

A simple way to modify the homotopy-approximations is to use a better initial approximation $x_0(\tau)$. Note that we have great freedom to construct the homotopy of equations $\tilde{E}(q) : \mathcal{E}_0 \sim \mathcal{E}_1$, which governs the homotopy of functions $\tilde{x}(\tau; q) : x_0(\tau) \sim x(\tau)$. Note also that, in theory, β in the initial approximation can be an *arbitrary* real number! So, using the same initial approximation $x_0(\tau) = \beta + (x^* - \beta) \cos \tau$ but regarding β as an unknown constant, we can get γ_0 by (2.68) that is now dependent on the unknown constant β . Then, the discrete squared residual E_0 given by the above initial approximation $x_0(\tau)$ is a function of β . Obviously, the “best” or “optimal” initial approximation $x_0(\tau)$ is given by the minimum of E_0 at $\beta = \beta^*$, where β^* is called the “best” or “optimal” value of β .

In case of $\lambda = -32$, $\varepsilon = 2$ and $x^* = 1$, by means of the above-mentioned approach and using the Mathematica command `Minimize` under the condition $\beta > 1$, we obtain the minimum of the squared residual of governing equation for the initial approximation at $\beta^* = 2.9116$, which gives us the optimal initial approximation

$$x_0(\tau) = 2.9116 - 1.9116 \cos \tau.$$

Similarly, using this optimal initial approximation, we can obtain the corresponding discrete squared residual $E_5(c_0)$ versus c_0 , as shown in Fig. 2.18, which suggests us an optimal convergence-control parameter $c_0^* = -2/125$. By means of the above optimal initial approximation and the optimal convergence-control parameter $c_0^* = -2/125$, the corresponding homotopy-series of $\gamma = \omega^2$ converges much faster than that given by the normal initial approximation $x_0 = 4 - 3 \cos \tau$, as shown in Table 2.12. Note that, the corresponding $[m, m]$ homotopy-Padé approximations converge faster, too, as shown in Table 2.13. Besides, the corresponding homotopy-approximations of $x(\tau)$ agree well with the numerical ones, as shown in Fig. 2.19. Note that the optimal initial approximation

$$x_0(\tau) = 2.9116 - 1.9116 \cos \tau$$

is more close to the exact ones than the normal initial approximation

$$x_0(\tau) = 4 - 3 \cos \tau.$$

According to Tables 2.11 and 2.12, the discrete squared residual given by the normal initial approximation $x_0(\tau) = 4 - 3 \cos \tau$ is 27.5 times larger than that given by the optimal ones $x_0(\tau) = 2.9116 - 1.9116 \cos \tau$. This is the essential reason why the homotopy-series given by the optimal initial approximation converges much faster.

Table 2.11 The discrete squared residual $E_m(c_0)$ and homotopy-approximations of $\gamma = \omega^2$ in case of $\lambda = -32$, $\varepsilon = 2$ and $x^* = 1$ by $\beta = 4$ and the optimal convergence-control parameter $c_0^* = -1/125$.

m , order of approx.	$E_m(c_0^*)$	$\gamma = \omega^2$
0	18046	77.5
5	127.2	32.849
10	30.1	33.227
15	10.1	31.489
20	2.5	31.945
25	1.4	31.348
30	0.3	31.643
40	5.8×10^{-2}	31.537
50	2.2×10^{-2}	31.491
60	1.2×10^{-2}	31.468

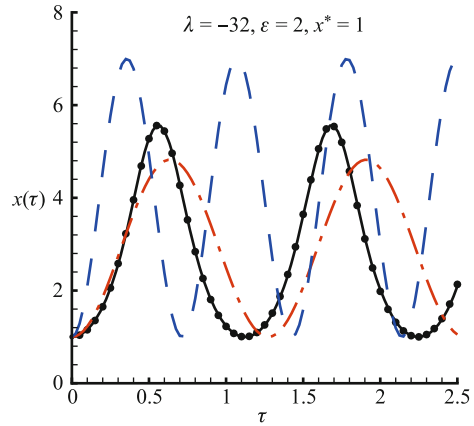
Table 2.12 The discrete squared residual $E_m(c_0)$ and homotopy-approximations of $\gamma = \omega^2$ in case of $\lambda = -32$, $\varepsilon = 2$ and $x^* = 1$ by $\beta^* = 2.9116$ and the optimal convergence-control parameter $c_0^* = -2/125$.

m , order of approx.	$E_m(c_0^*)$	$\gamma = \omega^2$
0	649.3	24.346
5	6.0	30.889
10	0.29	31.354
15	3.3×10^{-2}	31.359
20	5.8×10^{-3}	31.396
25	1.2×10^{-3}	31.410
30	2.7×10^{-4}	31.415
35	6.5×10^{-5}	31.418
40	1.6×10^{-5}	31.419
45	4.3×10^{-6}	31.420
50	1.2×10^{-6}	31.420
60	8.8×10^{-8}	31.420

Table 2.13 The $[m, m]$ homotopy-Padé approximations of $\gamma = \omega^2$ in case of $\lambda = -32$, $\varepsilon = 2$ and $x^* = 1$ by different initial approximations x_0 and different optimal convergence-control parameter.

m	$x_0 = 4 - 3 \cos \tau$, $c_0^* = -1/125$	$x_0^* = 2.9116 - 1.9116 \cos \tau$, $c_0^* = -2/125$
2	35.576	31.632
4	32.222	31.564
6	31.585	31.449
8	31.451	31.423
10	31.426	31.420
12	31.421	31.420
14	31.420	31.420
16	31.420	31.420
18	31.420	31.420
20	31.420	31.420

Fig. 2.19 Comparison of the numerical result with the homotopy-approximations in case of $\lambda = -32$, $\varepsilon = 2$ and $x^* = 1$ by means of $c_0 = -2/125$ and the optimal $\beta^* = 2.9116$. Symbols: numerical result; Solid line: 15th-order homotopy-approximation; Dash-dotted line: initial guess x_0 given by $\beta^* = 2.9116$; Dashed line: initial guess x_0 given by $\beta = 4$.



The optimization of initial approximations has general meanings in the frame of the HAM. In general, better homotopy-approximations are obtained by better initial approximations, if others are the same. Besides, as shown later in Chapter 8, using the great freedom on the choice of initial approximations, one can obtain multiple solutions of some nonlinear differential equations by means of the HAM.

2.3.7 Convergence acceleration by iteration

As mentioned above, the HAM provides us great freedom to choose the initial approximation. It is due to this freedom that we can choose an optimal initial approximation to greatly accelerate the convergence of homotopy-series, as shown in Sect. 2.3.6. Here, we further illustrate that, by means of the freedom on the choice of initial approximation, an iteration approach can be introduced in the frame of the HAM, which can greatly accelerate the convergence of the homotopy-series, too.

Note that any a real function, which satisfies the initial condition $x(0) = x^*$ and $x'(0) = 0$, can be used as an initial approximation $x_0(\tau)$ in the zeroth-order deformation equation (2.40). In Sect. 2.3.6, we illustrate that the convergence of the homotopy-series can be greatly accelerated by means of an optimal initial approximation. Obviously, the better the initial approximations, the faster the convergence of the corresponding homotopy-series. Clearly, the M th-order homotopy-approximation

$$\check{x}(\tau) \approx x_0(\tau) + \sum_{k=0}^M x_k(\tau) \quad (2.90)$$

satisfies the initial condition $x(0) = x^*$ and $x'(0) = 0$, and is often better than the initial approximation $x_0(\tau)$, if the convergence-control parameter c_0 is properly chosen. So, it is natural to use the above M th-order homotopy-approximation as a new initial approximation x_0 , i.e. $x_0(\tau) = \check{x}(\tau)$. In this way, a better homotopy-

approximation can be obtained in general. This provides us an iteration approach in the frame of the HAM. For simplicity, we call the above iteration approach as the M th-order homotopy-iteration.

Theoretically speaking, the periodic oscillation governed by (2.32) should be expressed by an infinite series

$$x(\tau) = \sum_{k=0}^{+\infty} a_k \cos(k\tau).$$

However, the first few terms are much more important than the high frequency terms, because the high frequency terms contribute very little to the accuracy of $x(\tau)$. So, in practice, a truncated expression

$$x(\tau) \approx \sum_{k=0}^N a_k \cos(k\tau) \quad (2.91)$$

is accurate enough for a large enough value of N . In other words, our homotopy-approximations contain the first $(N+1)$ terms, and all other terms with higher frequency, such as $\cos(N+1)\tau$, $\cos(N+2)\tau$ and so on, are neglected. To do so, we simply delete all higher-frequency terms of δ_{n-1} in the high-order deformation equation (2.55) before we solve it. Mathematically speaking, we approximate δ_{n-1} by

$$\delta_{n-1} \approx \sum_{k=0}^N A_{n,k} \cos(k\tau),$$

where

$$A_{n,0} = \frac{1}{2\pi} \int_0^{2\pi} \delta_{n-1}(\tau) d\tau, \quad A_{n,k} = \frac{1}{\pi} \int_0^{2\pi} \delta_{n-1}(\tau) \cos(k\tau) d\tau, \quad 1 \leq k \leq N.$$

In this way, the homotopy-approximation at each iteration can be expressed by (2.91) with a finite number of terms. This strategy of truncation is necessary for the iteration approach of the HAM. Otherwise, the length of homotopy-approximations increases exponentially in a few iterations so that the computational efficiency quickly becomes unacceptable. The corresponding Mathematica code (with iteration approach) is free available at

<http://numericaltank.sjtu.edu.cn/HAM.htm>

For example, let us consider again the case $\lambda = -32$, $\varepsilon = 2$ and $x^* = 1$. To compare the homotopy-approximations by the iteration approach with those given in Sect. 2.3.6, we use here the same optimal convergence-control parameter $c_0^* = -1/125$ and the same initial approximation $x_0 = 4 - 3 \cos \tau$. Without loss of generality, let us first use $M = 3$, i.e. a 3rd-order homotopy-iteration approach. Besides, we use $N = 21$, say, $x(\tau)$ is approximated by the first 21 low-frequency terms (from $\cos \tau$ to $\cos 21\tau$) only. The homotopy-approximations at each iteration and the corresponding discrete squared residual are listed in Table 2.14. Comparing these re-

sults with those in Table 2.11, the approximations given by the homotopy-iteration approach converge much faster than the normal HAM: the 11st iteration gives the exact value $\gamma = 31.420$ in only 4.8 seconds by a laptop (MacBook Pro, 2.8 GHz Inter Core 2 CPU, 4 GB EMS memory). Note that, the normal approach takes about 46.3 seconds, i.e. more than 9 times longer, to get the 60th-order approximation $\gamma = 31.468$, which still has a small difference from the exact result $\gamma = 31.420$.

Table 2.14 The discrete squared residual and homotopy-approximations of $\gamma = \omega^2$ in case of $\lambda = -32$, $\varepsilon = 2$ and $x^* = 1$ by the 3rd-order homotopy-iteration approach with the normal initial approximation $x_0 = 4 - 3 \cos \tau$ and the corresponding optimal convergence-control parameter $c_0^* = -1/125$.

m , iteration times	$E_m(c_0^*)$	$\gamma = \omega^2$
1	286.5	43.850
2	34.3	32.818
3	9.8	31.789
4	2.9	31.527
6	0.2	31.427
8	1.7×10^{-2}	31.419
10	1.2×10^{-3}	31.419
11	3.1×10^{-4}	31.420
12	8.3×10^{-5}	31.420
14	5.8×10^{-6}	31.420
16	4.0×10^{-7}	31.420
18	2.7×10^{-8}	31.420
20	1.9×10^{-9}	31.420

Let us compare the accuracy and used CPU time of the iteration approach and the normal HAM without iteration in details. It is found that the CPU time increases *linearly* with the iteration times for the iteration approach, but increases *exponentially* with the order of approximation for the non-iteration approach, as shown in Fig. 2.20. Thus, the iteration approach is computationally more efficient. More importantly, for the iteration approach, the squared residual of the homotopy-approximations decreases *exponentially* as the iteration times m increases, as shown in Fig. 2.21, until the minimum of the squared residual $E_{\min}(N) = 5.2 \times 10^{-21}$ is arrived. However, for normal approach without iteration, the squared residual decreases *algebraically* as the order of approximation m increases, as shown in Fig. 2.21. Thus, the approximations given by iteration approach converges much faster.

The discrete squared residual versus the CPU time of two approaches are given in Fig. 2.22, which indicates that the squared residual given by the iteration approach decreases *exponentially* with the CPU time until its minimum value $E_{\min}(N)$, whereas the squared residual given by the non-iteration approach decreases much more slowly. That means, using the same CPU time, one can obtain much more accurate homotopy-approximations by means of the iteration approach. All of the above conclusions have general meanings: they are correct for different-order of

Fig. 2.20 CPU time versus m in case of $\lambda = -32$, $\varepsilon = 2$ and $x^* = 1$ by means of $x_0 = 4 - 3 \cos \tau$ and $c_0 = -1/125$. Solid line: 3rd-order homotopy-iteration approach (m denotes the iteration times); Dashed line: normal approach without iteration (m denotes the order of homotopy-approximation).

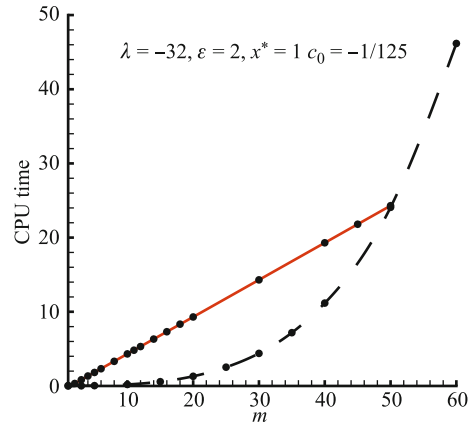


Fig. 2.21 Squared residual versus m in case of $\lambda = -32$, $\varepsilon = 2$ and $x^* = 1$ by means of $x_0 = 4 - 3 \cos \tau$ and $c_0 = -1/125$. Solid line: 3rd-order homotopy-iteration approach (m denotes the iteration times); Dashed line: normal approach without iteration (m denotes the order of homotopy-approximation).

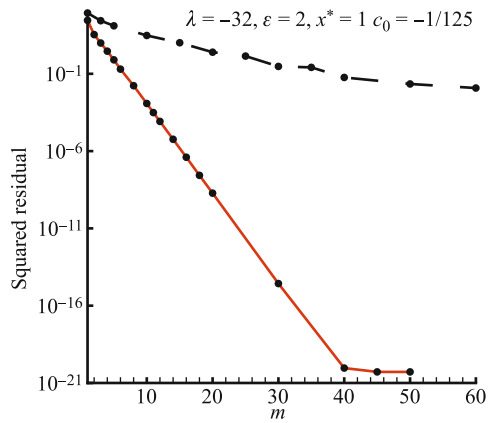
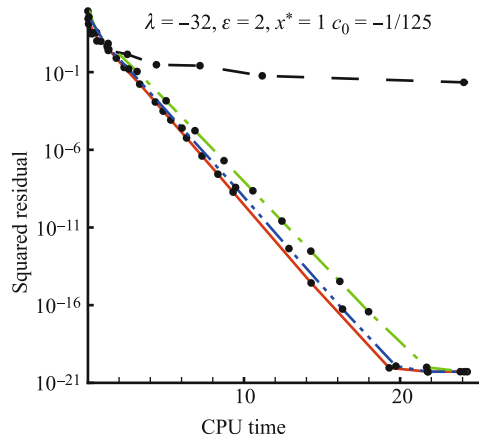


Fig. 2.22 Squared residual versus CPU time in case of $\lambda = -32$, $\varepsilon = 2$ and $x^* = 1$ by means of $x_0 = 4 - 3 \cos \tau$ and $c_0 = -1/125$. Dashed line: normal approach without iteration. Solid line: 3rd-order homotopy-iteration approach; Dash-dotted line: 2nd-order homotopy-iteration approach; Dash-double-dotted line: 4th-order homotopy-iteration approach.



homotopy-iteration approaches, as shown in Fig. 2.22. Note that the approximations given by the 3rd-order homotopy-iteration approach converges a little faster than those given by the 2nd and 4th-order homotopy-iteration approaches, whereas the approximations given by the 4th-order homotopy-iteration approach converge a little faster than those given by the 2nd-order. In practice, the 2nd, 3rd and 4th-order homotopy-iteration approaches are good enough, and higher-order iteration formulas are often unnecessary.

As shown in Figs. 2.21 and 2.22, when the above-mentioned minimum of the squared residual $E_{\min} = 5.2 \times 10^{-21}$ is arrived, one can not get better homotopy-approximations by more iterations. This is mainly because we use finite number (denoted by N) of terms in (2.91) to approximate $x(\tau)$. Theoretical speaking, the minimum squared residual $E_{\min}$ of the iteration approach is dependent upon N , and it should decrease if more terms in (2.91) are employed. This is indeed true in practice, as shown in Table 2.15, which illustrates that $E_{\min}$ is dependent upon N , the number of terms in (2.91), but has nothing to do with M , the order of homotopy-approximation formula in (2.90) at each iteration.

Let a_n denote the coefficient of the term $\cos(n\tau)$ in (2.91). It is found that each coefficient a_n ($0 \leq n \leq 21$) is modified at every homotopy-iteration, as listed in Table 2.16. Note that, as the iteration times increase, each coefficient a_n ($0 \leq n \leq 21$) converges quickly to a fixed value. For example, the coefficients a_0, a_1, a_5, a_{10} and a_{21} converge to $2.8027, -2.1860, -3.87 \times 10^{-3}, 1.33 \times 10^{-6}$ and -3.22×10^{-14} , respectively. Obviously, the coefficients for other higher-frequency terms are rather small, and thus can be neglected without loss of the accuracy of the homotopy-approximation $x(\tau)$.

To confirm the generality of the homotopy-iteration approach, we further apply it to all cases mentioned above in Sect. 2.3. The 3rd-order homotopy-iteration formula is used with the normal initial approximations $x_0 = \beta + (x^* - \beta) \cos \tau$. At the beginning of iteration, the convergence-control parameter c_0 is unknown. An optimal value of c_0 is determined at the first iteration by the minimum of the squared residual of the 3rd-order homotopy-approximation, and this optimal value is used for all other iterations. It is found that, for all of these cases, the approximations given by the homotopy-approach converge rather quickly, as shown in Tables 2.17 to 2.19. Note that, we obtain exact values of $\gamma = \omega^2$ mostly by two or three iterations. And the squared residual decreases rather quickly. Note that, in case of $\lambda = 9/4, \varepsilon = 1, x^* = 1$ and $\lambda = 4, \varepsilon = -1, x^* = -1$, the discrete squared residual stops decreasing at the 5th iteration, mainly because we approximate $x(\tau)$ by the first 21 low-frequency terms. All of these illustrate that, like the homotopy approach using the optimal initial approximation, the homotopy-iteration approach gives quickly convergent results with rather high computational efficiency in general.

The results given by the homotopy-iteration approach is still analytic approximations, because, different from numerical solutions, our homotopy-approximations are expressed by the continuous base functions $\cos(n\tau)$. Therefore, it is easy for us to give *arbitrary* order derivatives of $x(\tau)$ at *any* time $0 \leq \tau < +\infty$. This is impossi-

ble for any numerical methods. So, in essence, the homotopy-approximations given by the iteration approach are still analytic results.

The homotopy-iteration approach has general meanings and can be widely applied to greatly accelerate the convergence of homotopy-approximations, as shown later in this book (see Chapter 8, Chapter 9 and so on).

Table 2.15 The minimum of the squared residual $E_{\min}$ in case of $\lambda = -32, \varepsilon = 2, x^* = 1$ with $c_0 = -1/125$ and $x_0 = 4 - 3 \cos \tau$ by means of the different order homotopy-iteration approaches and the different number of terms N in (2.91).

N	2nd-order iteration formula ($M = 2$)	3rd-order iteration formula ($M = 3$)	4th-order iteration formula ($M = 4$)
5	0.389	0.389	0.389
10	5.51×10^{-7}	5.51×10^{-7}	5.51×10^{-7}
15	2.96×10^{-13}	2.96×10^{-13}	2.96×10^{-13}
21	5.2×10^{-21}	5.2×10^{-21}	5.2×10^{-21}
23	1.4×10^{-23}	1.4×10^{-23}	1.4×10^{-23}

Table 2.16 The modification of the coefficient a_n of (2.91) at the m times iteration in case of $\lambda = -32, \varepsilon = 2$ and $x^* = 1$ by the 3rd-order homotopy-iteration approach with the normal initial approximation $x_0 = 4 - 3 \cos \tau$ and the corresponding optimal convergence-control parameter $q_0^* = -1/125$.

m	a_0	a_1	a_5	a_{10}	a_{21}
1	2.9863	-2.2963	-3.60×10^{-4}	0	0
3	2.8236	-2.1977	-2.02×10^{-3}	1.52×10^{-7}	-4.65×10^{-17}
5	2.8063	-2.1863	-3.17×10^{-3}	6.10×10^{-7}	-2.57×10^{-15}
10	2.8027	-2.1861	-3.83×10^{-3}	1.26×10^{-7}	-2.44×10^{-14}
15	2.8027	-2.1860	-3.87×10^{-3}	1.33×10^{-6}	-3.15×10^{-14}
20	2.8027	-2.1860	-3.87×10^{-3}	1.33×10^{-6}	-3.22×10^{-14}
25	2.8027	-2.1860	-3.87×10^{-3}	1.33×10^{-6}	-3.22×10^{-14}
30	2.8027	-2.1860	-3.87×10^{-3}	1.33×10^{-6}	-3.22×10^{-14}

Table 2.17 The discrete squared residual and homotopy-approximations of $\gamma = \omega^2$ in case of $\lambda = -9/4, \varepsilon = 1$ and $x^* = 1$ by the 3rd-order homotopy-iteration approach with the normal initial approximation $x_0 = (3 - \cos \tau)/2$ and the corresponding optimal convergence-control parameter $c_0^* = -0.1632$.

m , iteration times	$E_m(c_0^*)$	$\gamma = \omega^2$
1	5.0×10^{-4}	3.9932
2	1.1×10^{-6}	3.9278
3	5.9×10^{-9}	3.9278
4	3.3×10^{-11}	3.9278
5	1.9×10^{-13}	3.9278

Table 2.18 The homotopy-approximations of $\gamma = \omega^2$ in three different cases by the 3rd-order homotopy-iteration approach with the normal initial approximation and the corresponding optimal convergence-control parameter c_0^* .

iteration times	$\lambda = 9/4, \varepsilon = 1,$ $x^* = 1,$ $c_0^* = -0.3333$	$\lambda = 0, \varepsilon = 1,$ $x^* = 1,$ $c_0^* = -1.3314$	$\lambda = 4, \varepsilon = -1,$ $x^* = -1,$ $c_0^* = -0.3075$
1	2.9921875001	0.7187500657	3.2427884644
2	2.9921730367	0.7177745854	3.2427770917
3	2.9921730364	0.7177700220	3.2427770917
4	2.9921730364	0.7177700111	3.2427770917
5	2.9921730364	0.7177700110	3.2427770917

Table 2.19 The discrete squared residual $E_m(c_0^*)$ in three different cases by the 3rd-order homotopy-iteration approach with the normal initial approximation and the corresponding optimal convergence-control parameter c_0^* .

iteration times	$\lambda = 9/4, \varepsilon = 1$ $x^* = 1,$ $c_0^* = -0.3333$	$\lambda = 0, \varepsilon = 1$ $x^* = 1,$ $c_0^* = -1.3314$	$\lambda = 4, \varepsilon = -1$ $x^* = -1,$ $c_0^* = -0.3075$
1	6.0×10^{-10}	2.2×10^{-6}	4.0×10^{-10}
2	5.7×10^{-20}	1.7×10^{-11}	5.8×10^{-22}
3	4.0×10^{-31}	9.5×10^{-17}	6.9×10^{-33}
4	3.3×10^{-38}	4.9×10^{-22}	6.6×10^{-39}
5	3.3×10^{-38}	7.4×10^{-26}	6.6×10^{-39}

2.3.8 Flexibility on the choice of auxiliary linear operator

In addition, we have great freedom to choose the auxiliary linear operator $\mathcal{L}$ in the so-called zeroth-order deformation equation (2.42). As pointed out in Sect. 2.3.2, the linear operator $\mathcal{L}_0(x) = x'' + \lambda x$, which is exactly the linear part of the original governing equation, can not give periodic approximations when $\lambda < 0$ or $\lambda = 0$. However, due to the freedom on the choice of the auxiliary linear operator $\mathcal{L}$, we can simply choose the auxiliary linear operator $\mathcal{L}(x) = x'' + x$ in the zeroth-order deformation equation (2.42) for *all* possible physical parameters related to periodic solutions, even including $\lambda \leq 0$. So, different from perturbation techniques, whose auxiliary linear operators are closely connected to the small/large physical parameters and types of considered equations, we have rather large freedom to choose a proper auxiliary linear operator to get accurate approximations by means of the HAM, as shown in Sect. 2.3.

Besides, due to the great freedom on constructing homotopy of equations and functions, we introduce the convergence-control parameter c_0 in the zeroth-order deformation equation (2.42). As shown in Sect. 2.3.3 and Sect. 2.3.4, the convergence-control parameter c_0 provides us a convenient way to guarantee the convergence of homotopy-series solution. Thus, the convergence-control parameter c_0 has an essen-

tial role in the frame of the HAM. It is interesting that, if we define such an auxiliary linear operator $\bar{\mathcal{L}} = \mathcal{L}/c_0$, then the zeroth-order deformation equation (2.42) can be rewritten as

$$(1 - q)\bar{\mathcal{L}}[\tilde{x}(\tau; q) - x_0(\tau)] = q [\tilde{\gamma}(q) \tilde{x}''(\tau; q) + \lambda \tilde{x}(\tau; q) + \varepsilon \tilde{x}^3(\tau; q)].$$

So, mathematically, the convergence-control parameter c_0 can be regarded as a part of the auxiliary linear operator $\mathcal{L}$. In essence, different values of the convergence-control parameter c_0 correspond to different auxiliary linear operators $\bar{\mathcal{L}} = \mathcal{L}/c_0$. Especially, the optimal convergence-control parameter c_0^* gives the optimal auxiliary linear operator $\bar{\mathcal{L}}$. So, in essence, it is due to the freedom on the choice of the auxiliary linear operator that we introduce the so-called convergence-control parameter c_0 in the zeroth-order deformation equation (2.42). Note that the concept of convergence-control is a key of the HAM.

Liao and Tan (2007) illustrated that, in the frame of the HAM, we have much larger freedom on the choice of the auxiliary linear operator $\mathcal{L}$ than we had thought traditionally. For example, let us consider the zeroth-order deformation equation (2.42). Using the freedom on the choice of the auxiliary linear operator $\mathcal{L}$ in (2.42), we can choose such an auxiliary linear operator

$$\check{\mathcal{L}}(x) = \frac{\partial^{2\kappa} x}{\partial \tau^{2\kappa}} + (-1)^{\kappa+1} x, \quad \kappa = 1, 2, 3, \dots, \quad (2.92)$$

where $\kappa \geq 1$ is an *arbitrary* positive integer. When $\kappa = 1$, we have $\check{\mathcal{L}}(x) = x'' + x$, so that $\check{\mathcal{L}}$ is exactly the same as the 2nd-order auxiliary linear operator $\mathcal{L}$ defined by (2.63). Thus, the above auxiliary linear operator is more general than (2.63). Note that all mathematical formulas in Sect. 2.3.2 and Sect. 2.3.3, such as the high-order deformation equations (2.55) and so on, keep the same in form, except that $\mathcal{L}$ defined by (2.63) is replaced by $\check{\mathcal{L}}$ defined by (2.92), a more general auxiliary linear operator. Note that we search for periodic solution expressed by (2.28). Thus, as mentioned in Sect. 2.3.7, δ_{n-1} in (2.55) can be expressed as a sum of some cosine functions. Thus, by means of the general auxiliary linear operator defined by (2.92), the high-order deformation equation (2.55) has a special solution

$$x_n^s(\tau) = \chi_n x_{n-1}(\tau) + c_0 \check{\mathcal{L}}^{-1}[\delta_{n-1}(X_{n-1}, \Gamma_{n-1})], \quad (2.93)$$

where the inverse operator $\check{\mathcal{L}}^{-1}$ is defined by

$$\check{\mathcal{L}}^{-1}[\cos(m\tau)] = \frac{(-1)^{\kappa+1} \cos(m\tau)}{(1 - m^{2\kappa})}, \quad m \neq 1, \quad (2.94)$$

and

$$\check{\mathcal{L}}^{-1}(C) = (-1)^{\kappa+1} C \quad (2.95)$$

for any constant C .

When $\kappa = 2$, the auxiliary linear operator (2.92) becomes $\check{\mathcal{L}}(x) = x^{(4)} - x$, where $u^{(4)}$ denotes the 4th-order differentiation with respect to τ . This 4th-order linear

operator has the property

$$\mathcal{L}(C_1 \cos \tau + C_2 \sin \tau + C_3 e^\tau + C_4 e^{-\tau}) = 0 \quad (2.96)$$

for any constant coefficients C_1, C_2, C_3 and C_4 . So, when $\kappa = 2$, the general solution of the high-order deformation equation (2.55) reads

$$x_n(\tau) = \chi_n x_{n-1}(\tau) + x_n^s(\tau) + C_1 \cos \tau + C_2 \sin \tau + C_3 e^\tau + C_4 e^{-\tau}, \quad (2.97)$$

where the special solution $x_n^s(\tau)$ is given by (2.93). Note that there are *four* integral coefficients C_1, C_2, C_3 and C_4 , but we have only *two* initial conditions defined by (2.56). Note that we search for periodic solutions of (2.32), expressed by (2.28). However, the terms e^τ and $e^{-\tau}$ are *not* periodic functions, and thus *disobey* the solution expression (2.28). Therefore, they can not appear in the expression of $x_n(\tau)$. So, in order to obtain periodic solution of $x_n(\tau)$, we must enforce $C_3 = C_4 = 0$. In other words, the solution expression (2.28) implies one additional period condition

$$x_n(\tau) = x_n(\tau + 2\pi),$$

which enforces $C_3 = C_4 = 0$. This condition comes from the periodic property of $x(\tau)$, i.e.

$$x(\tau) = x(\tau + 2\pi).$$

Thereafter, C_1 and C_2 are uniquely determined by the two initial conditions defined by (2.56).

Similarly, when $\kappa = 3$, the auxiliary linear operator (2.92) becomes

$$\mathcal{L}(x) = \frac{d^6 x}{d\tau^6} + x,$$

and the general solution of the corresponding high-order deformation equation (2.55) reads

$$x_n(\tau) = \chi_n x_{n-1}(\tau) + x_n^s(\tau) + C_1 \cos \tau + C_2 \sin \tau + e^{\sqrt{3}\tau/2}(C_3 \cos \tau + C_4 \sin \tau) + e^{-\sqrt{3}\tau/2}(C_5 \cos \tau + C_6 \sin \tau), \quad (2.98)$$

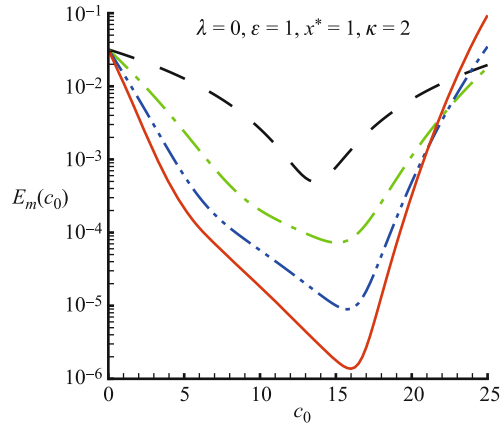
where the special solution $x_n^s(\tau)$ is given by (2.93), and C_i is a constant coefficient. Similarly, to obtain periodic solution of $x_n(\tau)$, we must enforce

$$C_3 = C_4 = C_5 = C_6 = 0.$$

The left two constants C_1 and C_2 are uniquely determined by the two initial conditions defined by (2.56).

Similarly, for any positive integers $\kappa \geq 1$, we can always obtain a *periodic* solution $x_n(\tau)$ of the high-order deformation equations (2.55) and (2.56) by means of the (2κ) th-order auxiliary linear operator $\mathcal{L}$ defined by (2.92). Here, it should be emphasized that, although e^τ , $e^\tau \cos \tau$ and $e^\tau \sin \tau$, which tend to infinity as $\tau \rightarrow +\infty$,

Fig. 2.23 Squared residual versus c_0 in case of $\lambda = 0$, $\varepsilon = 1$ and $x^* = 1$ by means of the 4th-order auxiliary linear operator $\mathcal{L}(x) = x'''' - x$ (corresponding to $\kappa = 2$). Dashed line: 1st-order approx.; Dash-dotted line: 3rd-order approx.; Dash-double-dotted line: 5th-order approx.; Solid line: 7th-order approx.



are the so-called secular terms, the terms $e^{-\tau}$, $e^{-\tau} \cos \tau$, $e^{-\tau} \sin \tau$ tend to zero as $\tau \rightarrow +\infty$ and thus do not belong to the traditional definition of secular term. So, using the traditional ideas of “avoiding the secular terms”, we can *not* avoid the appearance of the non-periodic terms $e^{-\tau}$, $e^{-\tau} \cos \tau$, $e^{-\tau} \sin \tau$ in $x_n(\tau)$. Therefore, the concept of *solution expression* is more general than the traditional ideas of “avoiding secular terms”. Indeed, the concept of solution expression has a very important role in the frame of the HAM.

Without loss of generality, let us consider here the case $\lambda = 0$, $\varepsilon = 1$ and $x^* = 1$. In this case, the corresponding initial approximation reads $x_0 = \cos \tau$. First of all, let us consider the case of $\kappa = 2$, i.e. the 4th-order differential operator $\mathcal{L}(x) = x'''' - x$ is used as the auxiliary linear operator. In a similar way as mentioned in Sect. 2.3.3, we can obtain the curves of the discrete squared residual $E_m(c_0)$ versus the convergence-control parameter c_0 , as shown in Fig. 2.23. The discrete squared residual at the 7th-order of approximation arrives the minimum at $c_0 = 15.97$. So, we choose the optimal convergence-control parameter $c_0^* = 16$. It is found that the discrete squared residual decreases monotonously as m , the order of approximation, increases, and $\gamma = \omega^2$ converges to the exact value 0.7177700110, as shown in Table 2.20. Besides, the corresponding $[m, m]$ homotopy-Padé approximations of $\gamma = \omega^2$ also converge to the exact value 0.7177700110, as shown in Table 2.21. Furthermore, even the 5th-order approximation is rather accurate: the 5th-order approximation of γ has only 0.006% relatively error, and the 5th-order approximation of $x(\tau)$ agrees well with the numerical ones, as shown in Fig. 2.24. All of these indicate that, the homotopy-approximations given by means of the 4th-order auxiliary linear operator

$$\mathcal{L}(x) = x'''' - x,$$

corresponding to $\kappa = 2$ in (2.92), are indeed convergent. Therefore, we can apply the HAM to obtain convergent homotopy-series solution of the original 2nd-order nonlinear differential equation (2.32) even by means of the 4th-order differential

Table 2.20 The discrete squared residual and homotopy-approximations of $\gamma = \omega^2$ in case of $\lambda = 0$, $\varepsilon = 1$ and $x^* = 1$ by the 4th-order ($\kappa = 2$) auxiliary linear operator $\mathcal{L}(x) = x'''' - x$ with the normal initial approximation $x_0 = \cos \tau$ and the corresponding optimal convergence-control parameter $c_0^* = 16$.

m , order of approx.	$E_m(c_0^*)$	$\gamma = \omega^2$
3	8.3×10^{-5}	0.7172586169
5	9.1×10^{-6}	0.7177245364
10	1.5×10^{-7}	0.7178419180
20	1.1×10^{-9}	0.7177729536
30	2.4×10^{-11}	0.7177703504
40	1.2×10^{-12}	0.7177700725
50	9.0×10^{-14}	0.7177700257
60	8.7×10^{-15}	0.7177700152
70	1.1×10^{-15}	0.7177700124
80	1.6×10^{-16}	0.7177700115
90	2.8×10^{-17}	0.7177700112
100	5.4×10^{-18}	0.7177700111

Table 2.21 $[m, m]$ homotopy-Padé approximations of $\gamma = \omega^2$ in case of $\lambda = 0$, $\varepsilon = 1$ and $x^* = 1$ by means of the 4th-order auxiliary linear operator $\mathcal{L}(x) = x'''' - x$ (corresponding to $\kappa = 2$).

m	$\gamma = \omega^2$ given by the $[m, m]$ homotopy-Padé approx.
5	0.7177585790
10	0.7177700704
15	0.7177700117
20	0.7177700111
25	0.7177700110
30	0.7177700110
35	0.7177700110
40	0.7177700110
45	0.7177700110
50	0.7177700110

operator as the auxiliary linear operator $\mathcal{L}$. This indicates that the HAM indeed provides us extremely large freedom to choose the auxiliary linear operator.

Similarly, we can further investigate the 6th-order auxiliary linear operator, corresponding to $\kappa = 3$ in (2.92). It is found that a convergent homotopy-series solution can be obtained by means of this 6th-order auxiliary linear operator, too. Therefore, one can obtain convergent series solution by the κ th-order auxiliary linear operator (2.92) for an *arbitrary* positive integer $\kappa \geq 1$. However, it is found that the homotopy-series given by the 6th-order auxiliary linear operator converges more slowly than that given by the 4th-order ones, as shown in Fig. 2.25. Besides, the homotopy-series given by the 2nd-order auxiliary linear operator converges faster than that given by the 4th and 6th-order ones, as shown in Fig. 2.25. In general, the larger the value of κ , the more slowly the corresponding homotopy-series con-

Fig. 2.24 Comparison of the homotopy-approximation with the numerical result in case of $\lambda = 0$, $\varepsilon = 1$, $x^* = 1$ by means of $c_0^* = 16$ and the 4th-order auxiliary linear operator $\mathcal{L}(x) = x'''' - x$ (corresponding to $\kappa = 2$). Solid line: 5th-order approximation; Symbols: numerical result.

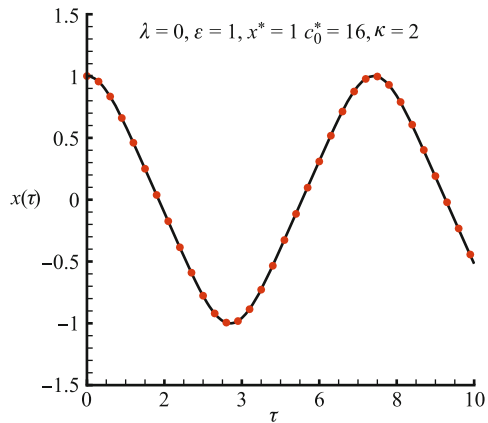
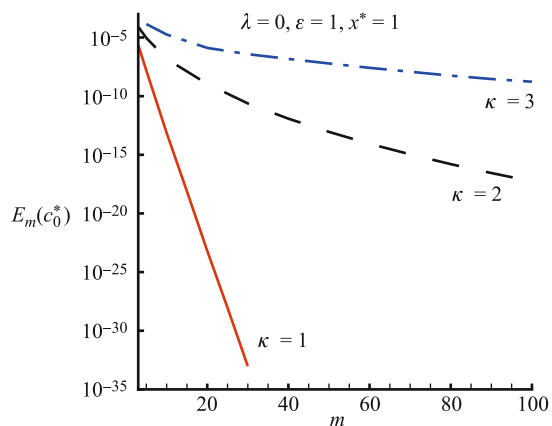


Fig. 2.25 Discrete squared residual E_m versus the order m of homotopy-approximation in case of $\lambda = 0$, $\varepsilon = 1$, $x^* = 1$ by means of different auxiliary linear operators defined by (2.92). Solid line: $\kappa = 1$ with $c_0^* = -4/3$; Dashed line: $\kappa = 2$ with $c_0^* = 16$; Dash-dotted line: $\kappa = 3$ with $c_0^* = -160$.



verges. Therefore, the 2nd-order auxiliary linear operator $\mathcal{L}(x) = x'' + x$ may be the best among the infinite number of auxiliary linear operators defined by (2.92).

It is found that the general auxiliary linear operator defined by (2.92) only works for the cases with $x = 0$ being a stable equilibrium point. This fact warns us that such kind of freedom is valuable only when it can be properly used. In practice, if such kind of freedom is used properly, one can solve some nonlinear problems in a very simple way. For example, as shown in Chapter 14, a 2nd-order nonlinear differential equation (Gelfand equation) can be successfully solved in a rather simple way by means of a 4th-order auxiliary linear operator.

In summary, in the frame of the HAM, the 2nd-order nonlinear differential equation (2.32) can be even transferred into an infinite number of 4th or 6th-order linear differential equations. To the best of my knowledge, this is against the mainstream of the traditional thought: it is often believed that one m th-order nonlinear differential equation should be transferred into an infinite number of linear differential equations with the same or lower order. So, although this simple example has no practical meaning (because the 2nd-order auxiliary linear operator $\mathcal{L}(x) = x'' + x$ seems to

be the best for this nonlinear oscillation problem), it shows in theory that we indeed have much larger freedom to solve nonlinear problems than we thought and believed before. I personally believe, if this kind of extremely large freedom is properly used, more accurate and better solutions of some difficult nonlinear problems should be found. Indeed, “the essence of mathematics lies entirely in its freedom”, as pointed out by Georg Cantor (1845 — 1918).

2.4 Concluding remarks and discussions

In this chapter, the basic ideas of the HAM are described by means of two simple examples: one is the nonlinear algebraic equation $f(x) = 0$, the other is the 2nd-order nonlinear differential equation (2.32) for periodic oscillations of a body. Although these two examples are rather simple, all methods and concepts mentioned in this chapter have general meanings.

The HAM has a solid base, i.e. the homotopy in topology, which connects two different things with a few same mathematical aspects by a continuous mapping. Given an equation that has at least one solution, denoted by $\mathcal{E}_1$, one first finds a proper, much simpler equation with a known solution, denoted by $\mathcal{E}_0$, and then constructs such a homotopy of equation $\mathcal{E}(q) : \mathcal{E}_0 \sim \mathcal{E}_1$ that it deforms (or varies) *continuously* from the initial equation $\mathcal{E}_0$ to the original equation $\mathcal{E}_1$ when the homotopy-parameter $q \in [0, 1]$ increases from 0 to 1. The so-called zeroth-order deformation equation defined by such kind of continuous deformation (or variation) of equations is a base of the HAM. In theory, given a nonlinear equation $\mathcal{E}_1$ having at least one solution, one has extremely large freedom to construct such a kind of zeroth-order deformation equation in many different ways. Besides, it has nothing to do with the existence of any small/large physical parameters: one can always construct such a zeroth-order deformation equation, even if the given equation $\mathcal{E}_1$ does not contain any small/large physical parameters. In essence, it is the extremely large freedom on constructing a homotopy of equations that makes the HAM rather powerful for strongly nonlinear problems and quite different from other traditional analytic techniques.

First of all, it has nothing to do with the existence of any small/large physical parameters to construct a homotopy of equations, i.e. the zeroth-order deformation equation. So, different from perturbation techniques, the HAM works even if a given nonlinear problem does not contain any small/large physical parameters, called traditionally perturbation quantities. So, in theory, the HAM can be applied much more widely than perturbation techniques, especially for those with strong nonlinearity.

Secondly, it is due to the freedom on constructing the zeroth-order deformation equation that the so-called convergence-control parameter c_0 can be introduced, which provides a convenient way to guarantee the convergence of homotopy-series. In other words, by means of properly choosing the value of such a non-zero auxiliary parameter c_0 , one can *control* the convergence of homotopy-series and give an optimal approximation with high accuracy. Thus, the convergence-control is a

fundamental concept of the HAM. In fact, it is the so-called convergence-control parameter c_0 that makes the HAM differs from all other analytic techniques.

Thirdly, it is due to the freedom on constructing the zeroth-order deformation equation that we have extremely large freedom to choose the so-called auxiliary linear operator $\mathcal{L}$. In general, using such kind of freedom, one can obtain better approximations by choosing a better auxiliary linear operator. For example, it is based on such kind of freedom that we choose the auxiliary linear operator $\mathcal{L}(x) = x'' + x$ to gain periodic solutions for all possible physical parameters of the nonlinear oscillation equation (2.32), even including $\lambda = 0$ and $\lambda < 0$. Note that, when $\lambda \leq 0$, the auxiliary linear operator $\mathcal{L}(x) = x'' + x$ is qualitatively quite different from the linear part $x'' + \lambda x$ of the original nonlinear equation (2.32). Such kind of freedom is so large that, in Sect. 2.3.8, the convergent homotopy-series solution of the 2nd-order nonlinear differential equation (2.32) can be obtained even by means of the 4th-order auxiliary linear operator

$$\check{\mathcal{L}}(x) = x'''' - x$$

and the 6th-order auxiliary linear operator

$$\check{\mathcal{L}}(x) = \frac{d^6 x}{d\tau^6} + x,$$

respectively! This illustrates that we indeed have extremely large freedom on the choice of the auxiliary linear operator $\mathcal{L}$.

Besides, it is due to the freedom on constructing the zeroth-order deformation equation that we can regard some unknown constants, such as $\gamma = \omega^2$ in (2.32), as a function of the homotopy-parameter $q \in [0, 1]$, as illustrated in Sect. 2.3.2. In this way, the so-called secular terms are easily avoided and the accurate approximations of the periodic solutions are obtained.

Finally, it is due to the freedom on constructing the zeroth-order deformation equation that we have freedom to choose different initial approximations. Based on such kind of freedom, the optimal initial approximation approach in Sect. 2.3.6 and the iteration approach in Sect. 2.3.7 are developed, which can greatly accelerate the convergence of homotopy-series. In practice, the optimal initial approximation, and especially the iteration approach, are strongly suggested for problems with strong nonlinearity.

Note that, the convergence-control is a fundamental concepts of the HAM. As revealed in Theorem 2.3, a non-zero auxiliary parameter can fantastically modify the convergence of an infinite series. In the frame of the HAM, such kind of auxiliary parameter is introduced in a natural way to approximations of nonlinear problems. Without such kind of guarantee of convergence of homotopy series, the freedom on the choice of the auxiliary linear operator and initial approximations has no meanings at all: a divergent series is useless. In essence, the extremely large freedom of constructing the zeroth-order deformation equation is based on the concept of the convergence-control by means of the convergence-control parameter c_0 . As shown later in this book, one can introduce more such kind of unknown

convergence-control parameters so as to enhance the ability of convergence-control. Since the introduction of the convergence-control parameter (it is previously called $\bar{h}$ -parameter), it has been widely applied to obtain accurate approximations of many nonlinear problems (Abbasbandy, 2006; Hayat and Sajid, 2007; Liang and Jeffrey, 2009; Niu and Wang, 2010; Van Gorder and Vajravelu, 2008; Wu and Cheung, 2009; Yabushita et al., 2007).

The solution expression is another fundamental concept of the HAM, especially for nonlinear problems with periodic solutions. For example, it is due to the periodic solution expression (2.28) that an additional algebraic equation for γ_n is given and the secular terms are avoided. In addition, it is also due to the periodic solution expression (2.28) that the additional integral coefficients of the 4th and 6th-order auxiliary linear operators are enforced to be zero so that the 2nd-order differential equation (2.32) can be solved by the rather general auxiliary linear operator (2.92). As mentioned in Sect. 2.3.7, the concept of solution expression has more general meanings than the idea of avoiding the secular terms, because it provides more restrictions. Note that the periodic solution expression (2.28) is based on the physical analysis on the nonlinear oscillation system (2.32). In other words, before solving this equation, we know from physical viewpoints that the solutions are periodic near some stable equilibrium points, but otherwise are non-periodic. So, like the idea of “avoiding secular terms”³, the solution expression (2.28) contains some empirical knowledge. This is easy for an applied mathematicians who have solid physical background, but is difficult for a pure mathematician. However, for any given equations, one can find some mathematical properties, especially some asymptotic properties, by analyzing it in details. Such kind of properties are helpful to determine a set of proper base functions for the unknown solution. So, the more such kind of properties, the better. Fortunately, a solution can be often expressed by different base functions.

In this chapter, we attempt to give the general and accurate definitions of some important concepts in the frame of the HAM. This is necessary for the development of the HAM, because these concepts are the base of it. Note that, the symbol $\bar{h}$ was used to denote the same non-zero auxiliary parameter in the so-called zeroth-order deformation equation when Liao (1997) first introduced it into the HAM. Besides, the so-called $\bar{h}$ -curves were often used to determine a interval of the auxiliary parameter $\bar{h}$ corresponding to convergent homotopy-series solutions. Since $\bar{h}$ has a very special meaning in quantum mechanics, we suggest to replace $\bar{h}$ by c_0 so as to avoid the possible confusion. Besides, since the essence of this auxiliary parameter c_0 is to control the convergence of homotopy-series, we rename it the convergence-control parameter. Due to the extremely large freedom on constructing a homotopy of equations, more convergence-control parameters can be introduced into zeroth-order deformation equations, as shown later in this book. However, c_0 is the most important, and thus can be regarded as a basic convergence-control parameter. In 2007, Yabushita et al. (2007) first used an optimal value of the auxiliary parameter in the frame of the HAM, which was determined by the minimum of

³ When one talks about avoidance of secular terms, it implies that she or he must know that the unknown solution is periodic before she or he solves it successfully.

the squared residual of two governing equations. This idea has general meanings, because we can always calculate the squared residual of governing equations no matter whether there exist closed-form analytic solutions or not. More importantly, the curves of squared residual versus the convergence-control parameter c_0 not only give the effective-region of the convergence-control parameter c_0 , but also its optimal value. Therefore, the optimal convergence-control parameter c_0^* determined by the minimum of squared residual of governing equations is strongly suggested to use in practice. For many nonlinear problems, it is enough to obtain an accurate enough homotopy-approximation by means of an optimal convergence-control parameter c_0 . However, for some problems with strong nonlinearity, one should accelerate convergence of the homotopy-series by means of the homotopy-Padé technique, or the optimal initial approximation, and especially the homotopy-iteration approach.

Note that, like perturbation techniques (Cole, 1992; Hilton, 1953; Hinch, 1991; Murdock, 1991; Nayfeh, 1973, 2000), the HAM can give accurate but very simple approximations for some nonlinear problems. For example, the 1st-order homotopy-approximation (2.73) of $x(\tau)$, the 2nd-order homotopy-approximations (2.74) and (2.75) of $x(\tau)$ are rather simple but very accurate, as shown in Fig. 2.10. Besides, the 2nd-order homotopy-approximation (2.79) of the period T is very simple, but agrees quite well with the exact period $T = 7.4163/\sqrt{\varepsilon}$ in the *whole* region $0 < \varepsilon < +\infty$, as shown in Fig. 2.12. Furthermore, the $[1,1]$ homotopy-Padé approximation (2.85) of $x(\tau)$ is very simple but accurate, as shown in Fig. 2.17. Especially, the homotopy-approximation (2.88) of $x(t)$ and the homotopy-approximation (2.89) of its period T are very simple but accurate for *all* physical parameter $0 < \varepsilon < +\infty$ and *all* initial position $-\infty < x^* < +\infty$ in case of $\lambda = 0$. In general, by means of the optimal convergence-control parameter c_0 , one often obtains accurate enough approximations in a few terms by means of the HAM. However, as the nonlinearity of equations becomes stronger and the complexity of problems increases, more and more terms are needed. Indeed, by means of the HAM together with computer algebra system, one often obtains high-order homotopy-approximations in a few seconds. However, this does not mean that the HAM always *had to* use many terms: this is a big misunderstanding to the HAM. On the other hand, for complicated problems with strong nonlinearity, it is often impossible to get an accurate enough approximation by a few terms. For these difficult problems, the HAM is more powerful, because it is essentially a method for the times of computer and internet.

In summary, the HAM is in essence based on the extremely large freedom of constructing a homotopy of equations. It is such kind of fantastic freedom, combined with more and more powerful computer algebra systems, that makes the HAM rather general and valid for problems with strong nonlinearity. In essence, it is due to such kind of fantastic freedom that the HAM greatly differs from other traditional analytic methods.

Acknowledgements In a private discussion, Dr. Pradeep Siddheshwar (Bangalore University, India) suggested me to rename c_0 the convergence-control parameter, which was called the auxiliary parameter in my previous articles (Liao, 1997, 1999a,b; Liao and Campo, 2002; Liao, 2003a, 2004,

2005, 2006, 2009) and in my book “Beyond Perturbation” (Liao, 2003b). Thanks to Dr. Zhiliang Lin (Shanghai Jiao Tong University, China) for his help on plotting Fig. 2.3.

Appendix 2.1 Derivation of δ_n in (2.57)

The definition of δ_n in (2.57) reads

$$\delta_n = \mathcal{D}_n [\tilde{\gamma}(q) \tilde{x}''(\tau; q) + \lambda \tilde{x}(\tau; q) + \varepsilon \tilde{x}^3(\tau; q)].$$

It holds according to the definition (2.50) that

$$\begin{aligned} & \mathcal{D}_n [\tilde{\gamma}(q) \tilde{x}''(\tau; q)] \\ &= \frac{1}{n!} \left\{ \frac{\partial^n}{\partial q^n} [\tilde{\gamma}(q) \tilde{x}''(\tau; q)] \right\} \Big|_{q=0} \\ &= \frac{1}{n!} \left[\sum_{k=0}^n \binom{n}{k} \frac{d^k \tilde{\gamma}(q)}{dq^k} \frac{\partial^{n-k} \tilde{x}''(\tau; q)}{\partial q^{n-k}} \right] \Big|_{q=0} \\ &= \left\{ \sum_{k=0}^n \left[\frac{1}{k!} \frac{d^k \tilde{\gamma}(q)}{dq^k} \right] \left[\frac{1}{(n-k)!} \frac{\partial^{n-k} \tilde{x}''(\tau; q)}{\partial q^{n-k}} \right] \right\} \Big|_{q=0} \\ &= \sum_{k=0}^n \left[\frac{1}{k!} \frac{d^k \tilde{\gamma}(q)}{dq^k} \right] \Big|_{q=0} \frac{\partial^2}{\partial \tau^2} \left\{ \left[\frac{1}{(n-k)!} \frac{\partial^{n-k} \tilde{x}(\tau; q)}{\partial q^{n-k}} \right] \Big|_{q=0} \right\} \\ &= \sum_{k=0}^n \gamma_k x''_{n-k}(\tau). \end{aligned}$$

Similarly, we have according to the definition (2.50) that

$$\mathcal{D}_n [\lambda \tilde{x}(\tau; q)] = \frac{1}{n!} \left\{ \frac{\partial^n}{\partial q^n} [\lambda \tilde{x}(\tau; q)] \right\} \Big|_{q=0} = \lambda x_n(\tau)$$

and

$$\begin{aligned} & \mathcal{D}_n [\varepsilon \tilde{x}^3(\tau; q)] \\ &= \frac{1}{n!} \left\{ \frac{\partial^n}{\partial q^n} [\varepsilon \tilde{x}^3(\tau; q)] \right\} \Big|_{q=0} \\ &= \frac{\varepsilon}{n!} \left[\sum_{k=0}^n \binom{n}{k} \frac{\partial^{n-k} \tilde{x}(\tau; q)}{\partial q^{n-k}} \frac{\partial^k \tilde{x}^2(\tau; q)}{\partial q^k} \right] \Big|_{q=0} \\ &= \varepsilon \left[\sum_{k=0}^n \frac{1}{k!(n-k)!} \frac{\partial^{n-k} \tilde{x}(\tau; q)}{\partial q^{n-k}} \sum_{j=0}^k \binom{k}{j} \frac{\partial^{k-j} \tilde{x}(\tau; q)}{\partial q^{k-j}} \frac{\partial^j \tilde{x}(\tau; q)}{\partial q^j} \right] \Big|_{q=0} \end{aligned}$$

$$\begin{aligned}
&= \varepsilon \sum_{k=0}^n \left[\frac{1}{(n-k)!} \frac{\partial^{n-k} \tilde{x}(\tau; q)}{\partial q^{n-k}} \Big|_{q=0} \right] \\
&\quad \cdot \sum_{j=0}^k \left[\frac{1}{(k-j)!} \frac{\partial^{k-j} \tilde{x}(\tau; q)}{\partial q^{k-j}} \Big|_{q=0} \right] \left[\frac{1}{j!} \frac{\partial^j \tilde{x}(\tau; q)}{\partial q^j} \Big|_{q=0} \right] \\
&= \varepsilon \sum_{k=0}^n x_{n-k}(\tau) \sum_{j=0}^k x_{k-j}(\tau) x_j(\tau).
\end{aligned}$$

Substituting the above expressions into the definition of δ_n , we have

$$\delta_n = \sum_{k=0}^n \gamma_k x''_{n-k}(\tau) + \lambda x_n(\tau) + \varepsilon \sum_{k=0}^n x_{n-k}(\tau) \sum_{j=0}^k x_{k-j}(\tau) x_j(\tau). \quad (2.99)$$

Note that, directly using the related Theorems proved in Chapter 4, one can obtain the same result.

Appendix 2.2 Derivation of (2.55) by the 2nd approach

As pointed out by Hayat and Sajid (2007), directly substituting the homotopy-Maclaurin series (2.48) and (2.49) into the zeroth-order deformation equations (2.42) and (2.43), and equating the coefficients of the like power of q , one can gain exactly the *same* equations as (2.55) and (2.56).

Using the homotopy-Maclaurin series (2.48), we have

$$\begin{aligned}
&(1-q)\mathcal{L}[\tilde{x}(\tau; q) - x_0(\tau)] \\
&= (1-q)\mathcal{L}\left[\sum_{k=0}^{+\infty} x_k(\tau) q^k - x_0(\tau)\right] \\
&= (1-q)\mathcal{L}\left[\sum_{k=1}^{+\infty} x_k(\tau) q^k\right] \\
&= \sum_{k=1}^{+\infty} \mathcal{L}[x_k(\tau)] q^k - \sum_{k=1}^{+\infty} \mathcal{L}[x_k(\tau)] q^{k+1} \\
&= \sum_{k=1}^{+\infty} \mathcal{L}[x_k(\tau)] q^k - \sum_{k=2}^{+\infty} \mathcal{L}[x_{k-1}(\tau)] q^k \\
&= \mathcal{L}[x_1(\tau)] q + \sum_{k=2}^{+\infty} \mathcal{L}[x_k(\tau) - x_{k-1}(\tau)] q^k \\
&= \sum_{n=1}^{+\infty} \mathcal{L}[x_n(\tau) - \chi_n x_{n-1}(\tau)] q^n, \tag{2.100}
\end{aligned}$$

where χ_n is defined by (2.58).

Substituting the homotopy-Maclaurin series (2.48) and (2.49) into the definition (2.35), we have

$$\begin{aligned}
 & \tilde{\gamma}(q) \tilde{x}''(\tau; q) + \lambda \tilde{x}(\tau; q) + \varepsilon \tilde{x}^3(\tau; q) \\
 &= \left(\sum_{i=0}^{+\infty} \gamma_i q^i \right) \left[\sum_{j=0}^{+\infty} x_j''(\tau) q^j \right] + \lambda \sum_{n=0}^{+\infty} x_n(\tau) q^n \\
 &+ \varepsilon \left[\sum_{i=0}^{+\infty} x_i(\tau) q^i \right] \left[\sum_{j=0}^{+\infty} x_j(\tau) q^j \right] \left[\sum_{l=0}^{+\infty} x_l(\tau) q^l \right] \\
 &= \sum_{n=0}^{+\infty} \left[\sum_{k=0}^n \gamma_k x_{n-k}''(\tau) \right] q^n + \lambda \sum_{n=0}^{+\infty} x_n(\tau) q^n \\
 &+ \varepsilon \sum_{n=0}^{+\infty} \left[\sum_{k=0}^n x_{n-k}(\tau) \sum_{j=0}^k x_j(\tau) x_{k-j}(\tau) \right] q^n \\
 &= \sum_{n=0}^{+\infty} \left[\sum_{k=0}^n \gamma_k x_{n-k}''(\tau) + \lambda x_n(\tau) + \varepsilon \sum_{k=0}^n x_{n-k}(\tau) \sum_{j=0}^k x_{k-j}(\tau) x_j(\tau) \right] q^n.
 \end{aligned}$$

According to (2.57), the above expression reads

$$\tilde{\gamma}(q) \tilde{x}''(\tau; q) + \lambda \tilde{x}(\tau; q) + \varepsilon \tilde{x}^3(\tau; q) = \sum_{n=0}^{+\infty} \delta_n(X_n, \Gamma_n) q^n. \quad (2.101)$$

Substituting (2.100) and (2.101) into the zeroth-order deformation equations (2.42), we have

$$\sum_{n=1}^{+\infty} \mathcal{L}[x_n(\tau) - \chi_n x_{n-1}(\tau)] q^n = c_0 \sum_{n=0}^{+\infty} \delta_n(X_n, \Gamma_n) q^{n+1}.$$

Equating the coefficient of like-power of q in above equation, we have

$$x_n(\tau) - \chi_n x_{n-1}(\tau) = c_0 \delta_{n-1}(X_{n-1}, \Gamma_{n-1}),$$

which is exactly the *same* as the high-order deformation equation (2.55).

Besides, substituting the homotopy-Maclaurin series (2.48) into the initial conditions $\tilde{x}(0; q) = 1$ and $\tilde{x}'(0; q) = 0$, equating the coefficient of the like power of q , we have

$$x_k(0) = 0, \quad x_k'(0) = 0, \quad k = 1, 2, 3, \dots,$$

which is exactly the *same* as the initial conditions (2.56).

Therefore, no matter whether one regards the homotopy-parameter q as a small parameter or not, one should always obtain the *same* high-order deformation equa-

tions from the *same* zeroth-order deformation equations, as proved in general by Hayat and Sajid (2007). This is easy to understand, because, according to the fundamental theorem in calculus (Fitzpatrick, 1996), the Taylor series of a real function is unique.

Appendix 2.3 Proof of Theorem 2.3

Proof. Define

$$\xi = 1 + c_0 + c_0 z,$$

which gives since $c_0 \neq 0$ that

$$\frac{1}{1+z} = -\frac{c_0}{(1-\xi)}.$$

Enforcing $|\xi| = |1 + c_0 + c_0 z| < 1$, we have by Newtonian binomial theorem that

$$\frac{1}{1+z} = -\frac{c_0}{1-\xi} = -c_0 (1 + \xi + \xi^2 + \xi^3 + \cdots) = -c_0 \sum_{n=0}^{+\infty} (1 + c_0 + c_0 z)^n.$$

In other words, it holds

$$\frac{1}{1+z} = \lim_{m \rightarrow +\infty} \left[-c_0 \sum_{n=0}^m (1 + c_0 + c_0 z)^n \right], \quad \text{when } |1 + c_0 + c_0 z| < 1.$$

For real numbers $z \in (-\infty, +\infty)$ and $c_0 \neq 0$, we have

$$\begin{aligned} & -c_0 \sum_{n=0}^m (1 + c_0 + c_0 z)^n \\ &= -c_0 \sum_{n=0}^m \sum_{k=0}^n \binom{n}{k} (1 + c_0)^{n-k} (c_0 z)^k \\ &= -c_0 \sum_{k=0}^m \sum_{n=k}^m \binom{n}{k} (1 + c_0)^{n-k} c_0^k z^k \\ &= \sum_{k=0}^m (-1)^k z^k (-c_0)^{k+1} \sum_{i=0}^{m-k} \binom{k+i}{k} (1 + c_0)^i \\ &= \sum_{k=0}^m (-z)^k \left[(-c_0)^{k+1} \sum_{i=0}^{m-k} \binom{k+i}{i} (1 + c_0)^i \right] \\ &= \sum_{n=0}^m (-z)^n \left[(-c_0)^{n+1} \sum_{i=0}^{m-n} \binom{n+i}{i} (1 + c_0)^i \right] \end{aligned}$$

$$= \sum_{n=0}^m \left[\mu_0^{m+1,n+1}(c_0) \right] (-z)^n,$$

where

$$\mu_0^{m,n}(c_0) = (-c_0)^n \sum_{i=0}^{m-n} \binom{n-1+i}{i} (1+c_0)^i.$$

Besides, $|1 + c_0 + c_0 z| < 1$ gives

$$-1 < 1 + c_0 + c_0 z < 1,$$

i.e.

$$-2 - c_0 < c_0 z < -c_0,$$

which leads to either

$$-1 < z < \frac{2}{|c_0|} - 1, \quad \text{when } c_0 < 0,$$

or

$$-\frac{2}{c_0} - 1 < z < -1, \quad \text{when } c_0 > 0.$$

Thus, it holds

$$\frac{1}{1+z} = \lim_{m \rightarrow +\infty} \sum_{n=0}^m \mu_0^{m+1,n+1}(c_0) (-z)^n,$$

in the region

$$-1 < z < \frac{2}{|c_0|} - 1, \quad \text{when } c_0 < 0,$$

or

$$-\frac{2}{c_0} - 1 < z < -1, \quad \text{when } c_0 > 0.$$

Appendix 2.4 Mathematica code (without iteration) for Example 2.2

Periodic solutions of nonlinear oscillation equation

$$x'' + \lambda x + \varepsilon x^3 = 0, \quad x(0) = x^*, x'(0) = 0$$

is solved by means of the HAM without iteration approach, where λ and ε are physical parameters, x^* is the starting position of oscillation, respectively. This Mathematica code is free available at

<http://numericaltank.sjtu.edu.cn/HAM.htm>

A Simple Users Guide

Input data:

lambda : physical parameter λ ;
 epsilon : physical parameter ε ;
 xstart : start-position $x^* = x(0)$;
 c0 : convergence-control parameter c_0 ;
 KAPPA : value of κ of the auxiliary linear operator (2.92).

Control parameter:

xOptimal : Optimal initial approximation is used when xOptimal = 1.

Calculated results:

X[k] : k th-order homotopy-approximation of $x(\tau)$;
 u[k] : k th-order homotopy-approximation of $x(t)$;
 GAMMA[k] : k th-order homotopy-approximation of γ ;
 Err[k] : squared residual E_k defined by (2.72).

Main codes:

ham[1,11] : First, gain 1st to 11th-order homotopy-approximation;
 ham[12,35] : Then, gain 12th to 35th-order homotopy-approximation;
 GetErr[k] : Gain squared residual E_k defined by (2.72).

Mathematica code (without iteration) for Example 2.2

by Shijun LIAO
 Shanghai Jiao Tong University
 June 2010

```

<<Calculus`Pade`;
<<Graphics`Graphics`;

(*****
(* Define physical and control parameters                               *)
(*****
KAPPA    =    1;
lambda   =   -9/4;
epsilon  =    1;
c0       =   -17/100;
xstart   =    1;

(*****
(* Using optimal initial approximation when xOptimal = 1                *)
(*****
xOptimal =    0;
If[epsilon > 0 && lambda >= 0, beta = 0];
  
```

```

If[epsilon > 0 && lambda < 0 && xstart > 0,
    beta = N[Sqrt[Abs[lambda/epsilon]],100]];
If[epsilon > 0 && lambda < 0 && xstart < 0,
    beta = -N[Sqrt[Abs[lambda/epsilon]],100]];
If[epsilon < 0 && lambda >= 0 &&
    Abs[xstart] < Sqrt[Abs[lambda/epsilon]], beta = 0];
If[!NumberQ[beta//N],Print[" There is no periodic solution
    for these input data !"]];
If[xOptimal == 1, beta = . ];

(*****
(* Define initial guess *)
(*****
x[0] = beta + (xstart - beta)*Cos[t];
X[0] = x[0];
u[0] = X[0] /. t->Sqrt[GAMMA[0]]*t ;

(*****
(* Define the function chi[k] *)
(*****
chi[k_] := If[ k <= 1, 0, 1 ];

(*****
(* Define inverse operator of auxiliary linear operator L *)
(*****
Linu[Cos[m_*t]] := (-1)^(KAPPA+1)*Cos[m*t]/(1-m^(2*KAPPA));

(*****
(* The linear property of the inverse operator Linv *)
(* Linv[f_+g_] := Linv[f]+Linv[g]; *)
(*****
Linv[p_Plus] := Map[Linv,p];
Linv[c_*f_] := c*Linv[f] /. FreeQ[c,t];
Linv[c_] := (-1)^(KAPPA+1)*c /. FreeQ[c,t];

(*****
(* Define GetDelta[k] *)
(* Calculate the right-hand side term delta[k] *)
(*****
GetDelta[k_] := Module[{temp},
temp[1] = xttgamma[k]//TrigReduce;
temp[2] = xxx[k]//TrigReduce;
temp[3] = temp[1] + lambda*x[k] + epsilon * temp[2];
Delta[k] = Expand[temp[3]];
];

(*****
(* Define GetxAll *)
(*****
GetxAll[k_] := Module[{temp},
xtt[k] = Expand[D[x[k],{t,2}]];
temp = Sum[x[j]*x[k-j],{j,0,k}]/Expand;
xx[k] = TrigReduce[temp];

```

```

temp    = Sum[x[j]*xx[k-j],{j,0,k}]/Expand;
xxx[k]  = TrigReduce[temp];
xttgamma[k] = Sum[gamma[j]*xtt[k-j],{j,0,k}]/Expand;
];

(*****
(* Define Getgamma                                     *)
(*****
Getgamma[k_] := Module[{temp, eq},
temp[1] = Delta[k]/TrigReduce;
temp[2] = Expand[temp[1]];
eq      = Coefficient[temp[2], Cos[t]];
temp[3] = Solve[eq == 0, gamma[k]];
gamma[k] = temp[3][[1, 1, 2]];
];

(*****
(* Define Getx                                         *)
(* Get solution of high-order deformation equation    *)
(*****
Getx[k_] := Module[{temp},
temp[1] = xSpecial + chi[k]*x[k-1];
temp[2] = temp[1] /. t -> 0;
x[k]    = temp[1] - temp[2]*Cos[t];
];

(*****
(* Define GetErr                                       *)
(* Gain squared residual of governing equation        *)
(*****
GetErr[k_] := Module[{temp, Xtt, error, delt, tt, Npoint, sum, i},
Xtt = D[X[k], {t, 2}];
error = GAMMA[k]*Xtt + lambda * X[k] + epsilon * X[k]^3;
Npoint = 50;
sum     = 0;
delt    = N[2*Pi/Npoint, 24];
For[i = 0, i <= Npoint, i=i+1,
  tt = i*delt;
  temp = error/.t -> tt //Expand;
  sum = sum + temp^2 //Expand;
];
Err[k] = sum/(Npoint+1)
];

(*****
(* Main Code                                           *)
(*****
ham[m0_, m1_] := Module[{temp, k, j, zz},
For[k=Max[1, m0], k <= m1, k=k+1,
  Print[" k = ", k];
  GetxAll[k-1];
  GetDelta[k-1];

```

```

RHS[k] = c0*Delta[k-1]//Expand;
Getgamma[k-1];
GAMMA[k-1] = Expand[Sum[gamma[j],{j,0,k-1}]];
T[k-1] = 2*Pi/Sqrt[GAMMA[k-1]];
If[NumberQ[GAMMA[k-1]],
Print[k-1,"th approx. of gamma = ",N[GAMMA[k-1],10],
" variation = ",gamma[k-1]//N ];
Print[k-1,"th approx. of T = ",N[T[k-1],10] ];
];
If[k == 1 && xOptimal == 1,
GetErr[0];
If[ xstart > 0,
temp = Minimize[{Err[0], beta > xstart},beta] ];
If[ xstart < 0,
temp = Minimize[{Err[0], beta < xstart},beta] ];
zz = beta/.temp[[2]];
beta = N[IntegerPart[zz*10000]/10000,100];
Print[" Optimal initial approx. is used with beta = ",
beta//N];
];
temp = RHS[k]/.Cos[t]->0//Expand;
xSpecial = Linv[temp];
Getx[k];
X[k] = X[k-1] + x[k]//Expand;
u[k] = X[k] /. t-> Sqrt[GAMMA[k-1]]*t;
];
Print["Successful !"];
];

(*****
(* Print physical and contyrol parameters *)
(*****
Print[" lambda = ",lambda];
Print[" epsilon = ",epsilon];
Print[" x* = ",xstart];
Print[" beta = ",beta];
Print[" c0 = ",c0];
Print[" KAPPA = ",KAPPA];
Print[" xOptimal = ",xOptimal];

(* Gain 10th-order homotopy-approximation *)
ham[1,10];

```

Appendix 2.5 Mathematica code (with iteration) for Example 2.2

Periodic solutions of nonlinear oscillation equation

$$x'' + \lambda x + \varepsilon x^3 = 0, \quad x(0) = x^*, \quad x'(0) = 0$$

is solved by means of the HAM with iteration approach, where λ and ε are physical parameters, x^* is the starting position of oscillation, respectively. This Mathematica code is free available at

<http://numericaltank.sjtu.edu.cn/HAM.htm>

A Simple Users Guide

Input data:

lambda : physical parameter λ ;
 epsilon : physical parameter ε ;
 xstart : start-position $x^* = x(0)$.

Control parameter:

IterOrder : value of M , the order of iteration formula (2.90);
 Nterms : value of N in the truncated expression (2.91).

Calculated results:

X[k] : $x(\tau)$ given by the k th-iteration ;
 u[k] : $x(t)$ given by the k th-iteration;
 GAMMA[k] : γ given by the k th-iteration;
 Err[k] : squared residual at the k th-iteration.

Main codes:

ham[1,11] : First, gain 1st to 11th-iteration approximations;
 ham[12,21] : Then, gain 12nd to 21st-iteration approximations;
 GetErr[k] : Gain squared residual of the k th-iteration approximation.

Mathematica code (with iteration) for Example 2.2

by Shijun LIAO
 Shanghai Jiao Tong University
 June 2010

```
<<Calculus`Pade`;
<<Graphics`Graphics`;

(*****
(* Physical and control parameters *)
(*****
lambda   =      0;
epsilon  =      1;
xstart   =      1;
If[NumberQ[c0], c0 = N[c0, 100]];

(*****
```

```

(* Control parameters for iteration approach *)
(*****)
IterOrder = 3;
Nterms = 21;

(*****)
(* Define initial approximation *)
(*****)
If[epsilon > 0 && lambda >= 0, beta = 0];
If[epsilon > 0 && lambda < 0 && xstart > 0,
  beta = N[Sqrt[Abs[lambda/epsilon]],100]];
If[epsilon > 0 && lambda < 0 && xstart < 0,
  beta = -N[Sqrt[Abs[lambda/epsilon]],100]];
If[epsilon < 0 && lambda >= 0 &&
  Abs[xstart] < Sqrt[Abs[lambda/epsilon]], beta = 0];
If[!NumberQ[beta/N],
  Print[" No periodic solution for the input data !"]];
x[0] = beta + (xstart - beta)*Cos[t];
X[0] = x[0];
u[0] = X[0] /. t->Sqrt[GAMMA[0]]*t ;

(*****)
(* Define the function chi[k] *)
(*****)
chi[k_] := If[ k <= 1, 0, 1 ];

(*****)
(* Define inverse operator of auxiliary linear operator *)
(*****)
Linv[Cos[m*t]] := Cos[m*t]/(1-m^2);

(*****)
(* The linear property of the inverse operator Linv *)
(* Linv[f_+g_] := Linv[f]+Linv[g]; *)
(*****)
Linv[p_Plus] := Map[Linv,p];
Linv[c_*f_] := c*Linv[f] /. FreeQ[c,t];
Linv[c_] := c /. FreeQ[c,t];

(*****)
(* Define GetDelta[k] *)
(*****)
GetDelta[k_] := Module[{temp,n},
temp[1] = xttgamma[k]//TrigReduce;
temp[2] = xxx[k]//TrigReduce;
temp[3] = temp[1] + lambda*x[k] + epsilon * temp[2];
Delta[k] = temp[3]//Expand;
];

(*****)
(* Define GetxAll *)
(*****)
GetxAll[k_] := Module[{temp},
xtt[k] = Expand[D[x[k],{t,2}]]];

```

```

temp    = Sum[x[j]*x[k-j],{j,0,k}]/Expand;
xx[k]   = TrigReduce[temp];
temp    = Sum[x[j]*xx[k-j],{j,0,k}]/Expand;
xxx[k]  = TrigReduce[temp];
xttgamma[k] = Sum[gamma[j]*xtt[k-j],{j,0,k}]/Expand;
];

(*****
(* Define Getgamma                                     *)
(*****
Getgamma[k_] := Module[{temp, eq},
temp[1] = Delta[k]/TrigReduce;
temp[2] = Expand[temp[1]];
eq      = Coefficient[temp[2], Cos[t]];
temp[3] = Solve[eq == 0, gamma[k]];
gamma[k] = temp[3][[1,1,2]];
];

(*****
(* Define Getx                                         *)
(*   Get solution of high-order deformation equation *)
(*****
Getx[k_] := Module[{temp},
temp[1] = xSpecial + chi[k]*x[k-1];
temp[2] = temp[1] /. t -> 0;
x[k]    = temp[1] - temp[2]*Cos[t];
];

(*****
(* Define GetErr                                       *)
(*   Get squared residual of governing equation      *)
(*****
GetErr[iter_] := Module[{temp, Xtt, error, delt, t, Npoint, sum, i},
Xtt = D[X[iter], {t, 2}];
error = GAMMA[iter]*Xtt + lambda*X[iter] + epsilon*X[iter]^3;
Npoint = 50;
sum     = 0;
delt    = N[2*Pi/Npoint, 100];
For[ i = 0, i <= Npoint, i=i+1,
tt = i*delt;
temp = error/.t -> tt //Expand;
sum = sum + temp^2 //Expand;
];
Err[iter] = sum/(Npoint+1);
];

(*****
(* Define Truncation                                   *)
(*   This module neglects high-order terms           *)
(*****
Truncation[k_, Nterms_] := Module[{temp, coef, i, j},
temp[1] = RHS[k] //TrigReduce;
temp[2] = temp[1] //Expand;
coef[0] = temp[2] /. Cos[_] -> 0;

```

```

For[i = 2, i <= Nterms, i = i + 1,
  coef[i] = Coefficient[temp[2], Cos[i*t]];
];
temp[3] = coef[0] + Sum[coef[j]*Cos[j*t], {j, 2, Nterms}];
RHS[k] = temp[3]//Expand;
];

(*****
(*                               Main Code                               *)
(*****
ham[m0_, m1_] := Module[{temp, k, j, zz},
For[iter = Max[1, m0], iter <= m1, iter = iter + 1,
  For[k = 1, k <= IterOrder, k = k+1,
    If[k == 1, Clear[gamma]];
    GetxAll[k-1];
    GetDelta[k-1];
    RHS[k] = c0*Delta[k-1]//Expand;
    Getgamma[k-1];
    Truncation[k, Nterms];
    xSpecial = Linv[RHS[k]];
    Getx[k];
  ];
  X[iter] = Sum[x[i], {i, 0, IterOrder}];
  GAMMA[iter] = Expand[Sum[gamma[j], {j, 0, IterOrder-1}]];
  variation = GAMMA[iter] - chi[iter]*GAMMA[iter-1]//N;
  T[iter] = 2*Pi/Sqrt[GAMMA[iter]];
  u[iter] = X[IterOrder] /. t -> Sqrt[GAMMA[iter]]*t;
  GetErr[iter];
  If[ iter == 1 ,
    c0 = .;
    temp = Minimize[{Err[1], c0 < 0}, c0];
    zz = c0/.temp[[2]];
    c0 = N[IntegerPart[zz*10000]/10000, 100];
    Print["The optimal value of c0 = ", c0//N];
  ];
  Print[ iter, "th iteration: error = ", Err[iter]//N];
  Print[ "      gamma = ", N[GAMMA[iter], 24],
    "      variation = ", variation];
  If[ Err[iter] < 10^(-30) || Abs[variaion] < 10^(-30),
    Print[" Desired accuracy is arrived: iteration stops!"];
    Goto[end];
  ];
  x[0] = X[iter];
  Clear[gamma];
];
Label[end];
Print[" Successful ! "];
];

(*****
(* Print physical and control parameters                               *)
(*****
Print["      lambda      = ", lambda];
Print["      epsilon     = ", epsilon];

```

```

Print["      x*      = ",xstart];
Print["-----"];
Print[" Oder of iteration formula  = ",IterOrder];
Print[" N, the number of terms in delta = ",Nterms];

(* Gain 10th homotopy-iteration approximation *)
ham[1,10];

```

Problems

2.1. Non-periodic solution of a nonlinear dynamic system

Use the HAM to obtain the non-periodic solutions of a nonlinear differential equation

$$\ddot{x}(t) + \lambda x(t) + \varepsilon x^3(t) = 0, \quad x(0) = x^*, \quad \dot{x}(0) = 0,$$

where λ and ε are physical constants, x^* denotes the starting position.

2.2. Periodic solution of a nonlinear dynamic system

Use the HAM to obtain the periodic solution of a nonlinear differential equation

$$\ddot{x}(t) + \lambda x(t) + \varepsilon x^3(t) = 0, \quad x(0) = 0, \quad \dot{x}(0) = v_0,$$

where $\lambda \in (-\infty, +\infty)$ and $\varepsilon \in (-\infty, +\infty)$ are physical constants, v_0 denotes the starting velocity.

2.3. Periodic and non-periodic solution of a nonlinear dynamic system

Use the HAM to obtain the periodic and non-periodic solutions of a nonlinear differential equation

$$\ddot{x}(t) + \lambda x(t) + \varepsilon x^2(t) = 0, \quad x(0) = x^*, \quad \dot{x}(0) = 0,$$

where $\lambda \in (-\infty, +\infty)$ and $\varepsilon \in (-\infty, +\infty)$ are physical constants, x^* denotes the starting position.

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Chapter 3

Optimal Homotopy Analysis Method

Abstract In this chapter, we describe and compare the different optimal approaches of the homotopy analysis method (HAM). A generalized optimal HAM is proposed, which logically contains the basic optimal HAM with only one convergence-control parameter and also the optimal HAM with an infinite number of parameters. It is found that approximations given by the optimal HAMs converge fast in general. Especially, the basic optimal HAM mostly gives good enough approximations. Thus, the optimal HAMs with a couple of convergence-control parameters are strongly suggested in practice.

3.1 Introduction

Nonlinear equations are much more difficult to solve than linear ones, especially by means of analytic methods. In general, there are two standards for a satisfactory analytic method of nonlinear equations:

- (a) It can *always* provide analytic approximations *efficiently*.
- (b) It can ensure that analytic approximations are *accurate* enough for *all* physical parameters.

Using above two standards as criterion, let us compare different analytic techniques for nonlinear problems.

Perturbation techniques (Lindstedt, 1882; Cole, 1992; Von Dyke, 1975; Murdock, 1991; Kevorkian and Cole, 1995; Nayfeh, 2000) are widely applied in science and engineering. Perturbation techniques are based on small/large physical parameters (called perturbation quantities) in governing equations or initial/boundary conditions. In general, perturbation approximations are expressed in series of perturbation quantities, and a nonlinear equation is replaced by an infinite number of linear (sometimes even nonlinear) sub-problems, which are completely determined by the type of the original governing equation and especially by the place where perturbation quantities appear. Perturbation methods are simple, and easy to understand.

Especially, based on small physical parameters, perturbation approximations often have clear physical meanings. Unfortunately, *not* every nonlinear problem has such kind of perturbation quantity. Besides, even if there exists such a small physical parameter, the related sub-problem might have no solutions, or might be so complicated that only a few sub-problems can be solved. Thus, it is *not* guaranteed that one can *always* gain perturbation approximations for a given nonlinear problem. More importantly, it is well-known that most perturbation approximations are valid only for small physical parameters. So, it is *not* guaranteed that a perturbation result is valid in the whole region of *all* physical parameters. Thus, perturbation techniques do not satisfy not only the standard (a) but also the standard (b) mentioned above.

To overcome the restrictions of perturbation techniques, some traditional non-perturbation methods are developed, such as Lyapunov's artificial small parameter method (Lyapunov, 1992), the δ -expansion method (Karmishin et al., 1990; Awrejcewicz et al., 1998), Adomian decomposition method (Adomian, 1976; Adomian and Adomian, 1984; Cherruault, 1988; Adomian, 1991, 1994; Rach, 2008), and so on. In principle, all of these methods are based on a so-called artificial parameter, and approximation solutions are expanded into series of such kind of artificial parameter. The artificial small parameter is often used in such a way that one can always gain approximation solutions for a given nonlinear equation. Compared with perturbation techniques, this is indeed a great progress. However, in theory, one can put the artificial small parameter in many different ways, but unfortunately there are no theories to guide us how to put it in a better place so as to gain a better approximation. For example, Adomian decomposition method simply uses the linear operator d^k/dx^k in most cases, where k is the highest order of derivative of governing equations, and therefore it is rather easy to gain solutions of the corresponding sub-problems by means of integrating k times with respect to x . However, such simple linear operator gives approximations in power-series, which unfortunately has often a finite radius of convergence. Thus, Adomian decomposition method can not ensure the convergence of its approximation series. Generally speaking, all traditional non-perturbation methods, such as Lyapunov's artificial small parameter method (Lyapunov, 1992), the δ -expansion method (Karmishin et al., 1990; Awrejcewicz et al., 1998) and Adomian decomposition method (Adomian, 1976; Adomian and Adomian, 1984; Cherruault, 1988; Adomian, 1991, 1994; Rach, 2008), can *not* guarantee the convergence of approximation series. So, these traditional non-perturbation methods satisfy only the standard (a) but *not* the standard (b) mentioned above.

In 1992 Liao (1992) took the lead to apply the homotopy (Hilton, 1953), a basic concept in topology (Sen, 1983), to gain analytic approximations of nonlinear differential equations. The early homotopy analysis method (HAM) was first described by Liao (1992) in his PhD dissertation. For a given nonlinear differential equation

$$\mathcal{N}[u(x)] = 0, \quad x \in \Omega,$$

where $\mathcal{N}$ is a nonlinear operator and $u(x)$ is an unknown function, Liao (1992) constructed a *one-parameter* family of equations in the embedding parameter $q \in [0, 1]$, called the zeroth-order deformation equation

$$(1 - q)\mathcal{L}[\phi(x; q) - u_0(x)] + q\mathcal{N}[\phi(x; q)] = 0, \quad x \in \Omega, q \in [0, 1], \quad (3.1)$$

where $\mathcal{L}$ is an auxiliary linear operator and $u_0(x)$ is an initial guess. In theory, the concept of homotopy in topology provides us much larger freedom to choose both of the auxiliary linear operator $\mathcal{L}$ and the initial guess than the traditional non-perturbation methods mentioned above, as pointed out by Liao (1999, 2003, 2007) and shown later in this book. At $q = 0$ and $q = 1$, we have $\phi(x; 0) = u_0(x)$ and $\phi(x; 1) = u(x)$, respectively. So, as the embedding parameter $q \in [0, 1]$ increases from 0 to 1, the solution $\phi(x; q)$ of the zeroth-order deformation equations varies (or deforms) from the initial guess $u_0(x)$ to the exact solution $u(x)$ of the original nonlinear differential equation $\mathcal{N}[u(x)] = 0$. Such kind of continuous variation is called deformation in topology, and this is the reason why we call (3.1) the zeroth-order deformation equation. Since $\phi(x; q)$ is also dependent upon the embedding parameter $q \in [0, 1]$, we can expand it into the Maclaurin series with respect to q :

$$\phi(x; q) = u_0(x) + \sum_{n=1}^{+\infty} u_n(x) q^n, \quad (3.2)$$

called the homotopy-Maclaurin series. Note that we have extremely large freedom to choose the auxiliary linear operator $\mathcal{L}$ and the initial guess $u_0(x)$. Assuming that, the auxiliary linear operator $\mathcal{L}$ and the initial guess $u_0(x)$ are so properly chosen that the above homotopy-Maclaurin series converges at $q = 1$, we have the so-called homotopy-series solution

$$u(x) = u_0(x) + \sum_{n=1}^{+\infty} u_n(x), \quad (3.3)$$

which satisfies the original equation $\mathcal{N}[u(x)] = 0$, as proved by Liao (1999, 2003) in general. Here, $u_n(x)$ is governed by the so-called high-order deformation equation

$$\mathcal{L}[u_n(x) - \chi_n u_{n-1}(x)] = -\mathcal{D}_{n-1} \{ \mathcal{N}[\phi(x; q)] \}, \quad (3.4)$$

where χ_k equals to 1 when $k \geq 2$ but zero otherwise, $\mathcal{D}_k$ is the k th-order homotopy-derivative operator defined by

$$\mathcal{D}_k = \left. \frac{1}{k!} \frac{\partial^k}{\partial q^k} \right|_{q=0}.$$

The high-order deformation equation (3.4) is always linear with the known term on the right-hand side, therefore is easy to solve, as long as we choose the auxiliary linear operator $\mathcal{L}$ properly.

Unfortunately, Liao (1997, 2003) found that the early HAM mentioned above can not always guarantee the convergence of approximation series of nonlinear equations in general. To overcome this restriction, Liao (1997) introduced such a

non-zero auxiliary parameter c_0 to construct a *two-parameter* family of equations¹, i.e. the zeroth-order deformation equation

$$(1 - q)\mathcal{L}[\phi(x; q) - u_0(x)] = c_0 q \mathcal{N}[\phi(x; q)], \quad x \in \Omega, \quad q \in [0, 1]. \quad (3.5)$$

In this way, the homotopy-series solution (3.3) is not only dependent upon the physical variable x but also the auxiliary parameter c_0 . Mathematically, it was found (Liao, 1997, 1999, 2003) that the auxiliary parameter c_0 can adjust and control the convergence region and rate of homotopy-series solutions, although c_0 has no physical meanings at all. For detailed mathematical proofs, please refer to Chapter 5. In essence, the use of the auxiliary parameter c_0 introduces us one more “artificial” degree of freedom, which greatly improves the early HAM: it is the auxiliary parameter c_0 which provides us a convenient way to guarantee the convergence of homotopy-series solution. For example, Liang and Jeffrey (2009) illustrated that, when analytic approximations given by the other analytic method is divergent in the whole domain, one can gain convergent series solution simply by choosing a proper auxiliary parameter c_0 . This is the reason why we call c_0 the *convergence-control parameter*.

The use of the convergence-control parameter c_0 is indeed a great progress in the frame of the HAM. It seems that more “artificial” degrees of freedom imply larger possibility to gain better approximations by means of the homotopy analysis method. Thus, Liao (1999) further introduced more “artificial” degrees of freedom by using the zeroth-order deformation equation in a more general form:

$$[1 - \alpha(q)]\mathcal{L}[\phi(x; q) - u_0(x)] = c_0 \beta(q) \mathcal{N}[\phi(x; q)], \quad x \in \Omega, \quad q \in [0, 1], \quad (3.6)$$

where $\alpha(q)$ and $\beta(q)$ are the so-called *deformation functions*² satisfying

$$\alpha(0) = \beta(0) = 0, \alpha(1) = \beta(1) = 1, \quad (3.7)$$

whose Taylor series

$$\alpha(q) = \sum_{m=1}^{+\infty} \alpha_m q^m, \quad \beta(q) = \sum_{m=1}^{+\infty} \beta_m q^m, \quad (3.8)$$

are convergent for $|q| \leq 1$. In fact, the zeroth-order deformation equation (3.6) can be further generalized, as shown by Liao (2003, 2004, 2009a). Obviously, there are an infinite number of the deformation functions as defined above. Thus, the approximation series given by the HAM can contain so many “artificial” degrees of freedom that they provide us great possibility to guarantee the convergence of

¹ Liao (1997) originally used the symbol $\hbar$ to denote the non-zero auxiliary parameter. But, $\hbar$ is well-known as Planck’s constant in quantum mechanics. To avoid misunderstanding, we replace $\hbar$ by the symbol c_0 in this book, which denotes the “basic” convergence-control parameter.

² $\alpha(q)$ and $\beta(q)$ were called “approaching function” in some early articles about the homotopy-analysis method. In this book, we name them “deformation function”, which better reveals its relationship with the zeroth-order deformation equations.

homotopy-series solution. Note that $u_n(x)$ is always governed by the same auxiliary linear operator $\mathcal{L}$, and we have great freedom to choose $\mathcal{L}$ in such a way that $u_n(x)$ is easy to obtain. More importantly, for properly chosen auxiliary linear operator $\mathcal{L}$ and initial guess, we can gain convergent homotopy-series solution by choosing proper convergence-control parameter c_0 and proper deformation functions $\alpha(q)$ and $\beta(q)$. Inversely, the guarantee of the convergence of homotopy-series solutions also provides us freedom to choose the auxiliary linear operator $\mathcal{L}$ and initial guess. It is due to such kind of guarantee in the frame of the HAM that a nonlinear ODE with variable coefficients can be transferred into a sequence of linear ODEs with constant coefficients (Liao, 2009c), that a nonlinear PDE can be transferred into an infinite number of linear ODEs (Liao, 2006, 2009b), that several coupled nonlinear ODEs can be transferred into an infinite number of linear decoupled ODEs (Yang and Liao, 2006), and that a 2nd-order nonlinear PDE can be replaced even by an infinite number of 4th-order linear PDEs (Liao, 2007). In fact, it is such kind of guarantee for convergence of series solutions, together with the extremely large freedom in choice of the auxiliary linear operators, that greatly simplifies finding convergent series of nonlinear equations in the frame of the HAM, as illustrated in above-mentioned articles (Liao, 2006, 2007, 2009b,c; Yang and Liao, 2006). On the other hand, without such kind of guarantee of convergence, we have in practice no *true* freedom to choose the auxiliary linear operator $\mathcal{L}$, because the freedom to get a divergent series solution has no meanings at all! For example, Liang and Jeffrey (2009) pointed out that the series solution given by means of the so-called “homotopy perturbation method” (He, 1999) is divergent at all points except the initial guess, and thus has completely no scientific meanings. So, unlike perturbation techniques and the traditional non-perturbation methods mentioned above, the HAM satisfies both the standard (a) and (b).

How to find a proper convergence-control parameter c_0 so as to gain a convergent series solution? A straight-forward way to check the convergence of a homotopy-series solution is to substitute it into original governing equations and boundary/initial conditions and then to check the corresponding squared residual integrated in the whole region. However, when the approximations contain unknown convergence-control parameters and/or other unknown physical parameters, it is time-consuming to calculate the squared residual at high-order of approximations. To avoid the time-consuming computation, Liao (1997, 1999, 2003) suggested to investigate the convergence of some special quantities which often have important physical meanings. For example, one can consider the convergence of $u'(0)$ and $u''(0)$ of a nonlinear differential equation $\mathcal{N}[u(x)] = 0$. It is found by Liao (1997, 1999, 2003) that there often exists such an effective-region $\mathbf{R}_e$ that any $c_0 \in \mathbf{R}_e$ gives a convergent series solution of such kind of quantities. Besides, such kind of effective-region can be found, although approximately, by plotting the curves of these unknown quantities versus c_0 . For example, for a nonlinear differential equation $\mathcal{N}[u(x)] = 0$, one may approximately determine $\mathbf{R}_e$ by plotting curves $u'(0) \sim c_0$, $u''(0) \sim c_0$ and so on. These curves are called “ c_0 -curves” or “curves for

convergence-control parameter”³, which have been successfully applied to solve many nonlinear problems (Liao, 2003).

However, such kind of c_0 -curves can not tell us the best convergence-control parameter c_0 , which corresponds to the fastest convergent series. In 2007, Yabushita et al. (2007) applied the HAM to solve two coupled nonlinear ODEs. They suggested the so-called “optimization method” to find out the two optimal convergence-control parameters by means of the minimum of the squared residual of governing equations. Let

$$E_m = \int_{\Omega} \left\{ \mathcal{N} \left[\sum_{n=1}^m u_n(x) \right] \right\}^2 d\Omega$$

denote the squared residual of the m th-order approximation of the governing equation $\mathcal{N}(u) = 0$, integrated in the whole domain Ω . In theory, if the squared residual E_m tends to zero, then $\sum_{n=0}^{+\infty} u_n(x)$ is a series solution of the original equation $\mathcal{N}(u) = 0$. So, if there exists only one convergence-control parameter c_0 , the so-called effective-region $\mathbf{R}_c$ of the convergence-control parameter c_0 is defined by

$$\mathbf{R}_c = \left\{ c_0 \mid \lim_{m \rightarrow +\infty} E_m(c_0) = 0 \right\}.$$

Besides, at the given order of approximation, the minimum of the squared residual E_m corresponds to the optimal approximation. So, the curves of the squared residual E_m versus c_0 indicate not only the effective-region $\mathbf{R}_c$ of the convergence-control parameter c_0 , but also the optimal value of c_0 that corresponds to the minimum of E_m . Note that one can gain the squared residual of an equation at any order of approximations, even if the exact solutions are unknown. Therefore, it is a very good idea of Yabushita et al. (2007) to use the squared residual to find out the effective-region $\mathbf{R}_c$ and the optimal convergence-control parameters.

In 2008, Akyildiz and Vajravelu (2008) gained optimal convergence-control parameter by the minimum of squared residual of governing equation, and found that the corresponding homotopy-series solution converges very quickly.

In 2008, Marinca and Herişanu (2008) combined c_0 and $\beta(q)$ in the zeroth-order deformation equation (3.6) as one function $\check{\beta}(q) = c_0\beta(q)$ with $\check{\beta}(0) = 0$ but $\check{\beta}(1) \neq 1$, and considered such a family of equations

$$(1-q)\mathcal{L}[\phi(x;q) - u_0(x)] = \check{\beta}(q) \mathcal{N}[\phi(x;q)], \quad q \in [0, 1], \quad (3.9)$$

where the Taylor series

$$\check{\beta}(q) = \sum_{n=1}^{+\infty} c_n q^n$$

converges at $q = 1$. The above equation is a special case of (3.6), if we choose

³ The c_0 -curve was originally called the $\hbar$ -curve, and $\mathbf{R}_c$ was originally denoted by $\mathbf{R}_{\hbar}$.

$$\alpha(q) = q, \quad \beta(q) = \frac{1}{c_0} \sum_{n=1}^{+\infty} c_n q^n = \frac{\check{\beta}(q)}{c_0}, \quad c_0 = \sum_{n=1}^{+\infty} c_n \neq 0. \quad (3.10)$$

So, the so-called “optimal homotopy asymptotic method” (Marinca and Herişanu, 2008, 2009) is still in the frame of the HAM. Even so, Marinca and Herişanu’s approach (Marinca and Herişanu, 2008) is interesting, which has the advantage that $\check{\beta}(1) = 1$ is *unnecessary* so that we have *larger* freedom to choose the auxiliary parameters c_n : all of them become the so-called convergence-control parameters. Marinca and Herişanu (2008) developed the so-called “optimal homotopy asymptotic method” by minimizing the squared residual E_m : at the m th-order of approximation, a set of nonlinear algebraic equations about $c_1, c_2, \dots, c_m$ are solved so as to find their optimal values. In theory, the more the convergence-control parameters are used, the better approximation we should obtain by this optimal HAM. However, with too many unknown parameters, it is time-consuming to find out the optimal convergence-control parameters, especially at high-order of approximations for a complicated nonlinear problem. For example, Niu and Wang (2010a) illustrated that the optimal approach given by Marinca and Herişanu (2008, 2009) is time-consuming (Marinca and Herişanu, 2010; Niu and Wang, 2010b), although their optimal HAM (Marinca and Herişanu, 2008) is more rigorous in theory than Niu and Wang’s ones. It seems that one had to balance the rigorousness in theory against the computational efficiency in practice.

To increase the computational efficiency, Liao (2010b) developed in 2010 an optimal HAM with only three convergence-control parameters. Like Marinca and Herişanu’s approach (Marinca and Herişanu, 2008, 2009), this optimal HAM is also based on the zeroth-order deformation equation (3.6). However, two types of special deformation-functions are used, which are determined completely by only one characteristic parameter $|c_1| < 1$ and $|c_2| < 1$, respectively. In this way, there exist at most only three unknown convergence-control parameters c_0, c_1 and c_2 at *any* order of approximations. In addition, the discrete squared residual is first introduced by Liao (2010b) to efficiently find out the optimal convergence-control parameters.

In Sect. 3.2.1, the Blasius boundary-layer flow is used as an example to illustrate the basic ideas of the different approaches of the optimal HAM. The comparisons of the different optimal HAMs are given in Sect. 3.2.2. A systematic description of the optimal HAM is given in Sect. 3.3, followed by some concluding remarks in Sect. 3.4.

3.2 An illustrative description

3.2.1 Basic ideas

For the sake of simplicity, let us first describe the basic ideas of the optimal HAM via the so-called Blasius boundary-layer flows in fluid mechanics, governed by a nonlinear differential equation

$$f'''(\eta) + \frac{1}{2}f(\eta)f''(\eta) = 0, \quad f(0) = f'(0) = 0, f'(+\infty) = 1, \quad (3.11)$$

where η is a similarity variable, $f(\eta)$ is related to the stream-function, and the prime denotes the derivative with respect to η , respectively. Let $\lambda > 0$ denote a kind of spatial scale-parameter. By means of the transformation

$$f(\eta) = \lambda^{-1} u(\xi), \quad \xi = \lambda \eta. \quad (3.12)$$

the original equation (3.11) becomes

$$u'''(\xi) + \frac{1}{2\lambda^2} u(\xi) u''(\xi) = 0, \quad u(0) = u'(0) = 0, u'(+\infty) = 1, \quad (3.13)$$

where the prime denotes the derivative with respect to ξ . For the sake of comparison with the results given by the normal HAM, we choose the same value $\lambda = 4$ as that used by Liao (1999).

Mathematically, due to the boundary condition $u'(+\infty) = 1$, one has the asymptotic property $u \sim \xi$ as $\xi \rightarrow +\infty$. Physically, it is a common knowledge that velocity of boundary-layer flows mostly tends to mainstream flow exponentially. According to these mathematical and physical knowledge, $u(\xi)$ could be expressed in the form

$$u(\xi) = A_{0,0} + \xi + \sum_{m=1}^{+\infty} \sum_{n=0}^{+\infty} A_{m,n} \xi^n \exp(-m\xi), \quad (3.14)$$

where $A_{m,n}$ is a constant to be determined. It provides us the so-called solution-expression of $u(\xi)$, which plays a key role in the frame of the HAM.

Note that there exist three boundary conditions. According to the solution-expression (3.14), the simplest three terms of $u(\xi)$ are $A_{0,0}$, ξ and $A_{1,0} \exp(-\xi)$. So, we choose the initial approximation in the form

$$u_0(\xi) = \bar{A}_{0,0} + \xi + \bar{A}_{1,0} \exp(-\xi),$$

where $\bar{A}_{0,0}$ and $\bar{A}_{1,0}$ are unknown constants. Enforcing $u_0(\xi)$ to satisfy the three boundary conditions, we have $\bar{A}_{0,0} = -1$ and $\bar{A}_{1,0} = 1$, i.e.

$$u_0(\xi) = \xi - 1 + e^{-\xi}. \quad (3.15)$$

In addition, according to the solution-expression (3.14), we should choose such an auxiliary linear operator $\mathcal{L}$ that the simplest three terms of $u(\xi)$, i.e. $A_{0,0}$, ξ and $A_{1,0} \exp(-\xi)$, should be the common solutions of $\mathcal{L}(u) = 0$, say,

$$\mathcal{L}(C_0 + C_1 \xi + C_2 e^{-\xi}) = 0, \quad (3.16)$$

which uniquely determines the auxiliary linear operator

$$\mathcal{L}(u) = u''' + u'', \quad (3.17)$$

where the prime denotes the derivative with respect to ξ , C_0, C_1 and C_2 are integration coefficients.

Based on the governing equation (3.13), we define such a nonlinear operator

$$\mathcal{N}(u) = u'''(\xi) + \frac{1}{2\lambda^2} u(\xi) u''(\xi). \quad (3.18)$$

Let $\alpha(q)$ and $\beta(q)$ denote two deformation-functions, i.e.

$$\alpha(0) = \beta(0) = 0, \quad \alpha(1) = \beta(1) = 1,$$

whose Maclaurin series

$$\alpha(q) \sim \sum_{k=1}^{+\infty} \alpha_k q^k, \quad \beta(q) \sim \sum_{k=1}^{+\infty} \beta_k q^k$$

converge at $q = 1$, say,

$$\sum_{k=1}^{+\infty} \alpha_k = 1, \quad \sum_{k=1}^{+\infty} \beta_k = 1.$$

Let $q \in [0, 1]$ denote the embedding parameter, $c_0 \neq 0$ the convergence-control parameter, and $\phi(\xi; q)$ a kind of continuous mapping of $u(\xi)$, respectively. We construct the so-called zeroth-order deformation equation

$$[1 - \alpha(q)] \mathcal{L}[\phi(\xi; q) - u_0(\xi)] = c_0 \beta(q) \mathcal{N}[\phi(\xi; q)], \quad (3.19)$$

subject to the boundary conditions

$$\phi = 0, \quad \frac{\partial \phi}{\partial \xi} = 0, \quad \text{at } \xi = 0 \quad (3.20)$$

and

$$\frac{\partial \phi}{\partial \xi} = 1, \quad \text{as } \xi \rightarrow +\infty. \quad (3.21)$$

Obviously, as q increases from 0 to 1, $\phi(\xi; q)$ varies (or deforms) from the initial guess $u_0(\xi)$ to the exact solution $u(\xi)$.

The homotopy-series solution reads

$$u(\xi) = u_0(\xi) + \sum_{k=1}^{+\infty} u_k(\xi), \quad (3.22)$$

where, according to Theorem 4.18, $u_m(\xi)$ is governed by the m th-order deformation equation

$$\mathcal{L} \left[u_m(\xi) - \sum_{k=1}^{m-1} \alpha_{m-k} u_k(\xi) \right] = c_0 \sum_{k=1}^m \beta_k \delta_{m-k}(\xi), \quad (3.23)$$

subject to the boundary conditions

$$u_m(0) = u'_m(0) = 0, \quad u'_m(+\infty) = 0, \quad (3.24)$$

with the definition

$$\delta_k(\xi) = \mathcal{D}_k \{ \mathcal{N}[\phi(\xi; q)] \}.$$

According to Theorem 4.2, we have

$$\mathcal{D}_k \{ \mathcal{N}[\phi(\xi; q)] \} = \mathcal{D}_k [\phi'''(\xi; q)] + \frac{1}{2\lambda^2} \mathcal{D}_k [\phi''(\xi; q) \phi(\xi; q)],$$

which gives according to Theorem 4.3 and Theorem 4.6 that

$$\delta_k(\xi) = \mathcal{D}_k \{ \mathcal{N}[\phi(\xi; q)] \} = u_k'''(\xi) + \frac{1}{2\lambda^2} \sum_{j=0}^k u_j''(\xi) u_{k-j}(\xi). \quad (3.25)$$

Let $u_m^*(\xi)$ denote a special solution of (3.23) and $\mathcal{L}^{-1}$ the inverse operator of $\mathcal{L}$, respectively. We have

$$u_m^*(\xi) = \sum_{k=1}^{m-1} \alpha_{m-k} u_k(\xi) + c_0 \sum_{k=1}^m \beta_k S_{m-k}(\xi), \quad (3.26)$$

where

$$S_k(\xi) = \mathcal{L}^{-1} [\delta_k(\xi)]. \quad (3.27)$$

The common solution reads

$$u_m(\xi) = u_m^*(\xi) + B_0 + B_1 \xi + B_2 e^{-\xi},$$

where the integral coefficients

$$B_1 = 0, \quad B_2 = \left. \frac{du_m^*}{d\xi} \right|_{\xi=0}, \quad B_0 = -u_m^*(0) - B_2,$$

are determined by the boundary conditions (3.24).

The corresponding Mathematica code is given in the appendix and free available (Accessed 25 Nov 2011, will be updated in the future) at

<http://numericaltank.sjtu.edu.cn/HAM.htm>

3.2.2 Different types of optimal methods

At the m th-order of approximation, we define the squared residual of the governing equation (3.13), i.e.

$$E_m = \int_0^{+\infty} \left\{ \mathcal{N} \left[\sum_{i=0}^m u_i(\xi) \right] \right\}^2 d\xi. \quad (3.28)$$

An optimal homotopy-approximation is given by the minimum of the squared residual. There are different types of optimal methods, dependent upon the number of the convergence-control parameters.

It is found that much CPU time is needed to calculate the exact squared residual E_m , especially for large m , the order of homotopy-approximation. For example, even in case that only the basic convergence-control parameter c_0 is unknown, by means of a laptop (MacBook Pro, 2.8 GHz Inter Core 2 Due, 4 GB 1067 MHz DDR3), it needs 68.13 seconds, 272.7 seconds and 1089.5 seconds to calculate the corresponding exact squared residual (3.28) for $m = 6, 8$ and 10 , respectively. When there are more than one unknown parameters, the needed CPU time increases exponentially so that the exact squared residual (3.28) is often useless in practice. Thus, to greatly decrease the CPU time, Liao (2010b) suggested to use the so-called “discrete squared residual” defined by

$$E_m \approx \frac{1}{(N+1)} \sum_{j=0}^N \left\{ \mathcal{N} \left[\sum_{i=0}^m u_i(\xi_j) \right] \right\}^2, \quad (3.29)$$

where $\xi_j = j\Delta\xi$, $\Delta\xi = 0.5$ and $N = 20$ for the Blasius boundary-layer flow governed by (3.13). It is found that much less CPU time is needed by means of the discrete squared residual defined above. For example, when only the basic convergence-control parameter c_0 is unknown, it takes only 0.40 seconds, 0.87 seconds and 1.61 seconds to gain the discrete squared residual E_6, E_8 and E_{10} defined by (3.29), respectively, which are only 0.59%, 0.32% and 0.15% of the CPU time for the exact squared residual defined by (3.28). Therefore, in the following part of this chapter, the discrete squared residual (3.29) is used to find the optimal values of the unknown convergence-control parameters. Besides, for the sake of impartial comparisons, E_{10} , the discrete squared residual at the 10th-order homotopy-approximation, is employed in this chapter to search for the optimal unknown convergence-control parameters.

3.2.2.1 Basic optimal HAM

If only the basic convergence-control parameter c_0 is unknown, we have the basic optimal HAM. In this case, the deformation-functions $\alpha(q)$ and $\beta(q)$, the initial approximation $u_0(\xi)$ and the auxiliary linear operator $\mathcal{L}$ do not contain any unknown parameters. Since the squared residual E_m is dependent upon c_0 , the optimal homotopy-approximation is gained by

$$\frac{dE_m(c_0)}{dc_0} = 0. \quad (3.30)$$

Without loss of generality, let us choose the simplest deformation-functions, i.e. $\alpha(q) = q$ and $\beta(q) = q$. The curves of the exact squared residual E_m defined by (3.28) versus c_0 at different order of approximation $m = 6, 8$ and 10 are

Fig. 3.1 Exact squared residual (3.28) by means of the basic optimal HAM. Solid line: 10th-order approximation; Dashed line: 8th-order approximation; Dash-dotted line: 6th-order approximation.

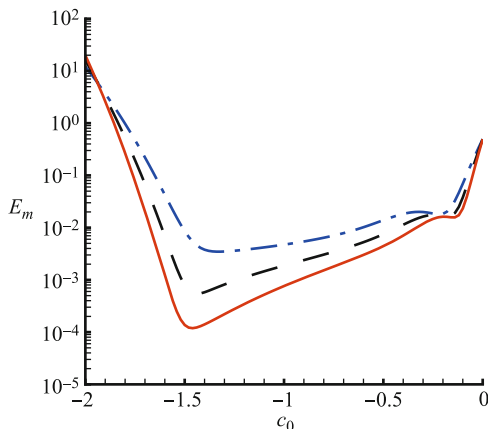
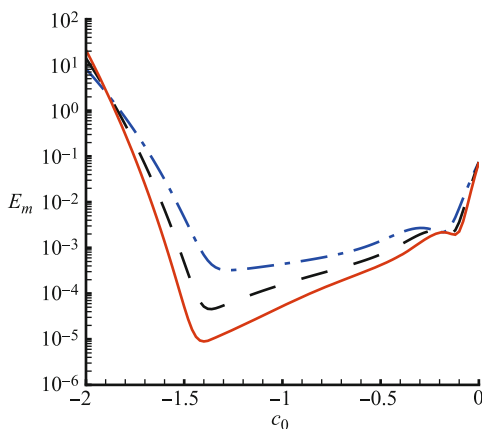


Fig. 3.2 Discrete squared residual (3.29) by means of the basic optimal HAM. Solid line: 10th-order approximation; Dashed line: 8th-order approximation; Dash-dotted line: 6th-order approximation.



shown in Fig. 3.1. Note that the exact squared residual decreases in the region $-1.8 \leq c_0 \leq -0.3$ as the order of approximation increases, which indicates that the homotopy-series converges for an arbitrary value $c_0 \in [-1.8, -0.3]$. More importantly, it indicates that the exact squared residual arrives its minimum at $c_0 \approx -3/2$, which suggests that the optimal value of c_0 is about $-3/2$. So, the curves of the exact squared residual versus c_0 provide us not only the effective-region $\mathbf{R}_c$ of the convergence-control parameter c_0 , but also the optimal value of c_0 which gives the optimal homotopy-series that converges fastest.

As shown in Fig. 3.2, the discrete squared residual (3.29) also decreases in the region $-1.8 \leq c_0 \leq -0.3$ as the order of approximation increases, so that the homotopy-series converges for arbitrary value $c_0 \in [-1.8, -0.3]$, too. More importantly, the optimal value of c_0 is also about $-3/2$. By means of the computer algebra system Mathematica, we directly employ the command `NMinimize` with

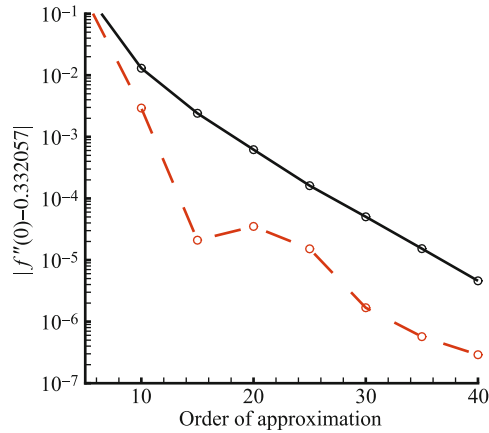
`WorkingPrecision->50`

to get the optimal value of the convergence-control parameter c_0 . It is found that the

Table 3.1 Minimum of the discrete squared residual E_m and the corresponding optimal value of c_0 given by the basic optimal HAM with the used CPU time.

m , order of approx.	Optimal value of c_0	Minimum of E_m	$ f'''(0) - 0.332057 $	CPU time (seconds)
2	-0.3897	3.46×10^{-3}	1.206	0.34
4	-1.0800	1.21×10^{-3}	0.0976	1.6
6	-1.2733	3.23×10^{-4}	0.0116	5.4
8	-1.3662	4.53×10^{-5}	0.00185	14.1
10	-1.4002	8.91×10^{-6}	9.71×10^{-4}	31.6
12	-1.4314	3.16×10^{-6}	2.94×10^{-4}	63.7
14	-1.4760	4.76×10^{-7}	1.89×10^{-4}	119.9
16	-1.4823	6.13×10^{-8}	8.84×10^{-5}	218.8
18	-1.4913	8.37×10^{-9}	1.02×10^{-5}	411.3
20	-1.4979	1.87×10^{-9}	8.08×10^{-6}	719.9
22	-1.5180	4.90×10^{-10}	1.06×10^{-5}	1190

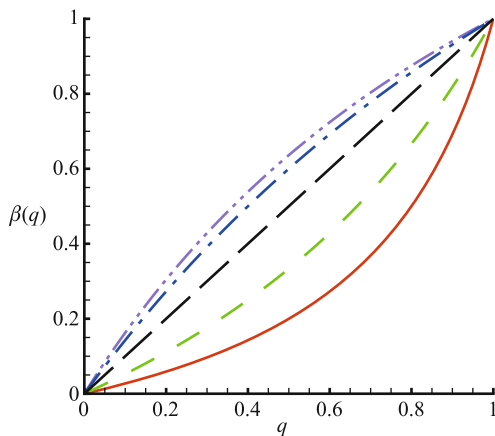
Fig. 3.3 The absolute error $|f'''(0) - 0.332057|$ at different order of approximations by means of the normal and the optimal HAM. Solid line: the normal HAM with $c_0 = -1$; Dashed line: the basic optimal HAM with $c_0 = -7/5$.



optimal value of c_0 given by the minimum of the discrete squared residual (3.29) is indeed more and more close to $-3/2$, as shown in Table 3.1. Thus, the discrete squared residual E_m defined by (3.29) can give good enough approximations of the optimal convergence-control parameter c_0 and its convergence-region.

According to Table 3.1, E_{10} has its minimum value 8.91×10^{-6} at $c_0 = -1.400$. By means of $c_0 = -7/5$, the value of $f'''(0)$ converges much faster to 0.332057 than the homotopy-series solution given by means of the normal HAM corresponding to $c_0 = -1$, as shown in Fig. 3.3. The corresponding discrete squared residual E_m decreases much more quickly than those given by the normal HAM with $c_0 = -1$, as shown in Fig. 3.5. So, even the basic optimal HAM can give much better approximations than the normal HAM. Therefore, it is strongly suggested that at least the basic optimal HAM should be used to gain the optimal homotopy-approximation.

Fig. 3.4 Deformation-function $\beta(q)$ defined by (3.31) and (3.32). Solid line: $c_1 = 3/4$; Dashed line: $c_1 = 1/2$; Long-dashed line: $c_1 = 0$; Dash-dotted line: $c_1 = -1/2$; Dash-double-dotted line: $c_1 = -3/4$.



3.2.2.2 Three-parameter optimal HAM

There are many different ways to introduce more convergence-control parameters in the frame of the HAM. For example, Liao (2010b) suggested the following one-parameter deformation-functions:

$$\alpha(q) = \sum_{m=1}^{+\infty} \alpha_m(c_2) q^m, \quad \beta(q) = \sum_{m=1}^{+\infty} \beta_m(c_1) q^m, \quad (3.31)$$

where

$$\alpha_1 = 1 - c_2, \quad \beta_m = 1 - c_1, \quad \alpha_m = (1 - c_2) c_2^{m-1}, \quad \beta_m = (1 - c_1) c_1^{m-1}, \quad m \geq 1 \quad (3.32)$$

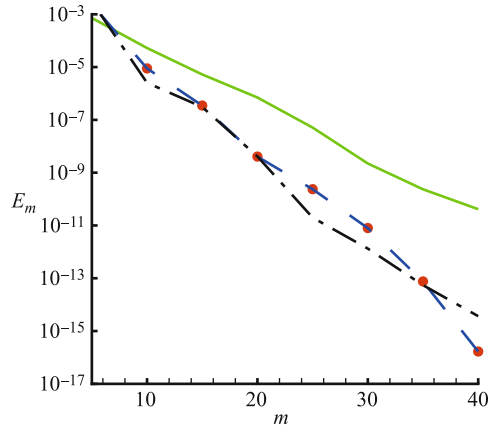
and $|c_1| < 1$ and $|c_2| < 1$ are the so-called convergence-control parameters. The different values of c_1 define different deformation-function $\beta(q)$, as shown in Fig. 3.4. In this case, the squared residual E_m contains at most three unknown convergence control parameters c_0 , c_1 and c_2 at any order of approximation. In theory, the more quickly E_m decreases to zero, the faster the corresponding homotopy-series solution converges. So, at the given order of approximation m , the corresponding optimal convergence-control parameters are determined by the minimum of E_m , corresponding to a set of three nonlinear algebraic equations

$$\frac{\partial E_m}{\partial c_0} = 0, \quad \frac{\partial E_m}{\partial c_1} = 0, \quad \frac{\partial E_m}{\partial c_2} = 0. \quad (3.33)$$

This provides us a three-parameter optimal HAM. In the special case of $c_1 = c_2$, we have only two unknown convergence-control parameters c_0 and c_1 , whose optimal values are given by

$$\frac{\partial E_m}{\partial c_0} = 0, \quad \frac{\partial E_m}{\partial c_1} = 0,$$

Fig. 3.5 Discrete squared residual (3.29) at different order of approximations by the normal HAM, the basic optimal HAM and the two-parameter optimal HAM. Solid line: the normal HAM with $c_0 = -1$; Symbols: the basic optimal HAM with $c_0 = -7/5$; Dashed line: the two-parameter optimal HAM with $c_0 = -1.0801$ and $c_1 = c_2 = -0.2964$; Dash-dotted line: the three-parameter optimal HAM with $c_0 = -1.7913$, $c_1 = 0.1647$ and $c_2 = 0.1075$.



corresponding to a two-parameter optimal HAM. When $c_1 = c_2 = 0$, we have $\alpha(q) = q$ and $\beta(q) = q$, so that only the basic convergence-control parameter c_0 is unknown: this is exactly the basic optimal HAM.

In case of $c_1 = c_2$, it is found that the discrete squared residual E_{10} arrives its minimum 8.91×10^{-6} at $c_0 = -1.0801$ and $c_1 = c_2 = -0.2964$. Using these two optimal convergence-control parameters, the corresponding squared residual E_m decreases much faster than that given by the normal HAM, but is almost the same as the basic optimal HAM, as shown in Fig. 3.5.

In case of $c_0 \neq c_1 \neq c_2$, the discrete squared residual E_{10} has the minimum 2.53×10^{-6} at

$$c_0 = -1.7913, \quad c_1 = 0.1647, \quad c_2 = 0.1075.$$

Using these three optimal convergence-control parameters, the corresponding squared residual decrease faster than that given by the normal HAM, but is not obviously better than the basic optimal HAM with $c_0 = -7/5$ and the two-parameter optimal HAM with $c_0 = -1.0801$ and $c_1 = c_2 = -0.2964$, as shown in Fig. 3.5.

Therefore, all of the optimal HAMs can greatly accelerate the convergence of series solution. However, for the Blasius flow, the optimal HAM with two or three unknown convergence-control parameters are *not* obviously better than the basic optimal HAM with only one unknown convergence-control parameter c_0 . So, it is strongly suggested to first use the basic optimal HAM in practice.

3.2.2.3 Infinite-parameter optimal HAM

If we choose the deformation-functions $\alpha(q) = q$ and

$$\beta(q) = \frac{1}{c_0} \sum_{n=1}^{+\infty} c_n q^n,$$

where

$$c_0 = \sum_{n=1}^{+\infty} c_n \neq 0,$$

then the zeroth-order deformation equation reads

$$(1-q)\mathcal{L}[\phi(\xi;q) - u_0(\xi)] = \left(\sum_{n=1}^{+\infty} c_n q^n \right) \mathcal{N}[\phi(\xi;q)],$$

and the corresponding m th-order deformation equation becomes

$$\mathcal{L}[u_m(\xi) - \chi_m u_{m-1}(\xi)] = \sum_{n=1}^m c_n \delta_{m-n}(\xi),$$

subject to the boundary conditions

$$u_m(0) = 0, \quad u'_m(0) = 0, \quad u'_m(+\infty) = 0,$$

where $\chi_1 = 0$ and $\chi_m = 1$ for $m \geq 2$. Note that the m th-order homotopy-approximation

$$u(\xi) \sim u_0(\xi) + \sum_{n=1}^m u_n(\xi)$$

contains the m unknown convergence-control parameters

$$c_1, c_2, c_3, \dots, c_m.$$

Therefore, in theory, there exist an *infinite* number of unknown convergence-control parameters

$$c_1, c_2, c_3, \dots,$$

as $m \rightarrow +\infty$. In this case, the optimal m th-order homotopy-approximation is given by a set of m nonlinear algebraic equations

$$\frac{\partial E_m}{\partial c_n} = 0, \quad 1 \leq n \leq m. \quad (3.34)$$

The above optimal HAM was suggested by Marinca and Herişanu (2008) in the so-called “optimal homotopy asymptotic method”.

Obviously, at the m th-order of approximation, the optimal HAM suggested by Marinca and Herişanu (2008) contains m unknown convergence-control parameters. It is found that, as the order of approximation increases, the squared residual of the optimal homotopy-approximation decreases faster than the basic optimal HAM that contains only one unknown convergence-control parameter c_0 , as shown in Table 3.2 and Fig. 3.6. Note that, the basic optimal HAM is a special case of the infinite-parameter HAM when $c_1 = c_0 \neq 0$ and $c_k = 0$ for $k > 1$. So, it is easy to understand that, at the *same* order of approximation, the infinite-parameter optimal HAM gives

Table 3.2 Minimum of the discrete squared residual E_m , the used CPU time (seconds) and the absolute error $|f''(0) - 0.332057|$ of the corresponding optimal homotopy-approximations given by the infinite-parameter optimal HAM suggested by Marinca and Herişanu (2008).

m , order of approx.	E_m	$ f''(0) - 0.332057 $	CPU time (seconds)
2	1.34×10^{-3}	0.705	0.43
3	4.94×10^{-4}	0.504	2.2
4	2.16×10^{-4}	0.305	4.3
5	9.92×10^{-5}	0.235	8.0
6	4.22×10^{-5}	0.128	28.4
7	1.31×10^{-5}	0.103	52.3
8	2.32×10^{-6}	0.0348	93.2
9	2.12×10^{-7}	0.0430	235.0
10	1.19×10^{-8}	0.0192	1387.3

better optimal homotopy-approximation than the basic optimal HAM. However, the number of the unknown convergence-control parameters of optimal HAM suggested by Marinca and Herişanu (2008) increases linearly with the order of approximation, so that the used CPU time increases exponentially, as shown in Table 3.2. So, if we consider the minimum of E_m versus the used CPU time, it is found that the discrete squared residual E_m given by the basic optimal HAM decreases faster than the infinite-parameter optimal HAM, as shown in Fig. 3.7. For example, using the basic optimal HAM, we spend 719.9 seconds CPU time to gain the 20th-order optimal homotopy-approximation with the squared residual 1.87×10^{-9} , as shown in Table 3.1. However, using the infinite-parameter optimal HAM, we spend 1387.3 seconds CPU time to obtain the 10th-order optimal homotopy-approximation with the squared residual 1.19×10^{-8} only, as shown in Table 3.2. Besides, the basic optimal HAM gives more accurate approximation of $f''(0)$ than the infinite-parameter optimal HAM, as shown in Fig. 3.8. So, the basic optimal HAM is computationally more efficient than the infinite-parameter optimal HAM suggested by Marinca and Herişanu (2008). Therefore, although the infinite-parameter optimal HAM is more rigorous in theory, the basic optimal HAM is computationally more efficient in practice.

The similar conclusion was found by Niu and Wang (2010a), who solved some differential equations by means of the optimal method suggested by Marinca and Herişanu (2008) and reported that the CPU time increases exponentially as the order of approximation increases, so that the so-called “optimal homotopy asymptotic method” is time-consuming (Marinca and Herişanu, 2010; Niu and Wang, 2010b), especially for high-order approximations. In theory, this is easy to understand: if we express the solution in the form

$$u(x) = \sum_{n=1}^{+\infty} a_n e_n(x),$$

Fig. 3.6 Comparison of the discrete squared residual E_m of the m th-order optimal homotopy-approximation. Solid line: basic optimal HAM; Dashed line: the infinite-parameter optimal HAM suggested by Marinca and Herişanu (2008).

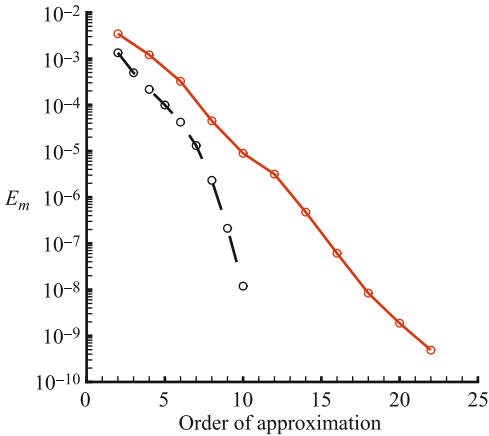


Fig. 3.7 Comparison of the discrete squared residual E_m of the m th-order optimal homotopy-approximation versus the used CPU time. Solid line: basic optimal HAM; Dashed line: the infinite-parameter optimal HAM suggested by Marinca and Herişanu (2008).

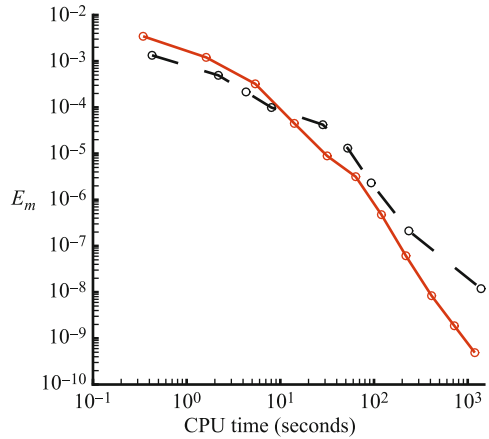
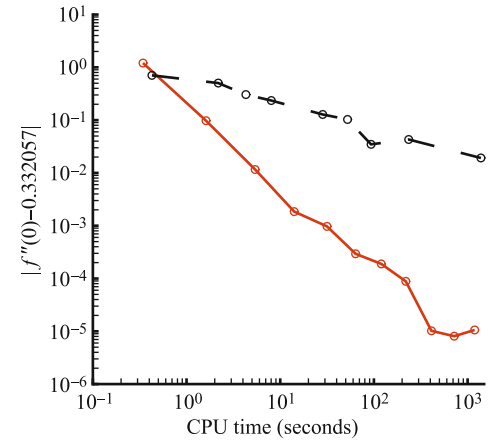


Fig. 3.8 Comparison of the absolute error $|f''(0) - 0.332057|$ of the m th-order optimal homotopy-approximation versus the used CPU time. Solid line: basic optimal HAM; Dashed line: the infinite-parameter optimal HAM suggested by Marinca and Herişanu (2008).



where $e_n(x)$ is a base function and a_n is unknown coefficient, and then use the method of least squares (Björck, 1996), we also had to solve a set of nonlinear algebraic equations with large number of unknowns, which is however rather time-consuming. To overcome this disadvantage of “optimal homotopy asymptotic method” suggested by Marinca and Herişanu (2008), Niu and Wang (2010a) developed the so-called “one-step optimal homotopy analysis method”: the optimal value of the first convergence-control parameter c_1 is approximately determined by solving only one algebraic equation

$$\frac{dE_1}{dc_1} = 0,$$

then keeping this known value of c_1 as a good approximation of its optimal value forever, one can similarly obtain c_2 by solving only one algebraic equation

$$\frac{dE_2}{dc_2} = 0,$$

and so on. In this way, one can obtain the “optimal” values $c_1, c_2, c_3, \dots$ up to any order of approximation by solving only one algebraic equation

$$\frac{dE_k}{dc_k} = 0, \quad k = 1, 2, 3, \dots$$

each time. However, rigorously speaking, $c_1, c_2, c_3, \dots$ obtained in this way are *not* the optimal ones in theory.

3.2.2.4 Finite-parameter optimal HAM

As shown above, it is time-consuming if an optimal HAM contains infinite number of convergence-control parameters. To overcome this disadvantage, we modify the so-called “optimal homotopy asymptotic method” (Marinca and Herişanu, 2008) by using only finite number of convergence-control parameters in this book.

If we choose $\alpha(q) = q$ and such a special deformation-function

$$\beta(q) = \frac{1}{c_0} \sum_{n=1}^{\kappa} c_n q^n,$$

where $\kappa \geq 1$ is a positive integer and

$$c_0 = \sum_{n=1}^{\kappa} c_n \neq 0,$$

then the zeroth-order deformation equation reads

$$(1-q)\mathcal{L}[\phi(\xi;q) - u_0(\xi)] = \left(\sum_{n=1}^{\kappa} c_n q^n \right) \mathcal{N}[\phi(\xi;q)],$$

and the corresponding m th-order deformation equation becomes

$$\mathcal{L}[u_m(\xi) - \chi_m u_{m-1}(\xi)] = \sum_{n=1}^{\min\{m, \kappa\}} c_n \delta_{m-n}(\xi),$$

subject to the boundary conditions

$$u_m(0) = 0, \quad u'_m(0) = 0, \quad u'_m(+\infty) = 0,$$

where $\chi_1 = 0$ and $\chi_m = 1$ for $m \geq 2$. Note that the m th-order homotopy-approximation

$$u(\xi) \sim u_0(\xi) + \sum_{n=1}^m u_n(\xi)$$

contains at most the κ unknown convergence-control parameters

$$c_1, c_2, c_3, \dots, c_{\kappa}.$$

Therefore, in theory, there exist a *finite* number of unknown convergence-control parameters

$$c_1, c_2, c_3, \dots, c_{\kappa}$$

even as $m \rightarrow +\infty$. In this case, the optimal m th-order homotopy-approximation is given by a set of $\min\{m, \kappa\}$ nonlinear algebraic equations

$$\frac{\partial E_m}{\partial c_n} = 0, \quad 1 \leq n \leq \min\{m, \kappa\}. \quad (3.35)$$

The above optimal HAM becomes exactly the so-called “optimal homotopy asymptotic method” suggested by Marinca and Herişanu (2008), if $\kappa \rightarrow \infty$. Besides, when $c_1 = c_0$ and $c_n = 0$ for $n > 1$, it becomes the basic optimal HAM. Therefore, this optimal HAM is more general.

Let us first consider the case of two convergence-control parameters c_1 and c_2 , i.e. $\kappa = 2$. Regarding c_1 and c_2 as unknown parameters, we first gain the m th-order homotopy-approximation and then obtain the optimal convergence-control parameters c_1 and c_2 by the minimum of E_m . It is found that, as the order of approximation increases, the used CPU time increases exponentially. The minimum of E_m , the used CPU time and the absolute error $|f''(0) - 0.332057|$ of the corresponding optimal m th-order homotopy-approximations are given in Table 3.3. Considering the squared residual versus the used CPU time, it is found that the optimal homotopy-approximations given by the basic optimal HAM (corresponding to $\kappa = 1$) converges faster than those by the finite-parameter optimal HAM when $\kappa = 2$, as shown in Figs. 3.9 and 3.10. The same conclusion is obtained in case of $\kappa = 3$, as shown in Table 3.4, Figs. 3.9 and 3.10. It is found that the squared residual E_m given by

Table 3.3 Minimum of the discrete squared residual E_m , the used CPU time (seconds) and the absolute error $|f''(0) - 0.332057|$ of the corresponding optimal homotopy-approximations given by the finite-parameter optimal HAM when $\kappa = 2$.

m , order of approx.	E_m	$ f''(0) - 0.332057 $	CPU time (seconds)
4	6.84×10^{-4}	0.440	2.23
6	4.74×10^{-4}	0.309	8.69
8	4.23×10^{-5}	1.85×10^{-3}	27.1
10	3.38×10^{-6}	2.61×10^{-3}	71.8
12	1.24×10^{-6}	1.56×10^{-3}	171.0
14	2.15×10^{-7}	9.55×10^{-5}	382.1
16	6.07×10^{-9}	2.76×10^{-4}	816.2
18	2.45×10^{-9}	1.62×10^{-4}	1637.8
20	1.78×10^{-9}	7.05×10^{-5}	3145.6

Table 3.4 Minimum of the discrete squared residual E_m , the used CPU time (seconds) and the absolute error $|f''(0) - 0.332057|$ of the corresponding optimal homotopy-approximations given by the finite-parameter optimal HAM when $\kappa = 3$.

m , order of approx.	E_m	$ f''(0) - 0.332057 $	CPU time (seconds)
4	5.21×10^{-4}	0.333	2.70
6	4.93×10^{-4}	0.508	11.2
8	2.57×10^{-5}	2.11×10^{-3}	38.0
10	2.52×10^{-6}	2.82×10^{-3}	107.1
12	2.50×10^{-6}	4.44×10^{-4}	279.5
14	1.51×10^{-7}	1.95×10^{-5}	686.2
16	4.67×10^{-9}	2.86×10^{-4}	1580.5
18	1.84×10^{-9}	7.07×10^{-5}	3399.2

two-parameter optimal HAM ($\kappa = 2$) decreases faster than those by three-parameter optimal HAM ($\kappa = 3$), as shown in Fig. 3.9. Notice that $\kappa = 1$ corresponds to the basic optimal HAM. So, this example illustrates that the basic optimal HAM with only one convergence-control parameter c_0 is computationally more efficient than other optimal HAM with more convergence-control parameters. Therefore, it is strongly suggested that the basic optimal HAM with only one convergence-control parameter c_0 should be used first in practice.

In practice, we can choose the optimal convergence-control parameters at a reasonably high order of approximation. For example, when $\kappa = 2$, the discrete squared residual at the 10th-order of approximation arrives its minimum 3.38×10^{-6} with the optimal convergence-control parameters $c_1 = -1.457$ and $c_2 = -0.0795$. Similarly, when $\kappa = 3$, the discrete squared residual at the 10th-order of approximation arrives its minimum 2.52×10^{-6} with the optimal convergence-control parameters $c_1 = -1.4870$, $c_2 = -0.0826$ and $c_3 = -0.0126$. It is found that the discrete squared residual of the homotopy-series given by the two-parameter optimal HAM ($\kappa = 2$) decreases a little more quickly than the basic optimal HAM ($\kappa = 1$) and even than

Fig. 3.9 Comparison of the discrete squared residual E_m of the m th-order optimal homotopy-approximation versus the used CPU time. Solid line: basic optimal HAM ($\kappa = 1$); Left-triangle: the finite-parameter optimal HAM when $\kappa = 2$; Right-triangle: the finite-parameter optimal HAM when $\kappa = 3$.

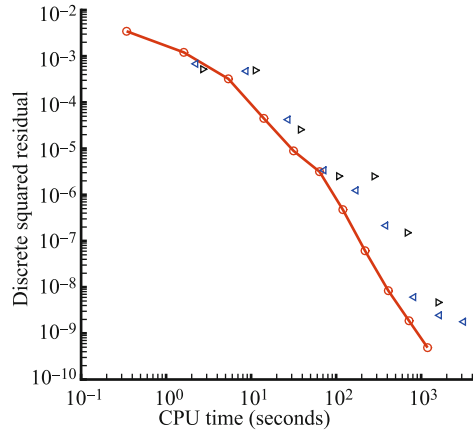
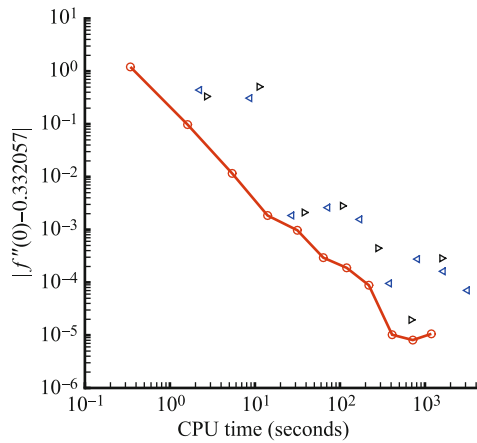


Fig. 3.10 Comparison of the absolute error $|f''(0) - 0.332057|$ of the m th-order optimal homotopy-approximation versus the used CPU time. Solid line: basic optimal HAM ($\kappa = 1$); Left-angle: the finite-parameter optimal HAM when $\kappa = 2$; Right-angle: the finite-parameter optimal HAM when $\kappa = 3$.



the three-parameter optimal HAM ($\kappa = 3$), as shown in Fig.3.11. However, it is found that $f''(0)$ given by the basic optimal HAM ($\kappa = 1$) converges to the exact value 0.332057 more quickly than the two-parameter ($\kappa = 2$) and three-parameter ($\kappa = 3$) optimal HAM, as shown in Fig. 3.12. Note that the approximations given by the three-parameter optimal HAM ($\kappa = 3$) are often worse than those given by the two-parameter optimal HAM ($\kappa = 2$), as shown in Figs. 3.11 and 3.12. Therefore, although the used CPU time is nearly the same for different optimal approaches ($\kappa = 1, 2, 3$) to gain results at the same order of approximation, the optimal HAMs with more convergence-control parameters do not give obviously better homotopy-approximations than the basic optimal HAM in general. Thus, the basic optimal HAM is strongly suggested to use in practice.

In summary, the approximations given by an optimal HAM converge much faster than the normal HAM in general. The example considered in this chapter suggests that the optimal HAMs with one or two convergence-control parameters are computationally most efficient and can give accurate enough approximations, but the optimal HAMs with too many convergence-control parameters are time-consuming.

Fig. 3.11 Discrete squared residual versus used CPU time. Solid line: the basic optimal HAM ($\kappa = 1$) with $c_0 = -7/5$; Left-triangle: two-parameter optimal HAM ($\kappa = 2$) with $c_1 = -1.4572$ and $c_2 = -0.0795$; Right-triangle: three-parameter optimal HAM ($\kappa = 3$) with $c_1 = -1.4870$, $c_2 = -0.0826$ and $c_3 = -0.0126$.

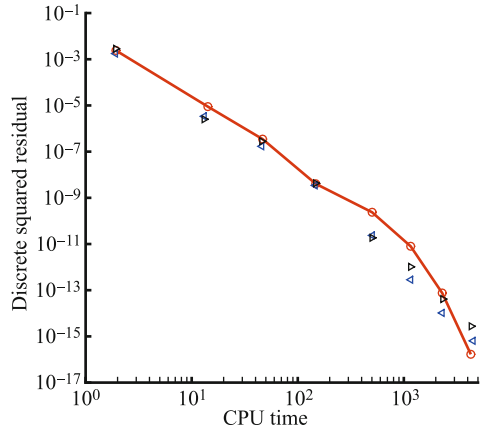
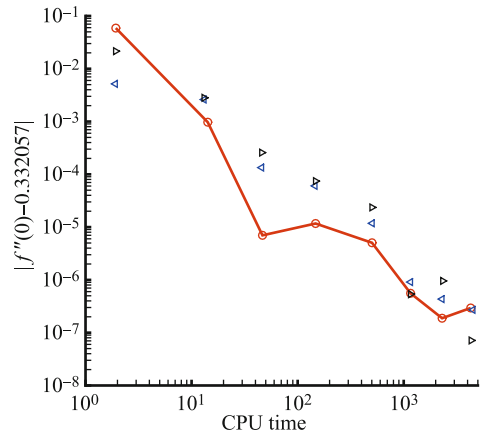


Fig. 3.12 Absolute error $|f''(0) - 0.332057|$ versus used CPU time. Solid line: the basic optimal HAM ($\kappa = 1$) with $c_0 = -7/5$; Left-triangle: two-parameter optimal HAM ($\kappa = 2$) with $c_1 = -1.4572$ and $c_2 = -0.0795$; Right-triangle: three-parameter optimal HAM ($\kappa = 3$) with $c_1 = -1.4870$, $c_2 = -0.0826$ and $c_3 = -0.0126$.



3.3 Systematic description

The definition of the discrete squared residual (3.29) and the above-mentioned optimal HAMs have general meanings. Thus, they can be applied to solve different types of nonlinear equations with strong nonlinearity. Here, we give a brief description in general.

Given a nonlinear differential equation

$$\mathcal{N}[u(\mathbf{x}, t)] = 0, \quad (3.36)$$

where $u(\mathbf{x}, t)$ is an unknown function, $\mathbf{x}$ and t denote spatial and temporal independent variables, respectively, we can choose a proper initial guess $u_0(\mathbf{x}, t)$ and a proper auxiliary linear operator $\mathcal{L}$ to construct the zeroth-order deformation equation

$$[1 - \alpha(q)] \mathcal{L}[\phi(\mathbf{x}, t; q) - u_0(\mathbf{x}, t)] = c_0 \beta(q) \mathcal{N}[\phi(\mathbf{x}, t; q)], \quad (3.37)$$

where $q \in [0, 1]$ is an embedding parameter, $\alpha(q)$ and $\beta(q)$ are two deformation-functions with κ unknown convergence-control parameters

$$\mathbf{c} = \{c_1, c_2, \dots, c_\kappa\},$$

where κ may be an infinity, and the Maclaurin series of $\alpha(q), \beta(q)$ read

$$\alpha(q) \sim \sum_{n=1}^{+\infty} \alpha_n q^n, \quad \beta(q) \sim \sum_{n=1}^{+\infty} \beta_n q^n.$$

Assuming that the initial guess $u_0(\mathbf{x}, t)$, the auxiliary linear operator $\mathcal{L}$, and the $(\kappa + 1)$ convergence-control parameters $c_0, c_1, \dots, c_\kappa$ are so properly chosen that the homotopy-Maclaurin series

$$\phi(\mathbf{x}, t; q) = u_0(\mathbf{x}, t) + \sum_{m=1}^{+\infty} u_m(\mathbf{x}, t) q^m \quad (3.38)$$

converges at $q = 1$, we have the homotopy-series solution

$$u(\mathbf{x}, t) = u_0(\mathbf{x}, t) + \sum_{m=1}^{+\infty} u_m(\mathbf{x}, t). \quad (3.39)$$

Substituting (3.38) into the zeroth-order deformation equation (3.37) and then equating the coefficients of the like-power of the embedding parameter q , we have⁴ the m th-order deformation equation:

$$\mathcal{L} \left[u_m(\mathbf{x}, t) - \sum_{n=1}^{m-1} \alpha_{m-n} u_n(\mathbf{x}, t) \right] = c_0 \sum_{n=1}^m \beta_n \delta_{m-n}(\mathbf{x}, t), \quad (3.40)$$

where

$$\delta_j(\mathbf{x}, t) = \mathcal{D}_j \left\{ \mathcal{N} \left[\sum_{n=0}^{+\infty} u_n(\mathbf{x}, t) q^n \right] \right\}. \quad (3.41)$$

The definition of the j th-order homotopy-derivative operator $\mathcal{D}_j$ is given in Sect. 3.1. The special solution $u_m^*(\mathbf{x}, t)$ of (3.40) is given by

$$u_m^*(\mathbf{x}, t) = \sum_{n=1}^{m-1} \alpha_{m-n} u_n(\mathbf{x}, t) + c_0 \sum_{n=1}^m \beta_n S_{m-n}(\mathbf{x}, t), \quad (3.42)$$

where

$$S_n(\mathbf{x}, t) = \mathcal{L}^{-1} [\delta_n(\mathbf{x}, t)] \quad (3.43)$$

and $\mathcal{L}^{-1}$ is the inverse operator of $\mathcal{L}$. Then, $u_m(\mathbf{x}, t)$ is uniquely determined by the corresponding boundary/initial conditions.

⁴ Using the so-called m th-order homotopy-derivative operator $\mathcal{D}_m$ and related theorems proved in Chapter 4, one can obtain exactly the same m th-order deformation equation.

To avoid time-consuming computation for the exact squared residual, at the m th-order of approximation, we define a kind of discrete squared residual E_m in a similar way to (3.29).

Definition 3.1. Let

$$(\mathbf{x}_j, t_j) \in \Omega, \quad j = 0, 1, 2, \dots, N,$$

denote the properly chosen $N + 1$ points in the domain Ω on which a nonlinear equation $\mathcal{N}[u(\mathbf{x}, t)] = 0$ is defined. Then, the integral

$$E_m = \frac{1}{(N+1)} \sum_{j=0}^N \left\{ \mathcal{N} \left[\sum_{n=0}^m u_n(\mathbf{x}_j, t_j) \right] \right\}^2 \quad (3.44)$$

is called *the discrete squared residual* of $\mathcal{N}(u) = 0$ on the domain Ω .

Assume that the m th-order homotopy-approximation contains $\kappa' + 1$ unknown convergence-control parameters, where $\kappa' \leq \kappa$. Then, E_m contains $\kappa' + 1$ unknown convergence-control parameters $c_0, c_1, \dots, c_{\kappa'}$, whose optimal values are determined by the minimum of E_m , corresponding to a set of $\kappa' + 1$ nonlinear algebraic equations

$$\frac{\partial E_m}{\partial c_j} = 0, \quad 0 \leq j \leq \kappa' \leq \kappa.$$

The above approach gives the so-called finite-parameter optimal HAM when $\kappa \geq 1$ is a fixed finite integer. It gives the so-called infinite-parameter optimal HAM when there are an infinite number of convergence-control parameters, i.e. $\kappa \rightarrow +\infty$, say, the number of convergence-control parameter linearly increases with the order of approximation.

The above optimal HAMs are based on the deformation-functions with some unknown convergence-control parameters. There are an infinite number of such kind of deformation-functions satisfying the properties (3.7) and (3.8). For example, Liao (2010b) suggested the one-parameter deformation-function

$$\beta(q; c) = (1 - c) \sum_{n=1}^{+\infty} c^{n-1} q^n, \quad |c| < 1 \quad (3.45)$$

in the so-called three-parameter optimal HAM, which contains the special case $\beta(q; 0) = q$ (when $c = 0$) that is mostly used in the frame of the HAM. Besides, one can define the following one-parameter deformation-function

$$\bar{\beta}(q; c) = \frac{1}{\zeta(c)} \sum_{n=1}^{+\infty} \frac{q^n}{n^c}, \quad c > 1, \quad (3.46)$$

where $\zeta(c)$ is Riemann zeta function, and c is a convergence-control parameter. We call $\beta(q; c)$ and $\bar{\beta}(q; c)$ the first and second-type of one-parameter deformation functions, respectively.

Besides, any two different deformation-functions may create a new one. For example,

$$\alpha(q) = \beta(q; c_1) \bar{\beta}(q; c_2) \quad (3.47)$$

defines a deformation-function with two convergence-control parameters c_1 and c_2 . Zhao and Wong (preprint) defines a family of deformation-functions

$$A_{m+1}(q; \mathbf{c}_{m+1}) = \frac{c_{m+1} A_m(q; \mathbf{c}_m)}{1 + (c_{m+1} - 1) A_m(q; \mathbf{c}_m)}, \quad (3.48)$$

where $c_j \neq 0$ ($j = 1, 2, 3, \dots, m+1$) is convergence-control parameter, and $A_m(q; \mathbf{c}_m)$ is a m -parameter deformation-function with the m convergence-control parameters $\mathbf{c}_m = \{c_1, c_2, \dots, c_m\}$. To avoid singularity of the deformation-function defined above, we had to add such a restriction

$$1 + (c_{m+1} - 1) A_m(q; \mathbf{c}_m) \neq 0, \quad q \in [0, 1]. \quad (3.49)$$

In general, given any a convergent series

$$\Pi = \sum_{n=1}^{+\infty} c_n \neq 0,$$

we can always define a corresponding deformation-function

$$\beta(q; \mathbf{c}_\infty) = \frac{1}{\Pi} \sum_{n=1}^{+\infty} c_n q^n,$$

where

$$\mathbf{c}_\infty = \{c_1, c_2, \dots\}.$$

Especially, when $\Pi = \sum_{n=1}^{\kappa} c_n \neq 0$, we have a κ -parameter deformation-function

$$\beta(q; \mathbf{c}_\kappa) = \frac{1}{\Pi} \sum_{n=1}^{\kappa} c_n q^n.$$

In a special case of $\alpha(q) = q$ and

$$\beta(q) = \frac{1}{c_0} \sum_{n=1}^{\kappa} c_n q^n, \quad c_0 = \sum_{n=1}^{\kappa} c_n \neq 0,$$

where $\kappa \geq 1$ is either a finite integer or the infinity. The corresponding zeroth-order deformation equation reads

$$(1 - q) \mathcal{L} [\phi(\mathbf{x}, t; q) - u_0(\mathbf{x}, t)] = \left(\sum_{n=1}^{\kappa} c_n q^n \right) \mathcal{N} [\phi(\mathbf{x}, t; q)], \quad (3.50)$$

and the corresponding high-order deformation equation becomes

$$\mathcal{L}[u_m(\mathbf{x}, t) - \chi_m u_{m-1}(\mathbf{x}, t)] = c_0 \sum_{n=1}^{\min\{m, \kappa\}} \beta_n \delta_{m-n}(\mathbf{x}, t). \quad (3.51)$$

The corresponding optimal homotopy-approximation contains $\min\{m, \kappa\}$ convergence control parameters. Especially, when $\kappa = 1$, we have only one convergence-control parameter c_0 , which is exactly the basic optimal HAM. When $\kappa = +\infty$, it is the optimal HAM suggested by Marinca and Herişanu (2008). When $\kappa = 2$ or $\kappa = 3$, we have a finite-parameter optimal HAM proposed in this chapter. Thus, this optimal HAM logically contains the basic optimal HAM and the optimal approach suggested by Marinca and Herişanu (2008), and therefore is more general.

3.4 Concluding remarks and discussions

Based on the homotopy in topology, the HAM provides us extremely large freedom to choose the initial approximation, the auxiliary linear operator $\mathcal{L}$ and the deformation-functions to construct the so-called zeroth-order deformation equation. Especially, the so-called convergence-control parameter c_0 introduced by Liao (1997) provides us a convenient way to control and adjust the convergence of the homotopy-series: unlike other analytic techniques, the HAM can guarantee the convergence of solution series of nonlinear equations. In fact, it is the convergence-control parameter c_0 that differs the HAM from other analytic techniques. So, the introduction of the convergence-control parameter c_0 in the zeroth-order deformation equation (3.5) by Liao (1997) is a milestone of the HAM. In essence, the convergence-control parameter c_0 provides us one more “artificial” degree of freedom to guarantee the convergence of homotopy-series solution. Therefore, Liao (1999) further proposed a more generalized zeroth-order deformation equation (3.6), which provides us extremely large freedom to introduce more convergence-control parameters in theory. Then, the optimal values of the convergence-control parameters are determined by the minimum of the squared residual of governing equations, which give the fastest convergent series solution of a given nonlinear equation in general.

In this chapter, using the Blasius flow as an example, we describe the basic ideas of the different types of optimal HAM, and compare them. Using the deformation-functions

$$\alpha(q) = q, \quad \beta(q) = \frac{1}{c_0} \sum_{n=1}^{\kappa} c_n q^n, \quad c_0 = \sum_{n=1}^{\kappa} c_n \neq 0, \quad \kappa \geq 1,$$

we propose a more generalized optimal HAM: it becomes the basic optimal HAM when $\kappa = 1$, Marinca and Herişanu’s optimal HAM (Marinca and Herişanu, 2008)

when $\kappa \rightarrow +\infty$, and an finite-parameter optimal HAM when κ is a finite positive integer. Based on these computations, we have the following conclusions:

1. the homotopy-approximations given by the optimal convergence-control parameters are much more accurate in general;
2. the basic optimal HAM with one convergence-control parameter c_0 can give good enough approximations in most cases, and therefore is strongly suggested to use in practice at first;
3. more convergence-control parameters might give better approximations, but need much more CPU time. So, considering the accuracy versus CPU time, the basic optimal HAM with one convergence-control parameter and the optimal HAMs with two or three convergence-control parameters are strongly suggested in practice. In other words, we should consider the computational efficiency in practice.

Note that the computational efficiency of different types of optimal HAMs depends strongly on the method of searching for the minimum of squared residual. In this book, the command `NMinimize` with `WorkingPrecision->50` of the computer algebra system `Mathematica` is used to find out the minimum of squared residual and the corresponding optimal convergence-control parameters. The optimal HAMs mentioned in this chapter should be more powerful, if better methods of finding out minimum of squared residual are proposed in future.

Since there exist no rigorous theories up to now to guide us how to choose a good enough zeroth-order deformation equation for a given nonlinear equation in general, it is important in practice to provide a convenient way to guarantee the fast convergence of the homotopy-series. Our strategy is to introduce some unknown auxiliary-parameters without physical meanings (i.e. the convergence-control parameters) and then to determine their optimal values by the minimum of squared residual of governing equations. It should be emphasized that, in the frame of the HAM, we can introduce such kind of unknown auxiliary-parameters in many different ways. For example, if we choose such an initial approximation

$$u_0(\xi) = \xi + (1 + \mu) e^{-\xi} - \frac{\mu}{2} e^{-2\xi} - \left(1 + \frac{\mu}{2}\right) \quad (3.52)$$

for the Blasius boundary-layer flow considered in this chapter, we can regard the unknown μ as one convergence-control parameter, too. Indeed, the optimal HAMs provide us extremely large freedom to introduce different types of convergence-control parameters to gain accurate enough approximation.

Appendix 3.1 Mathematica code for Blasius flow

Blasius boundary-layer flow equation

$$f''' + \frac{1}{2}ff'' = 0, \quad f(0) = 0, \quad f'(0) = 0, \quad f'(+\infty) = 1$$

is solved by means of the different types of optimal HAM. This code is free available at

<http://numericaltank.sjtu.edu.cn/HAM.htm>

A Simple Users Guide

Input data:

c0: Basic convergence-control parameter c_0 ;

c1, c2: Convergence-control parameters c_1 and c_2 .

Control parameter:

OHAM : Basic or three-parameter optimal HAM when OHAM = 0;
 Infinite or finite-parameter optimal HAM when OHAM = 1;

Nstep: The value of the integer κ in (3.50);

PRN: Show the used CPU time when PRN = 1.

Calculated results:

f[k]: $f(\eta)$;

fx[k]: $f'(\eta)$;

fx0[k]: $f''(0)$;

delta[k]: $\delta_k(\xi)$ defined by (3.25);

S[k]: $S_k(\xi)$ defined by (3.27);

alpha[n]: α_n , the coefficient of Maclaurin series of $\alpha(q)$;

beta[n]: β_n , the coefficient of Maclaurin series of $\beta(q)$;

cc[n]: Set of parameters $\{c_1, c_2, \dots, c_n\}$, where $n = \min\{m, \kappa\}$;

u[k]: $u_k(\xi)$;

U[k]: k th-order approximation of $u(\xi)$, i.e. $\sum_{n=0}^k u_n(\xi)$;

Err[k]: Residual error square E_k defined by (3.29).

Main codes:

ham[1, 11] : First, gain 1st to 11th-order homotopy-approximation;

ham[12, 25] : Then, gain 12th to 25th-order homotopy-approximation;

GetErr[k] : Gain squared residual E_k defined by (3.29).

Mathematica code for Blasius flow

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June 2010

```
<<Calculus`Pade`;  
<<Graphics`Graphics`;  
  
(*****)
```

```

(* Define the initial guess *)
(*****)
u[0]      = y - 1 + Exp[-y];
U[0]      = u[0];
f[0]      = U[0]/lambda /. y-> lambda*x ;
fxx0[0]   = D[f[0],{x,2}] /.x->0 ;
lambda    = N[4,100];

(*****)
(* Define auxiliary linear operator L *)
(*****)
L[f_] := D[f,{y,3}] + D[f,{y,2}];

(*****)
(* Define inverse operator of auxiliary linear operator Linv *)
(* Gain the solution u of equation: L[u] = f *)
(*****)
Linv[f_] := Module[{temp},
temp = DSolve[ L[w[y]] == f, w[y], y];
temp[[1,1,2]] /. C[_]->0 // Expand
];

(*****)
(* The property of the inverse operator Linv *)
(* Linv[f_+g_] := Linv[f] + Linv[g] *)
(*****)
Linv[p_Plus] := Map[Linv,p];
Linv[c_*f_] := c*Linv[f] /. FreeQ[c,y];

(*****)
(* Define Getdelta[m] *)
(*****)
Getdelta[k_] := Module[{temp},
uyy[k] = D[u[k],{y,2}] //Expand ;
uyyy[k] = D[uyy[k],y] //Expand ;
temp[1] = Sum[u[k-n]*uyy[n],{n,0,k}]/2/lambda/lambda//Expand;
temp[2] = uyyy[k] + temp[1] //Expand;
delta[k] = Collect[temp[2], y^_.*Exp[_]];
];

(*****)
(* Define GetErr[m] *)
(* This module gives averaged squared residual *)
(*****)
GetErr[k_] := Module[{temp,dy,G,x,s},
Uyy[k] = D[U[k],{y,2}];
Uyyy[k] = D[Uyy[k],y];
error[k] = Uyyy[k] + U[k]*Uyy[k]/2/lambda/lambda;
Ymax = 10;
Nmax = 20;
dy = N[Ymax/Nmax,100];
s = 0;
For[j = 0, j <= Nmax, j = j + 1,
x = j*dy;

```

```

G[j]      = error[k] /. y -> x //Expand;
temp[2]   = G[j]^2;
s         = s + temp[2];
];
Err[k]    = s/(Nmax+1) /. {AA->1-c1, BB-> 1-c2};
If[NumberQ[Err[k]],Err[k]//N//Print];
];

(*****
(*                               Main Code                               *)
(*****
ham[begin_,end_] :=Block[{uSpecial,B0,B2,temp,z,s},
time[0] = SessionTime[];
For[k=begin,k<=end,k=k+1,
  If[k==1,
    Print["-----"]
  ];
  If[k == 1 && OHAM == 0,
    Nstep = Infinity;
    If[NumberQ[c1], beta[1] = 1-c1, beta[1] = AA ];
    If[NumberQ[c2], alpha[1] = 1-c2, alpha[1] = BB ];
    Print["FINITE-parameter optimal HAM"];
    Print["    c0 = ",c0];
    Print["    c1 = ",c1];
    Print["    c2 = ",c2];
    If[c1 ==0 && c2 ==0,
      Print["This is the BASIC optimal HAM ! "]
    ];
  ];
  If[ k > 1 && OHAM == 0 ,
    beta[k] = c1*beta[k-1];
    alpha[k] = c2*alpha[k-1];
  ];
  If[k == 1 && OHAM > 0,
    If[Nstep == Infinity,
      Print["INFINITE-parameter optimal HAM"],
      Print["FINITE-parameter optimal HAM"]
    ];
    alpha[1] = 1 - c2;
    c0 = 1;
    Print["    Nstep = ",Nstep];
    Print["    c0 = ",c0];
    Print["    c2 = ",c2];
  ];
  If[k > 1 && OHAM > 0, alpha[k] = c2*alpha[k-1] ];
  If[k==1,
    Print["-----"]
  ];
  If[OHAM > 0 && k == 1, cc = {beta[1]} ];
  If[OHAM > 0 && k > 1 && k <= Nstep,
    temp = Union[cc,{beta[k]}];
    cc = temp
  ];
  Print["k = ",k];

```

```

Getdelta[k-1];
temp = Linv[delta[k-1]];
S[k-1] = Collect[temp, y^_.*Exp[_.]];
temp = Sum[alpha[k-n]*u[n], {n, 1, k-1}];
uSpecial = temp+c0*Sum[beta[n]*S[k-n], {n, 1, Min[k, Nstep]}];
B2 = D[uSpecial, y] /. y->0;
B0 = -B2 - uSpecial /. y->0;
temp = uSpecial + B0 + B2*Exp[-y] // Expand;
u[k] = Collect[temp, y^_.*Exp[_.]];
U[k] = Expand[U[k-1] + u[k]];
f[k] = U[k]/lambda /. {y -> lambda*x, AA->1-c1, BB->1-c2};
fx[k] = D[f[k], x];
fxx0[k] = D[fx[k], x] /. x->0;
If[NumberQ[fxx0[k]], Print[" f''(0) = ", fxx0[k]/N ]];
If[IntegerQ[k/5] && PRN == 1,
  time[k] = SessionTime[];
  temp = time[k]-time[0];
  Print["Used CPU time = ", temp, " (seconds) "];
];
];
Print["successful "];
];

(*****
(* Define physical and control parameters *)
(*****
OHAM = 0;
Nstep = Infinity;
PRN = 1;
c0 = -7/5;
c1 = 0;
c2 = 0;

(* Gain the 10th-order homotopy-approximation *)
ham[1, 10];

```

Problems

3.1. Effect of convergence-control parameter in initial approximation

How to find the optimal homotopy-approximations if one uses the initial approximation $u_0(\xi)$ defined by (3.52) and regards μ as one of convergence-control parameters? How to introduce more such kind of convergence-control parameters in initial approximations?

3.2. Combination of the optimal HAM with iteration

How to combine the optimal HAMs with the iteration approach described in Chapter 2 so as to further accelerate the convergence of series solution of a nonlinear differential equation?

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Chapter 4

Systematic Descriptions and Related Theorems

Abstract In this chapter, the homotopy analysis method (HAM) is systematically described in details as a whole. Mathematical theorems related to the so-called homotopy-derivative operator and deformation equations are proved, which are helpful to gain high-order approximations. Some theorems of convergence are proved, and the methods to control and accelerate convergence are generally described. A few of open questions are discussed.

4.1 Brief frame of the homotopy analysis method

In Chapter 2, the basic ideas of the homotopy analysis method (HAM) (Liao, 1992, 1997, 1999a,b; Liao and Campo, 2002; Liao, 2003a,b, 2004, 2005, 2006, 2009a,b, 2010a,b; Liao and Tan, 2007; Li et al., 2010; Xu et al., 2010) are described by means of two simple examples. In this chapter, we systematically describe the HAM in a general way.

The starting-point of the homotopy analysis method is to construct the so-called zeroth-order deformation equation. Given an original nonlinear equation

$$\mathcal{N}[u(\mathbf{x}, t)] = 0$$

denoted by $\mathcal{E}_1$, which has at least one solution $u(\mathbf{x}, t)$, where $\mathcal{N}$ denotes a nonlinear operator, $\mathbf{x}$ is a vector of all spatial independent-variables, t denotes the temporal independent-variable, respectively. Assume that we can choose an initial equation $\mathcal{E}_0$ whose solution $u_0(\mathbf{x}, t)$ is easy to know, and that we can construct such a homotopy (Armstrong, 1983; Sen, 1983) of equations $\tilde{\mathcal{E}}(q) : \mathcal{E}_0 \sim \mathcal{E}_1$ that, as the homotopy-parameter $q \in [0, 1]$ increases from 0 to 1, $\tilde{\mathcal{E}}(q)$ deforms (or varies) *continuously* from the initial equation $\mathcal{E}_0$ to the original equation $\mathcal{E}_1$, while its solution exists for $q \in [0, 1]$ and besides varies *continuously* from the known solution $u_0(\mathbf{x}, t)$ of the initial equation $\mathcal{E}_0$ to the unknown solution $u(\mathbf{x}, t)$ of the original equation $\mathcal{E}_1$, i.e. $\mathcal{N}[u(\mathbf{x}, t)] = 0$. Such kind of homotopy of equations is called *the zeroth-order*

deformation equation, whose definition is given in Chapter 2. Given an original nonlinear equation $\mathcal{E}_1$, we have extremely large freedom to construct many different zeroth-order deformation equations, as pointed out in Chapter 2. In essence, it is such kind of freedom that differs the HAM from other analytic approximation techniques (Cole, 1992; Hilton, 1953; Hinch, 1991; Murdock, 1991; Nayfeh, 1973, 2000).

Assume that, for a given original equation $\mathcal{N}[u(\mathbf{x}, t)] = 0$, a zeroth-order deformation equation is constructed so properly that the solution $\phi(\mathbf{x}, t; q)$ exists in $q \in [0, 1]$ and is *analytic* at $q = 0$, therefore its Maclaurin series about the homotopy-parameter q exists. At $q = 0$, according to the definition of the zeroth-order deformation equation, we have the auxiliary equation $\mathcal{E}_0$, whose solution $u_0(\mathbf{x}, t)$ is easy to known, i.e.

$$\phi(\mathbf{x}, t; 0) = u_0(\mathbf{x}, t). \quad (4.1)$$

At $q = 1$, $\tilde{\mathcal{E}}(q)$ is equivalent to the original equation $\mathcal{N}[u(\mathbf{x}, t)] = 0$ so that we have

$$\phi(\mathbf{x}, t; 1) = u(\mathbf{x}, t). \quad (4.2)$$

Using (4.1), the Maclaurin series of $\phi(\mathbf{x}, t; q)$ with respect to q reads

$$\phi(\mathbf{x}, t; q) \sim u_0(\mathbf{x}, t) + \sum_{k=1}^{+\infty} u_k(\mathbf{x}, t) q^k, \quad (4.3)$$

where

$$u_k(\mathbf{x}, t) = \frac{1}{k!} \left. \frac{\partial^k \phi(\mathbf{x}, t; q)}{\partial q^k} \right|_{q=0} = \mathcal{D}_k[\phi(\mathbf{x}, t; q)].$$

Here, (4.3) is called the homotopy-Maclaurin series of $\phi(\mathbf{x}, t; q)$, $\mathcal{D}_k[\phi(\mathbf{x}, t; q)]$ is called the k th-order homotopy-derivative of $\phi(\mathbf{x}, t; q)$, $\mathcal{D}_k$ is called the k th-order homotopy-derivative operator, respectively. Especially, we have at $q = 1$ the homotopy-series

$$\phi(\mathbf{x}, t; 1) \sim u_0(\mathbf{x}, t) + \sum_{k=1}^{+\infty} u_k(\mathbf{x}, t). \quad (4.4)$$

If the above homotopy-series is convergent to $\phi(\mathbf{x}, t; 1)$, then, according to (4.2), we have the homotopy-series solution

$$u(\mathbf{x}, t) = u_0(\mathbf{x}, t) + \sum_{k=1}^{+\infty} u_k(\mathbf{x}, t). \quad (4.5)$$

In practice, only finite terms can be obtained, which give us the M th-order homotopy-approximation

$$u(\mathbf{x}, t) \approx u_0(\mathbf{x}, t) + \sum_{k=1}^M u_k(\mathbf{x}, t). \quad (4.6)$$

Note that the unknown term $u_k(\mathbf{x}, t)$ is governed by the so-called high-order deformation equation, which is completely determined by the zeroth-order deforma-

tion equation. In Sect. 4.3, different types of zeroth-order deformation equations and their high-order deformation equations are given. All of these high-order deformation equations are linear with respect to the unknown $u_k(\mathbf{x}, t)$, and thus are easy to solve by mean of computer algebra system such as Mathematica, Maple and so on. Besides, in Sect. 4.2, some theorems for the operator $\mathcal{D}_k$ of the homotopy-derivative are proved. Using these theorems, it is easy to gain the corresponding high-order deformation equations of different types of zeroth-order deformation equations. Furthermore, in Sect. 4.4, it is proved that all homotopy-series satisfy the original equation and thus are homotopy-series solution, as long as the zeroth-order deformation equation is so properly constructed that its solution exists and is analytic for q in the whole domain $q \in [0, 1]$.

Therefore, the key point of the HAM is to construct a proper zeroth-order deformation equation so that its solution exists and is analytic in the domain $q \in [0, 1]$.

4.2 Properties of homotopy-derivative

As mentioned in Chapter 2, the so-called *homotopy-derivative* is used in the frame of the HAM. Here, we first give a rigorous definition of the homotopy-derivative and then prove some of its properties in general. By means of these properties, it is easy to deduce the corresponding high-order deformation equations of any a given zeroth-order deformation equation.

Definition 4.1. Let ϕ be a function of the homotopy-parameter q , then

$$\mathcal{D}_m(\phi) = \frac{1}{m!} \frac{d^m \phi}{dq^m} \Big|_{q=0} \quad (4.7)$$

is called *the m th-order homotopy-derivative* of ϕ , where $m \geq 0$ is an integer, and $\mathcal{D}_m$ is called *the operator of the m th-order homotopy-derivative*.

Theorem 4.1. For two arbitrary homotopy-Maclaurin series

$$\phi = \sum_{k=0}^{+\infty} u_k q^k, \quad \psi = \sum_{k=0}^{+\infty} w_k q^k,$$

where ϕ and ψ are analytic in $q \in [0, a)$, it holds

$$(a) \quad \mathcal{D}_m(\phi) = u_m, \quad (4.8)$$

$$(b) \quad \mathcal{D}_m(q^k \phi) = \mathcal{D}_{m-k}(\phi) = \begin{cases} u_{m-k}, & \text{when } 0 \leq k \leq m, \\ 0, & \text{otherwise,} \end{cases} \quad (4.9)$$

$$(c) \quad \mathcal{D}_m(\phi \psi) = \sum_{k=0}^m \mathcal{D}_k(\phi) \mathcal{D}_{m-k}(\psi) = \sum_{k=0}^m u_k w_{m-k}$$

$$= \sum_{k=0}^m \mathcal{D}_{m-k}(\phi) \mathcal{D}_k(\psi) = \sum_{k=0}^m u_{m-k} w_k, \quad (4.10)$$

$$(d) \quad \mathcal{D}_m(\phi^{n+1}) = \sum_{k=0}^m \mathcal{D}_k(\phi) \mathcal{D}_{m-k}(\phi^n) = \sum_{k=0}^m \mathcal{D}_{m-k}(\phi) \mathcal{D}_k(\phi^n), \quad (4.11)$$

where $m \geq 0$, $n \geq 1$, and $0 \leq k \leq m$ are integers.

Proof. (a) According to Taylor theorem (Fitzpatrick, 1996), the unique coefficient u_m of the Maclaurin series $\phi = \sum_{m=0}^{+\infty} u_m q^m$ with respect to the homotopy-parameter q is given by

$$u_m = \frac{1}{m!} \left. \frac{\partial^m \phi}{\partial q^m} \right|_{q=0},$$

which gives (4.8) by means of the definition (4.7) of $\mathcal{D}_m(\phi)$.

(b) It holds

$$q^k \phi = q^k \sum_{j=0}^{+\infty} u_j q^j = \sum_{j=0}^{+\infty} u_j q^{j+k} = \sum_{m=k}^{+\infty} u_{m-k} q^m,$$

which gives by means of (4.8) that

$$\mathcal{D}_m(q^k \phi) = u_{m-k} = \mathcal{D}_{m-k}(\phi), \quad \text{when } 0 \leq k \leq m,$$

and

$$\mathcal{D}_m(q^k \phi) = 0, \quad \text{when } k > m.$$

(c) According to Leibnitz's rule for derivatives of product, it holds

$$\frac{\partial^m(\phi \psi)}{\partial q^m} = \sum_{k=0}^m \frac{m!}{k!(m-k)!} \frac{\partial^k \phi}{\partial q^k} \frac{\partial^{m-k} \psi}{\partial q^{m-k}} = \sum_{k=0}^m \frac{m!}{k!(m-k)!} \frac{\partial^k \psi}{\partial q^k} \frac{\partial^{m-k} \phi}{\partial q^{m-k}},$$

which gives according to (4.7) and (4.8) that

$$\begin{aligned} \mathcal{D}_m(\phi \psi) &= \frac{1}{m!} \left. \frac{\partial^m(\phi \psi)}{\partial q^m} \right|_{q=0} = \sum_{k=0}^m \left(\left. \frac{1}{k!} \frac{\partial^k \phi}{\partial q^k} \right|_{q=0} \right) \left[\left. \frac{1}{(m-k)!} \frac{\partial^{m-k} \psi}{\partial q^{m-k}} \right|_{q=0} \right] \\ &= \sum_{k=0}^m \mathcal{D}_k(\phi) \mathcal{D}_{m-k}(\psi) = \sum_{k=0}^m u_k w_{m-k}. \end{aligned}$$

Similarly, it holds

$$\mathcal{D}_m(\phi \psi) = \sum_{k=0}^m \mathcal{D}_k(\psi) \mathcal{D}_{m-k}(\phi) = \sum_{k=0}^m u_{m-k} w_k.$$

(d) Write $\psi = \phi^n$. According to (4.10), it holds

$$\mathcal{D}_m(\phi^{n+1}) = \mathcal{D}_m(\phi\psi) = \sum_{k=0}^m \mathcal{D}_k(\phi) \mathcal{D}_{m-k}(\psi) = \sum_{k=0}^m \mathcal{D}_k(\phi) \mathcal{D}_{m-k}(\phi^n).$$

Similarly, it holds

$$\mathcal{D}_m(\phi^{n+1}) = \sum_{k=0}^m \mathcal{D}_k(\phi^n) \mathcal{D}_{m-k}(\phi).$$

This ends the proof. $\square$

Theorem 4.2. *If $\phi = \sum_{k=0}^{+\infty} u_k q^k$ and $\psi = \sum_{k=0}^{+\infty} w_k q^k$ are two homotopy-Maclaurin series, where ϕ and ψ are analytic in $q \in [0, a)$, f and g are independent of the homotopy-parameter $q \in [0, 1]$, then it holds*

$$\mathcal{D}_m(f\phi + g\psi) = f\mathcal{D}_m(\phi) + g\mathcal{D}_m(\psi) = fu_m + gw_m. \quad (4.12)$$

Proof. Because $\mathcal{D}_m$ defined by (4.7) is a linear operator, and besides f and g are independent of q , it obviously holds

$$\mathcal{D}_m(f\phi + g\psi) = \mathcal{D}_m(f\phi) + \mathcal{D}_m(g\psi) = f\mathcal{D}_m(\phi) + g\mathcal{D}_m(\psi).$$

According to (4.8), we have $\mathcal{D}_m(\phi) = u_m$ and $\mathcal{D}_m(\psi) = w_m$. This ends the proof. $\square$

Theorem 4.3. *Let $\mathcal{L}$ denote a linear operator independent of the homotopy-parameter $q \in [0, 1]$. For two homotopy-Maclaurin series*

$$\phi = \sum_{k=0}^{+\infty} u_k q^k, \quad \psi = \sum_{k=0}^{+\infty} w_k q^k,$$

where ϕ and ψ are analytic in $q \in [0, a)$, it holds

$$\mathcal{D}_m[\mathcal{L}(\phi)] = \mathcal{L}[\mathcal{D}_m(\phi)] = \mathcal{L}(u_m), \quad (4.13)$$

and

$$\mathcal{D}_m[\psi \mathcal{L}(\phi)] = \sum_{n=0}^m \mathcal{D}_{m-n}(\psi) \mathcal{L}[\mathcal{D}_n(\phi)] = \sum_{n=0}^m w_{m-n} \mathcal{L}(u_n). \quad (4.14)$$

where $m \geq 0$ is an integer.

Proof. Since $\mathcal{L}$ is linear and independent of q , using Theorem 4.2, we have

$$\mathcal{L}(\phi) = \mathcal{L}\left(\sum_{k=0}^{+\infty} u_k q^k\right) = \sum_{k=0}^{+\infty} \mathcal{L}(u_k) q^k.$$

Using the statement (4.8), one has $\mathcal{D}_m[\mathcal{L}(\phi)] = \mathcal{L}(u_m)$. On the other side, according to statement (4.8), it holds $\mathcal{L}[\mathcal{D}_m(\phi)] = \mathcal{L}(u_m)$. Thus,

$$\mathcal{D}_m[\mathcal{L}(\phi)] = \mathcal{L}[\mathcal{D}_m(\phi)] = \mathcal{L}(u_m)$$

holds. Then, according to Theorem 4.1, we have

$$\begin{aligned} \mathcal{D}_m[\psi \mathcal{L}(\phi)] &= \sum_{n=0}^m \mathcal{D}_{m-n}(\psi) \mathcal{D}_n[\mathcal{L}(\phi)] \\ &= \sum_{n=0}^m \mathcal{D}_{m-n}(\psi) \mathcal{L}[\mathcal{D}_n(\phi)] = \sum_{n=0}^m w_{m-n} \mathcal{L}(u_n). \end{aligned}$$

This ends the proof. $\square$

Theorem 4.2 and Theorem 4.3 indicate the linear superposition and commutativity property of the operator $\mathcal{D}_m$ defined by (4.7), respectively.

Theorem 4.4. *For two homotopy-Maclaurin series*

$$\phi = \sum_{i=0}^{+\infty} u_i q^i, \quad \psi = \sum_{j=0}^{+\infty} w_j q^j,$$

where ϕ and ψ are analytic in $q \in [0, a)$, if $\phi = \psi$ in $q \in [0, a)$, then $u_m = w_m$ and $\mathcal{D}_m(\phi) = \mathcal{D}_m(\psi)$ for any integer $m \geq 0$ and a real number $a > 0$.

Proof. Since $\phi = \psi$, it holds

$$\sum_{k=0}^{+\infty} (u_k - w_k) q^k = 0.$$

The above expression holds at all points $q \in [0, a)$, if and only if

$$u_m = w_m, \quad m \geq 0,$$

which gives, due to (4.8), that

$$\mathcal{D}_m(\phi) = \mathcal{D}_m(\psi).$$

This ends the proof. $\square$

Theorem 4.5. *Let $f(\phi), g(\psi)$ denote two smooth functions. For two homotopy-Maclaurin series*

$$\phi = \sum_{i=0}^{+\infty} u_i q^i, \quad \psi = \sum_{j=0}^{+\infty} w_j q^j,$$

if $f(\phi) = g(\psi)$ in a domain $q \in [0, a)$, then

$$\mathcal{D}_m[f(\phi)] = \mathcal{D}_m[g(\psi)]$$

for any integer $m \geq 0$ and a real number $a > 0$.

Proof. Write

$$\Phi = f(\phi), \quad \Psi = g(\psi).$$

Then, using Theorem 4.4, we have

$$\mathcal{D}_m(\Phi) = \mathcal{D}_m(\Psi),$$

which gives

$$\mathcal{D}_m[f(\phi)] = \mathcal{D}_m[g(\psi)].$$

This ends the proof. $\square$

Theorem 4.4 and Theorem 4.5 indicate the uniqueness of the homotopy-derivatives of a homotopy-Maclaurin series $\phi = \sum_{k=0}^{+\infty} u_k q^k$ and a function $f(\phi)$. According to the fundamental theorems in calculus (Fitzpatrick, 1996), these theorems are obvious. Therefore, given a zeroth-order deformation equation, its m th-order high-order deformation equations is unique, no matter how one obtains it.

Theorem 4.6. *For an arbitrary homotopy-Maclaurin series $\phi = \sum_{k=0}^{+\infty} u_k q^k$, it holds*

$$(a) \quad \mathcal{D}_m(\phi^2) = \sum_{n=0}^m u_{m-n} u_n, \quad (4.15)$$

$$(b) \quad \mathcal{D}_m(\phi^3) = \sum_{n=0}^m u_{m-n} \sum_{k=0}^n u_{n-k} u_k, \quad (4.16)$$

$$(c) \quad \mathcal{D}_m(\phi^4) = \sum_{n=0}^m u_{m-n} \sum_{k=0}^n u_{n-k} \sum_{j=0}^k u_{k-j} u_j, \quad (4.17)$$

$$(d) \quad \mathcal{D}_m(\phi^5) = \sum_{n=0}^m u_{m-n} \sum_{k=0}^n u_{n-k} \sum_{j=0}^k u_{k-j} \sum_{i=0}^j u_{j-i} u_i, \quad (4.18)$$

$$(e) \quad \mathcal{D}_m(\phi^\sigma) = \sum_{r_1=0}^m u_{m-r_1} \sum_{r_2=0}^{r_1} u_{r_1-r_2} \sum_{r_3=0}^{r_2} u_{r_2-r_3} \cdots \sum_{r_{\sigma-1}=0}^{r_{\sigma-2}} u_{r_{\sigma-2}-r_{\sigma-1}} u_{r_{\sigma-1}}, \quad (4.19)$$

where $m \geq 0$ and $\sigma \geq 2$ are positive integer.

Proof. (a) According to (4.10) and (4.8), it holds

$$\mathcal{D}_m(\phi^2) = \sum_{n=0}^m \mathcal{D}_{m-n}(\phi) \mathcal{D}_n(\phi) = \sum_{n=0}^m u_{m-n} u_n.$$

(b) According to (4.15), we have

$$\mathcal{D}_n(\phi^2) = \sum_{k=0}^n u_{n-k} u_k.$$

Then, according to (4.10), it holds

$$\mathcal{D}_m(\phi^3) = \sum_{n=0}^m \mathcal{D}_{m-n}(\phi) \mathcal{D}_n(\phi^2) = \sum_{n=0}^m u_{m-n} \sum_{k=0}^n u_{n-k} u_k.$$

(c) According to (4.16), we have

$$\mathcal{D}_n(\phi^3) = \sum_{k=0}^n u_{n-k} \sum_{j=0}^k u_{k-j} u_j.$$

Then, according to (4.10), it holds

$$\mathcal{D}_m(\phi^4) = \sum_{n=0}^m \mathcal{D}_{m-n}(\phi) \mathcal{D}_n(\phi^3) = \sum_{n=0}^m u_{m-n} \sum_{k=0}^n u_{n-k} \sum_{j=0}^k u_{k-j} u_j.$$

(d) According to (4.17), we have

$$\mathcal{D}_n(\phi^4) = \sum_{k=0}^n u_{n-k} \sum_{j=0}^k u_{k-j} \sum_{i=0}^j u_{j-i} u_i.$$

Then, according to (4.10), it holds

$$\mathcal{D}_m(\phi^5) = \sum_{n=0}^m \mathcal{D}_{m-n}(\phi) \mathcal{D}_n(\phi^4) = \sum_{n=0}^m u_{m-n} \sum_{k=0}^n u_{n-k} \sum_{j=0}^k u_{k-j} \sum_{i=0}^j u_{j-i} u_i.$$

(e) This statement can be proved by the method of mathematical induction.

(i) According to (4.15), it is obvious that (4.19) holds when $\sigma = 2$.

(ii) Assume that the statement (4.19) holds for $\sigma = \kappa$, i.e.

$$\mathcal{D}_m(\phi^\kappa) = \sum_{r_1=0}^m u_{m-r_1} \sum_{r_2=0}^{r_1} u_{r_1-r_2} \sum_{r_3=0}^{r_2} u_{r_2-r_3} \cdots \sum_{r_{\kappa-1}=0}^{r_{\kappa-2}} u_{r_{\kappa-2}-r_{\kappa-1}} u_{r_{\kappa-1}},$$

where $m \geq 0$ and $\kappa \geq 2$ are integers. Replacing r_j by r'_{j+1} and m by r'_1 , the above expression reads

$$\mathcal{D}_{r'_1}(\phi^\kappa) = \sum_{r'_2=0}^{r'_1} u_{r'_1-r'_2} \sum_{r'_3=0}^{r'_2} u_{r'_2-r'_3} \sum_{r'_4=0}^{r'_3} u_{r'_3-r'_4} \cdots \sum_{r'_\kappa=0}^{r'_{\kappa-1}} u_{r'_{\kappa-1}-r'_\kappa} u_{r'_\kappa}.$$

Using the above expression and by means of (4.11) and (4.8), it holds

$$\begin{aligned} \mathcal{D}_m(\phi^{\kappa+1}) &= \sum_{r'_1=0}^m \mathcal{D}_{m-r'_1}(\phi) \mathcal{D}_{r'_1}(\phi^\kappa) \\ &= \sum_{r'_1=0}^m u_{m-r'_1} \sum_{r'_2=0}^{r'_1} u_{r'_1-r'_2} \sum_{r'_3=0}^{r'_2} u_{r'_2-r'_3} \sum_{r'_4=0}^{r'_3} u_{r'_3-r'_4} \cdots \sum_{r'_\kappa=0}^{r'_{\kappa-1}} u_{r'_{\kappa-1}-r'_\kappa} u_{r'_\kappa}. \end{aligned}$$

Therefore, (4.19) holds for $\sigma = \kappa + 1$.

(iii) According to (i) and (ii), the statement (4.19) holds for any positive integer $\sigma \geq 2$. This ends the proof. $\square$

Theorem 4.7. *For a homotopy-Maclaurin series*

$$\phi = \sum_{k=0}^{+\infty} u_k q^k,$$

it holds the recursion formulas

$$\begin{aligned} \mathcal{D}_0(e^{\alpha\phi}) &= e^{\alpha u_0}, \\ \mathcal{D}_m(e^{\alpha\phi}) &= \alpha \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) \mathcal{D}_{m-k}(\phi) \mathcal{D}_k(e^{\alpha\phi}) \\ &= \alpha \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) u_{m-k} \mathcal{D}_k(e^{\alpha\phi}), \end{aligned}$$

where $m \geq 1$ is an integer, and $\alpha \neq 0$ is independent of the homotopy-parameter q .

Proof. According to the definition (4.7) of the operator $\mathcal{D}_m$, it holds obviously

$$\mathcal{D}_0(e^{\alpha\phi}) = e^{\alpha u_0}.$$

Besides, since α is independent of q , one has

$$\frac{\partial e^{\alpha\phi}}{\partial q} = \alpha e^{\alpha\phi} \frac{\partial \phi}{\partial q}.$$

Thus, according to Leibnitz's rule for derivatives of product, it holds

$$\begin{aligned} \frac{1}{m!} \frac{\partial^m e^{\alpha\phi}}{\partial q^m} &= \frac{1}{m!} \frac{\partial^{m-1}}{\partial q^{m-1}} \left(\alpha e^{\alpha\phi} \frac{\partial \phi}{\partial q} \right) = \frac{\alpha}{m} \sum_{k=0}^{m-1} \frac{1}{k!(m-1-k)!} \frac{\partial^k e^{\alpha\phi}}{\partial q^k} \frac{\partial^{m-k}\phi}{\partial q^{m-k}} \\ &= \alpha \sum_{k=0}^{m-1} \frac{(m-k)}{m} \left[\frac{1}{(m-k)!} \frac{\partial^{m-k}\phi}{\partial q^{m-k}} \right] \left(\frac{1}{k!} \frac{\partial^k e^{\alpha\phi}}{\partial q^k} \right). \end{aligned}$$

Setting $q = 0$ in above expression and using the definition (4.7) and the statement (4.8), one has

$$\mathcal{D}_m(e^{\alpha\phi}) = \alpha \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) \mathcal{D}_{m-k}(\phi) \mathcal{D}_k(e^{\alpha\phi}) = \alpha \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) u_{m-k} \mathcal{D}_k(e^{\alpha\phi}),$$

where $m \geq 1$ is an integer. This ends the proof. $\square$

Theorem 4.8. *For a homotopy-Maclaurin series*

$$\phi = \sum_{k=0}^{+\infty} u_k q^k,$$

it holds the recursion formulas

$$\begin{aligned} \mathcal{D}_0(\sin \phi) &= \sin u_0, \quad \mathcal{D}_0(\cos \phi) = \cos u_0, \\ \mathcal{D}_m(\sin \phi) &= \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) \mathcal{D}_{m-k}(\phi) \mathcal{D}_k(\cos \phi) \\ &= \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) u_{m-k} \mathcal{D}_k(\cos \phi), \\ \mathcal{D}_m(\cos \phi) &= - \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) \mathcal{D}_{m-k}(\phi) \mathcal{D}_k(\sin \phi) \\ &= - \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) u_{m-k} \mathcal{D}_k(\sin \phi), \end{aligned}$$

where $m \geq 1$ is an integer.

Proof. According to the definition (4.7), it holds obviously

$$\mathcal{D}_0(\sin \phi) = \sin u_0, \quad \mathcal{D}_0(\cos \phi) = \cos u_0.$$

Write $i = \sqrt{-1}$. Using Euler formula and Theorem 4.2, it holds for an integer $m \geq 1$ that

$$\mathcal{D}_m(\sin \phi) = \mathcal{D}_m\left(\frac{e^{i\phi} - e^{-i\phi}}{2i}\right) = \frac{1}{2i} [\mathcal{D}_m(e^{i\phi}) - \mathcal{D}_m(e^{-i\phi})] \quad (4.20)$$

and

$$\mathcal{D}_m(\cos \phi) = \mathcal{D}_m\left(\frac{e^{i\phi} + e^{-i\phi}}{2}\right) = \frac{1}{2} [\mathcal{D}_m(e^{i\phi}) + \mathcal{D}_m(e^{-i\phi})]. \quad (4.21)$$

Using Theorem 4.7 and then Theorem 4.2, we have

$$\begin{aligned} \mathcal{D}_m(e^{i\phi}) &= \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) \mathcal{D}_k(e^{i\phi}) \mathcal{D}_{m-k}(i\phi) \\ &= i \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) \mathcal{D}_k(e^{i\phi}) \mathcal{D}_{m-k}(\phi) \end{aligned}$$

and similarly,

$$\mathcal{D}_m(e^{-i\phi}) = -i \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) \mathcal{D}_k(e^{-i\phi}) \mathcal{D}_{m-k}(\phi).$$

Substituting the above two expressions into (4.20) and (4.21), then using Theorem 4.2 and Euler formula, we have

$$\begin{aligned}
 \mathcal{D}_m(\sin \phi) &= \frac{1}{2} \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) \mathcal{D}_{m-k}(\phi) [\mathcal{D}_k(e^{i\phi}) + \mathcal{D}_k(e^{-i\phi})] \\
 &= \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) \mathcal{D}_{m-k}(\phi) \mathcal{D}_k \left(\frac{e^{i\phi} + e^{-i\phi}}{2} \right) \\
 &= \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) \mathcal{D}_{m-k}(\phi) \mathcal{D}_k(\cos \phi) \\
 &= \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) u_{m-k} \mathcal{D}_k(\cos \phi),
 \end{aligned}$$

and similarly

$$\begin{aligned}
 \mathcal{D}_m(\cos \phi) &= \frac{i}{2} \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) \mathcal{D}_{m-k}(\phi) [\mathcal{D}_k(e^{i\phi}) - \mathcal{D}_k(e^{-i\phi})] \\
 &= - \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) \mathcal{D}_{m-k}(\phi) \mathcal{D}_k \left(\frac{e^{i\phi} - e^{-i\phi}}{2i} \right) \\
 &= - \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) \mathcal{D}_{m-k}(\phi) \mathcal{D}_k(\sin \phi) \\
 &= - \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) u_{m-k} \mathcal{D}_k(\sin \phi).
 \end{aligned}$$

This ends the proof. $\square$

The statements (4.11), (4.15) and (4.19) were proved by Molabahrani and Khani (2009). Most of the others were proved by Liao (2009a). Using these theorems, one can calculate the homotopy-derivatives of any a given smooth function of a homotopy-Maclaurin series. For example, given a homotopy-Maclaurin series

$\phi = \sum_{k=0}^{+\infty} u_k q^k$, we have using Theorem 4.2 and the statement (4.10) that

$$\mathcal{D}_m \left(3\phi^2 + 4e^{-5\phi} \sin \phi \right) = 3 \sum_{k=0}^m u_k u_{m-k} + 4 \sum_{k=0}^m \mathcal{D}_k \left(e^{-5\phi} \right) \mathcal{D}_{m-k}(\sin \phi),$$

where $\mathcal{D}_k(e^{-5\phi})$ and $\mathcal{D}_{m-k}(\sin \phi)$ are given by Theorem 4.7 and Theorem 4.8, respectively. Following Molabahrani and Khani (2009) and Liao (2009a), Turkyilmazoglu (2010) proved the following theorem:

Theorem 4.9. *Define an operator*

$$\check{\mathcal{D}}_m(\phi) = \frac{1}{m!} \frac{\partial^m \phi}{\partial q^m}.$$

For a smooth function $f \in C^\infty(a, b)$ and a homotopy-Maclaurin series

$$\phi = \sum_{k=0}^{+\infty} u_k q^k,$$

it holds

$$\check{\mathcal{D}}_0[f(\phi)] = f(\phi), \quad (4.22)$$

$$\check{\mathcal{D}}_m[f(\phi)] = \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) \check{\mathcal{D}}_{m-k}(\phi) \frac{\partial}{\partial \phi} \{ \check{\mathcal{D}}_k[f(\phi)] \}, \quad (4.23)$$

and

$$\mathcal{D}_m[f(\phi)] = \{ \check{\mathcal{D}}_m[f(\phi)] \} \big|_{q=0}. \quad (4.24)$$

Proof. It is obvious that $\check{\mathcal{D}}_0[f(\phi)] = f(\phi)$ holds. In case of $m \geq 1$, we have by Leibnitz's rule for derivatives of product that

$$\begin{aligned} \check{\mathcal{D}}_m[f(\phi)] &= \frac{1}{m!} \frac{\partial^m f(\phi)}{\partial q^m} = \frac{1}{m!} \frac{\partial^{m-1}}{\partial q^{m-1}} \left[\frac{\partial \phi}{\partial q} \frac{\partial f(\phi)}{\partial \phi} \right] \\ &= \frac{1}{m!} \sum_{k=0}^{m-1} \frac{(m-1)!}{k! (m-1-k)!} \frac{\partial^{m-1-k}}{\partial q^{m-1-k}} \left(\frac{\partial \phi}{\partial q} \right) \frac{\partial^k}{\partial q^k} \left[\frac{\partial f(\phi)}{\partial \phi} \right] \\ &= \sum_{k=0}^{m-1} \frac{(m-k)}{m} \left[\frac{1}{(m-k)!} \frac{\partial^{m-k} \phi}{\partial q^{m-k}} \right] \left\{ \frac{1}{k!} \frac{\partial^k}{\partial q^k} \left[\frac{\partial f(\phi)}{\partial \phi} \right] \right\} \\ &= \sum_{k=0}^{m-1} \left(1 - \frac{k}{m} \right) \check{\mathcal{D}}_{m-k}(\phi) \check{\mathcal{D}}_k \left[\frac{\partial f(\phi)}{\partial \phi} \right]. \end{aligned} \quad (4.25)$$

Since

$$\check{\mathcal{D}}_k \left[\frac{\partial f(\phi)}{\partial \phi} \right] = \frac{\partial}{\partial \phi} \{ \check{\mathcal{D}}_k[f(\phi)] \},$$

we have

$$\check{\mathcal{D}}_m[f(\phi)] = \sum_{k=0}^{m-1} \left(1 - \frac{k}{m} \right) \check{\mathcal{D}}_{m-k}(\phi) \frac{\partial}{\partial \phi} \{ \check{\mathcal{D}}_k[f(\phi)] \}$$

for $m \geq 1$. Then, according to the definition of $\check{\mathcal{D}}_m$, it obviously holds

$$\mathcal{D}_m[f(\phi)] = \{ \check{\mathcal{D}}_m[f(\phi)] \} \big|_{q=0}.$$

This ends the proof. $\square$

The above theorem contains Theorem 4.7 and Theorem 4.8, and thus is more general, although it is not straightforward. Using the above theorem, we prove here the following theorem with the explicit expressions:

Theorem 4.10. *For a smooth function $f(u)$ and a homotopy-Maclaurin serie*

$$\phi = \sum_{k=0}^{+\infty} u_k q^k,$$

it holds

$$\mathcal{D}_0[f(\phi)] = f(u_0), \quad (4.26)$$

$$\mathcal{D}_m[f(\phi)] = \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) u_{m-k} \frac{\partial \{\mathcal{D}_k[f(\phi)]\}}{\partial u_0}, \quad (4.27)$$

and

$$\mathcal{D}_m[f(\phi)] = \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) u_{m-k} \mathcal{D}_k[f'(\phi)], \quad (4.28)$$

for $m \geq 1$.

Proof. According to the definition (4.7), it is obvious that the statement (4.32) is true. At $q = 0$, we have

$$\check{\mathcal{D}}_{m-k}(\phi) = \mathcal{D}_{m-k}(\phi) = u_{m-k}.$$

Besides, at $q = 0$, we have $\phi = u_0$ and therefore $\partial/\partial\phi = \partial/\partial u_0$. Then, according to Theorem 4.9, we have

$$\mathcal{D}_m[f(\phi)] = \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) u_{m-k} \frac{\partial \{\mathcal{D}_k[f(\phi)]\}}{\partial u_0}.$$

Setting $q = 0$ in (4.25), we obtain

$$\mathcal{D}_m[f(\phi)] = \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) u_{m-k} \mathcal{D}_k[f'(\phi)].$$

This ends the proof. □

Note that, Theorem 4.7 and Theorem 4.8 are special cases of the statement (4.28) of Theorem 4.10. By means of the recursion formula (4.27) of Theorem 4.10, it is easy to get the high-order homotopy-derivatives of an *arbitrary* smooth function of a given homotopy-Maclaurin series $\phi = \sum_{k=0}^{+\infty} u_k q^k$. For example, it gives when $f(\phi) = \phi^\alpha$ that

$$\mathcal{D}_0(\phi^\alpha) = u_0^\alpha,$$

$$\begin{aligned}
\mathcal{D}_1(\phi^\alpha) &= \alpha u_0^{\alpha-1} u_1, \\
\mathcal{D}_2(\phi^\alpha) &= \frac{1}{2} \alpha (\alpha-1) u_0^{\alpha-2} u_1^2 + \alpha u_0^{\alpha-1} u_2, \\
\mathcal{D}_3(\phi^\alpha) &= \frac{1}{6} \alpha (\alpha-1) (\alpha-2) u_0^{\alpha-3} u_1^3 + \alpha (\alpha-1) u_0^{\alpha-2} u_1 u_2 + \alpha u_0^{\alpha-1} u_3, \\
&\vdots
\end{aligned}$$

where $\alpha \neq 0$ is independent of the homotopy-parameter q . Similarly, it gives when $f(\phi) = \alpha^\phi$ that

$$\begin{aligned}
\mathcal{D}_0(\alpha^\phi) &= \alpha^{u_0}, \\
\mathcal{D}_1(\alpha^\phi) &= (\ln \alpha) \alpha^{u_0} u_1, \\
\mathcal{D}_2(\alpha^\phi) &= \frac{1}{2} (\ln \alpha)^2 \alpha^{u_0} u_1^2 + (\ln \alpha) \alpha^{u_0} u_2, \\
\mathcal{D}_3(\alpha^\phi) &= \frac{1}{6} (\ln \alpha)^3 \alpha^{u_0} u_1^3 + (\ln \alpha)^2 \alpha^{u_0} u_1 u_2 + (\ln \alpha) \alpha^{u_0} u_3, \\
&\vdots
\end{aligned}$$

where $\alpha > 0$ is independent of the homotopy-parameter q . These results are exactly the same as those given by Theorem 4.9. In general, by means of the recursion formula (4.27) of Theorem 4.10, we have for *any* given smooth function $f(\phi)$ that

$$\begin{aligned}
\mathcal{D}_0[f(\phi)] &= f(u_0), \\
\mathcal{D}_1[f(\phi)] &= f'(u_0) u_1, \\
\mathcal{D}_2[f(\phi)] &= \frac{1}{2} f''(u_0) u_1^2 + f'(u_0) u_2, \\
\mathcal{D}_3[f(\phi)] &= \frac{1}{6} f'''(u_0) u_1^3 + f''(u_0) u_1 u_2 + f'(u_0) u_3, \\
\mathcal{D}_4[f(\phi)] &= \frac{1}{24} f''''(u_0) u_1^4 + \frac{1}{2} f'''(u_0) u_1^2 u_2 + f''(u_0) \left(u_1 u_3 + \frac{u_2^2}{2} \right) + f'(u_0) u_4, \\
&\vdots
\end{aligned}$$

In practice, by means of computer algebra system like Mathematica and Maple, it is easy to get $\mathcal{D}_m[f(\phi)]$ for rather large m . For example, using the following Mathematica commands:

```

GetD[0] := f[u[0]];
GetD[m_] := Sum[(1-k/m)*u[m-k]*D[GetD[k], u[0]], {k, 0, m-1}];

```

we can obtain $\mathcal{D}_m[f(\phi)]$ for *any* a given smooth function $f(\phi)$ of a homotopy-Maclaurin series $\phi = \sum_{k=0}^{+\infty} u_k q^k$.

Therefore, for *any* given smooth function $f(\phi)$, where ϕ is a homotopy-Maclaurin series, we can always obtain its m th-order homotopy-derivative $\mathcal{D}_m[f(\phi)]$ by means of above theorems. Besides, the following theorems can be proved in a similar way.

Theorem 4.11. *For a smooth function $f(u, w)$ and two homotopy-Maclaurin serie*

$$\phi = \sum_{k=0}^{+\infty} u_k q^k, \quad \psi = \sum_{k=0}^{+\infty} w_k q^k,$$

it holds

$$\mathcal{D}_0[f(\phi, \psi)] = f(u_0, w_0), \quad (4.29)$$

$$\begin{aligned} \mathcal{D}_m[f(\phi, \psi)] &= \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) u_{m-k} \frac{\partial \{\mathcal{D}_k[f(\phi, \psi)]\}}{\partial u_0} \\ &\quad + \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) w_{m-k} \frac{\partial \{\mathcal{D}_k[f(\phi, \psi)]\}}{\partial w_0}, \end{aligned} \quad (4.30)$$

and

$$\begin{aligned} \mathcal{D}_m[f(\phi, \psi)] &= \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) u_{m-k} \mathcal{D}_k \left[\frac{\partial f(\phi, \psi)}{\partial \phi} \right] \\ &\quad + \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) w_{m-k} \mathcal{D}_k \left[\frac{\partial f(\phi, \psi)}{\partial \psi} \right] \end{aligned} \quad (4.31)$$

for $m \geq 1$.

Theorem 4.12. *For a smooth function $f(u, u', u'')$ and a homotopy-Maclaurin serie*

$$\phi = \sum_{k=0}^{+\infty} u_k(x) q^k$$

with the definition

$$\phi' = \sum_{k=0}^{+\infty} u'_k(x) q^k, \quad \phi'' = \sum_{k=0}^{+\infty} u''_k(x) q^k,$$

where the prime denotes the differentiation with respect to x , it holds

$$\mathcal{D}_0[f(\phi, \phi', \phi'')] = f(u_0, u'_0, u''_0), \quad (4.32)$$

$$\begin{aligned} \mathcal{D}_m[f(\phi, \phi', \phi'')] &= \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) u_{m-k} \frac{\partial \{\mathcal{D}_k[f(\phi, \phi', \phi'')]\}}{\partial u_0} \\ &\quad + \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) u'_{m-k} \frac{\partial \{\mathcal{D}_k[f(\phi, \phi', \phi'')]\}}{\partial u'_0} \end{aligned} \quad (4.33)$$

$$+ \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) u''_{m-k} \frac{\partial \{\mathcal{D}_k[f(\phi, \phi', \phi'')]\}}{\partial u''_0},$$

and

$$\begin{aligned} \mathcal{D}_m[f(\phi, \phi', \phi'')] &= \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) u_{m-k} \mathcal{D}_k \left[\frac{\partial f(\phi, \phi', \phi'')}{\partial \phi} \right] \\ &+ \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) u'_{m-k} \mathcal{D}_k \left[\frac{\partial f(\phi, \phi', \phi'')}{\partial \phi'} \right] \\ &+ \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) u''_{m-k} \mathcal{D}_k \left[\frac{\partial f(\phi, \phi', \phi'')}{\partial \phi''} \right] \end{aligned} \quad (4.34)$$

for $m \geq 1$.

Note that the above theorems do not hold for functions which explicitly contain the homotopy-parameter, such as $f(\phi, \psi, q)$. For such kind of complicated functions, it is efficient to directly expand it into a homotopy-Maclaurin series and then use the following theorem.

Theorem 4.13. Let $\mathcal{N}(\phi, \psi, q)$ denote a nonlinear operator, where $q \in [0, 1]$ is the homotopy-parameter,

$$\phi \sim \sum_{m=0}^{+\infty} u_m q^m, \quad \psi \sim \sum_{n=0}^{+\infty} w_n q^n,$$

are two homotopy-Maclaurin series, respectively, which are analytic in $q \in [0, a]$ for $a > 0$. Let

$$\mathcal{N}(\phi, \psi, q) \sim \sum_{k=0}^{+\infty} \delta_k q^k$$

denote the homotopy-Maclaurin series of $\mathcal{N}(\phi, \psi, q)$. Then, it holds

$$\mathcal{D}_k[\mathcal{N}(\phi, \psi, q)] = \delta_k = \left[\frac{1}{k!} \frac{d^k}{dq^k} \mathcal{N} \left(\sum_{m=0}^{+\infty} u_m q^m, \sum_{n=0}^{+\infty} w_n q^n, q \right) \right] \Big|_{q=0}. \quad (4.35)$$

Proof. The statement (4.35) holds obviously, according to the definition (4.7). $\square$

By means of Theorem 4.13, we can get homotopy-derivative of any a given nonlinear operator. For example, let $\phi \sim \sum_{n=0}^{+\infty} u_n q^n$ denote a homotopy-Maclaurin series. By means of computer algebra system such as Mathematica, it is easy to obtain the homotopy-Maclaurin series of $\sin(q\phi)/q$, i.e.

$$\frac{\sin(q\phi)}{q} \sim u_0 + u_1 q + \left(u_2 - \frac{1}{6}u_0^3\right) q^2 + \left(u_3 - \frac{1}{2}u_0^2 u_1\right) q^3$$

$$+ \left(u_4 - \frac{1}{2} u_0^2 u_2 - \frac{1}{2} u_0 u_1^2 + \frac{1}{120} u_0^5 \right) q^4 + \dots$$

Thus, we have

$$\mathcal{D}_0 \left[\frac{\sin(q\phi)}{q} \right] = u_0, \quad \mathcal{D}_1 \left[\frac{\sin(q\phi)}{q} \right] = u_1, \quad \mathcal{D}_2 \left[\frac{\sin(q\phi)}{q} \right] = u_2 - \frac{1}{6} u_0^3, \quad (4.36)$$

and so on. For details, please refer to Liao (2009b).

Using above-mentioned theorems, we can derive explicit formulas of homotopy-derivative of many complicated functions. For example, the alternative way to get $\mathcal{D}_m[\sin(q\phi)/q]$ is given by the following theorem.

Theorem 4.14. *For a homotopy-Maclaurin series*

$$\phi = \sum_{k=0}^{+\infty} u_k q^k,$$

where ϕ is analytic in $q \in [0, a)$ for $a > 0$, it holds

$$\mathcal{D}_m \left[\frac{\sin(q\phi)}{q} \right] = \mathcal{D}_{m+1} [\sin(q\phi)], \quad m \geq 0,$$

where

$$\begin{aligned} \mathcal{D}_0 [\sin(q\phi)] &= 0, \quad \mathcal{D}_0 [\cos(q\phi)] = 1, \\ \mathcal{D}_n [\sin(q\phi)] &= \sum_{k=0}^{n-1} \left(1 - \frac{k}{n} \right) \mathcal{D}_{n-1-k}(\phi) \mathcal{D}_k [\cos(q\phi)] \\ &= \sum_{k=0}^{n-1} \left(1 - \frac{k}{n} \right) u_{n-1-k} \mathcal{D}_k [\cos(q\phi)], \\ \mathcal{D}_n [\cos(q\phi)] &= - \sum_{k=0}^{n-1} \left(1 - \frac{k}{n} \right) \mathcal{D}_{n-1-k}(\phi) \mathcal{D}_k [\sin(q\phi)] \\ &= - \sum_{k=0}^{n-1} \left(1 - \frac{k}{n} \right) u_{n-1-k} \mathcal{D}_k [\sin(q\phi)] \end{aligned}$$

for $n \geq 1$.

Proof. According to the definition (4.7), it holds

$$\mathcal{D}_0 [\sin(q\phi)] = \sin 0 = 0, \quad \mathcal{D}_0 [\cos(q\phi)] = \cos 0 = 1.$$

According to the definition (4.7), we have

$$\sin(q\phi) = \sum_{k=1}^{+\infty} \mathcal{D}_k [\sin(q\phi)] q^k,$$

which gives

$$\frac{\sin(q\phi)}{q} = \sum_{k=1}^{+\infty} \mathcal{D}_k[\sin(q\phi)] q^{k-1} = \sum_{m=0}^{+\infty} \mathcal{D}_{m+1}[\sin(q\phi)] q^m$$

so that

$$\mathcal{D}_m \left[\frac{\sin(q\phi)}{q} \right] = \mathcal{D}_{m+1}[\sin(q\phi)].$$

According to Theorem 4.8, it holds

$$\mathcal{D}_n[\sin(q\phi)] = \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) \mathcal{D}_{n-k}(q\phi) \mathcal{D}_k[\cos(q\phi)]$$

which gives according to Theorem 4.1 that

$$\begin{aligned} \mathcal{D}_n[\sin(q\phi)] &= \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) \mathcal{D}_{n-1-k}(\phi) \mathcal{D}_k[\cos(q\phi)] \\ &= \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) u_{n-1-k} \mathcal{D}_k[\cos(q\phi)] \end{aligned}$$

for $n \geq 1$. Similarly, it holds

$$\mathcal{D}_n[\cos(q\phi)] = - \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) u_{n-1-k} \mathcal{D}_k[\sin(q\phi)], \quad n \geq 1.$$

This ends the proof. $\square$

Using Theorem 4.14, we gain exactly the same results as (4.36). Using Mathematica, it is easy to gain $\mathcal{D}_n[\sin(q\phi)/q]$, where $0 \leq n \leq 4$, by the following commands

```
Dsin[0] = 0;
Dcos[0] = 1;
Dsin[n_] := Sum[(1 - k/n)*u[n - 1 - k]*Dcos[k], {k, 0, n-1}];
Dcos[n_] := -Sum[(1 - k/n)*u[n - 1 - k]*Dsin[k], {k, 0, n-1}];
For[k = 0, k <= 4, k = k + 1, Dsin[k+1] // Print]
```

4.3 Deformation equations

4.3.1 A brief history

Nonlinear differential equations are much more difficult to solve than linear ones. Perturbation techniques are based on the expansion of small/large physical parameters, called perturbation quantity. Unfortunately, many nonlinear problems have no

such kind of small/large physical parameters. More importantly, perturbation techniques can not guarantee the convergence of approximation series. To overcome the dependence of perturbation techniques on the small/large physical parameters, the so-called “artificial small parameter” (Lyapunov, 1992) (denoted here by ε) was introduced in some different ways: when $\varepsilon = 0$, one has a much simpler equation whose solution is easy to obtain; when $\varepsilon = 1$, one has exactly the same original nonlinear equation whose solution is unknown. Then, expanding the unknown solution in a perturbation series with respect to the artificial parameter ε and then substituting it into the original nonlinear equation, one can get a power series of ε . Finally, setting $\varepsilon = 1$, one has an approximation of the solution of the original equation. For example, if a nonlinear differential equation

$$\mathcal{L}_0[u(\mathbf{x}, t)] + \mathcal{N}_0[u(\mathbf{x}, t)] = 0$$

does not contain any small/large physical parameter, where $\mathcal{L}_0$ and $\mathcal{N}_0$ are linear and nonlinear differential operators, corresponding to the linear and nonlinear parts of a given nonlinear equation, and $u(\mathbf{x}, t)$ denotes the unknown function with the independent spatial and temporal variables, respectively, then, one can introduce “an artificial small parameter” ε to construct such a family of equation

$$\mathcal{L}_0[u(\mathbf{x}, t)] + \varepsilon \mathcal{N}_0[u(\mathbf{x}, t)] = 0,$$

and in addition, gives its “perturbation approximations” by first regarding ε as a perturbation quantity and then setting $\varepsilon = 1$. This method, called “the artificial small parameter method” (Lyapunov, 1992), was first proposed in 1892 by the Russian mathematician Aleksandr Mikhailovich Lyapunov (1857 — 1918), but unfortunately did not become a mainstream of perturbation techniques, possibly because the meaning of the so-called “artificial small parameter” has not been understood in details and besides the use of the so-called “artificial small parameter” needs a rigorous theory.

On the other side, Jules Henri Poincaré (1854 — 1912), a French mathematician, proposed the concept of homotopy in topology (Armstrong, 1983; Sen, 1983). Topology becomes an important field of mathematics, and the concept of homotopy has been widely applied by pure mathematicians to prove the existence, uniqueness of solutions of equations, and so on. Besides, based on the concept of homotopy, some numerical techniques such as the homotopy continuation method (Alexander, 1978; Li and Yorke, 1978, 1980; Li, 1993) have been developed, which are rather powerful to find out zeros of a set of nonlinear algebraic equations. It is a pity that the concept of the homotopy was not introduced to gain analytic approximations of nonlinear differential equations.

The concept of homotopy was first introduced to gain analytic approximations of nonlinear differential equations by Shijun Liao (1992). For a given nonlinear differential equation

$$\mathcal{N}[u(\mathbf{x}, t)] = 0,$$

where $\mathcal{N}$ is a nonlinear differential operator, $u(\mathbf{x}, t)$ is an unknown function, $\mathbf{x}$ denotes a vector of spatial independent-variables, t denotes the temporal independent-variable, respectively, Liao (1992) applied the concept of homotopy in topology to construct such a kind of *one-parameter* family of equations in the embedding parameter $q \in [0, 1]$, namely the zeroth-order deformation equation

$$(1 - q)\mathcal{L}[\phi(\mathbf{x}, t; q) - u_0(\mathbf{x}, t)] + q\mathcal{N}[\phi(\mathbf{x}, t; q)] = 0, \quad q \in [0, 1], \quad (4.37)$$

where $\mathcal{L}$ is an auxiliary linear operator and $u_0(\mathbf{x}, t)$ is an initial approximation, that, as the embedding parameter $q \in [0, 1]$ increases from 0 to 1, $\phi(\mathbf{x}, t; q)$ varies from the initial approximation $u_0(\mathbf{x}, t)$ to the unknown exact solution $u(\mathbf{x}, t)$ of the original equation. This work reveals the relationship between Lyapunov's "artificial small parameter" (Lyapunov, 1992) and Poincaré's concept of homotopy (Armstrong, 1983; Sen, 1983). Since the embedding-parameter $q \in [0, 1]$ has no physical meanings, one can always constructs such a kind of zeroth-order deformation equation, no matter whether there exist small/large physical parameters or not. So, this approach is more general than perturbation techniques in theory. On the other hand, the concept of the homotopy, i.e. the continuous variation or deformation, is more general in theory, which provides much larger freedom to construct the so-called zeroth-order deformation equations than Lyapunov's artificial small parameter method (Lyapunov, 1992): for example, one has extremely large freedom to choose the auxiliary linear operator $\mathcal{L}$, which is unnecessary to be $\mathcal{L}_0$ that corresponds to the whole linear part of a given nonlinear differential equation.

Unfortunately, it was found that the homotopy-series given by the zeroth-order deformation equation (4.37) are sometimes divergent, especially for equations with strong nonlinearity. This is mainly because, like perturbation techniques, the early HAM based on (4.37) can not guarantee the convergence of the homotopy-series. Realizing that one has extremely large freedom to construct a homotopy of equations, Liao (1997) introduced a non-zero auxiliary parameter c_0 to construct a more generalized zeroth-order deformation equation

$$(1 - q)\mathcal{L}[\phi(\mathbf{x}, t; q) - u_0(\mathbf{x}, t)] = c_0 q \mathcal{N}[\phi(\mathbf{x}, t; q)], \quad q \in [0, 1], \quad (4.38)$$

such that, the corresponding homotopy-series is dependent upon not only the physical independent-variables $\mathbf{x}$ and t but also the non-zero auxiliary parameter c_0 , although c_0 has no physical meanings at all. In essence, the use of the auxiliary parameter c_0 introduces us one more "artificial" degree of freedom, which has no physical meaning but can greatly improve the early HAM: it is found that the non-zero auxiliary parameter c_0 indeed provides a convenient way to control the convergence of the homotopy-series, say, one can guarantee the convergence of homotopy series by means of choosing a proper value of c_0 . For example, by choosing a proper auxiliary parameter c_0 , Abbasbandy (2006) applied the HAM to obtain accurate approximations of a nonlinear heat transfer equation for *all* possible values of a physical parameter, even if the corresponding perturbation approximations are divergent for the large physical parameter. Especially, Liang and Jeffrey (2009) illustrated by a simple example that, the HAM can give convergent series solution by choosing a proper

value of c_0 , even when the series given by the other analytic method is divergent in the whole domain except at the initial/boundary condition. For this reason, we call c_0 *the convergence-control parameter* today, which provides us a new concept: *the convergence-control of series*. In essence, it is the so-called convergence-control parameter c_0 that differs the HAM from other analytic techniques, such as perturbation techniques (Cole, 1992; Hilton, 1953; Hinch, 1991; Murdock, 1991; Nayfeh, 1973, 2000), Lyapunov's artificial small parameter method (Lyapunov, 1992), Adomian decomposition method (Adomian, 1976, 1991, 1994), the δ -expansion method (Karmishin et al., 1990) and so on. So, the use of the convergence-control parameter c_0 is the most important step in the development of the HAM.

The use of the convergence-control parameter c_0 is indeed a great progress, which provides us one more "artificial" degree of freedom in essence. It seems that more "artificial" degrees of freedom imply larger possibility to get better approximations by means of the homotopy analysis method. Thus, Liao (1999a) further introduced more "artificial" degrees of freedom by constructing the zeroth-order deformation equation in a more general form:

$$[1 - \alpha(q)]\mathcal{L}[\phi(\mathbf{x}, t; q) - u_0(\mathbf{x}, t)] = c_0 \beta(q) \mathcal{N}[\phi(\mathbf{x}, t; q)], \quad q \in [0, 1], \quad (4.39)$$

where $\alpha(q)$ and $\beta(q)$ are the analytic functions satisfying

$$\alpha(0) = \beta(0) = 0, \quad \alpha(1) = \beta(1) = 1, \quad (4.40)$$

whose Maclaurin series

$$\alpha(q) \sim \sum_{m=1}^{+\infty} \alpha_m q^m, \quad \beta(q) \sim \sum_{m=1}^{+\infty} \beta_m q^m, \quad (4.41)$$

exist and are convergent for $|q| \leq 1$, i.e.

$$\alpha(1) = \sum_{k=1}^{+\infty} \alpha_k = 1, \quad \beta(1) = \sum_{k=1}^{+\infty} \beta_k = 1, \quad (4.42)$$

where α_k and β_k are constants. This kind of generalization provides us larger possibility to guarantee the convergence of homotopy-series of original nonlinear equations.

In addition, realizing the extremely large freedom of constructing a homotopy of equations, Liao (2003b) introduced a non-zero auxiliary function $H(\mathbf{x}, t)$ to further generalize the zeroth-order deformation equation, i.e.

$$[1 - \alpha(q)]\mathcal{L}[\phi(\mathbf{x}, t; q) - u_0(\mathbf{x}, t)] = c_0 H(\mathbf{x}, t) \beta(q) \mathcal{N}[\phi(\mathbf{x}, t; q)], \quad (4.43)$$

where the analytic functions $\alpha(q)$ and $\beta(q)$ satisfy (4.40) and their Maclaurin series satisfy (4.42), respectively. Besides, Liao (2003b) gave a more generalized zeroth-order deformation equation

$$\begin{aligned}
& [1 - \alpha(q)] \mathcal{L} [\phi(\mathbf{x}, t; q) - u_0(\mathbf{x}, t)] \\
& = c_0 H(\mathbf{x}, t) \beta(q) \mathcal{N} [\phi(\mathbf{x}, t; q)] + \mathcal{A} [\phi(\mathbf{x}, t; q), \mathbf{x}, t; q],
\end{aligned} \tag{4.44}$$

where $\mathcal{A}$ is a nonlinear operator, which equals to zero when $q = 0$ and $q = 1$, i.e.

$$\mathcal{A} [\phi(\mathbf{x}, t; q), \mathbf{x}, t; q] = 0, \quad \text{when } q = 0 \text{ and } q = 1. \tag{4.45}$$

Using the generalized zeroth-order deformation equations (4.43) and (4.44), Liao (2003b) proved that Lyapunov's artificial small parameter method (Lyapunov, 1992), Adomian decomposition method (Adomian, 1976, 1991, 1994) and the δ -expansion method (Karmishin et al., 1990) are only special cases of the HAM. Besides, Sajid and Hayat (2008) proved that the so-called "homotopy perturbation method" (He, 1998, 1999) proposed in 1998 is exactly the same as the early HAM (Liao, 1992) developed by Liao in 1992, and therefore the "homotopy perturbation method" contains "nothing new except its name" (Sajid and Hayat, 2008). This reveals the generality and validity of the HAM from another view-point. Obviously, there exist an infinite number of analytic functions $\alpha(q)$ and $\beta(q)$ satisfying (4.40) and (4.42). Besides, there exist an infinite number of nonlinear operators $\mathcal{A}$ satisfying (4.45). So, the zeroth-order deformation equation (4.44) is indeed rather general. Thus, the homotopy approximations given by the HAM contain so many "artificial" degrees of freedom that we have a lots of ways to guarantee the convergence of homotopy-series.

In 2008, Marinca and Herişanu (2008) combined c_0 and $\beta(q)$ in the zeroth-order deformation equation (4.39) as a function $\bar{\beta}(q)$ with $\bar{\beta}(0) = 0$ but $\bar{\beta}(1) = c_0 \neq 1$, and constructed such a homotopy of equations

$$(1 - q) \mathcal{L} [\phi(\mathbf{x}, t; q) - u_0(\mathbf{x}, t)] = \bar{\beta}(q) \mathcal{N} [\phi(\mathbf{x}, t; q)], \quad q \in [0, 1], \tag{4.46}$$

where the Taylor series

$$\bar{\beta}(q) = \sum_{k=1}^{+\infty} \bar{\beta}_k q^k$$

converges at $q = 1$. Note that the above equation is a special case of (4.39), if one chooses $\alpha(q) = q$ and

$$\bar{\beta}(q) = c_0 \beta(q), \quad \text{i.e.} \quad \bar{\beta}_k = c_0 \beta_k.$$

So, the so-called "homotopy asymptotic method" proposed by Marinca and Herişanu (2008) is in essence still in the frame of the HAM. Even so, Marinca and Herişanu (2008) suggested an interesting approach, which has the advantage that $\bar{\beta}(1) = 1$ is unnecessary so that one has more freedom to choose the parameters $\bar{\beta}_k$: each of them can be unknown now and thus can be regarded as a convergence-control parameter.

In summary, the development of the HAM is strongly related to the generalization of the so-called zeroth-order deformation equations. In the early zeroth-order deformation equation (4.37), the concept of homotopy in topology was first introduced into the analytic approximation methods. In the modified zeroth-order deformation

equation (4.38), the so-called convergence-control parameter was first introduced, which provides us not only a convenient way to control the convergence of homotopy series, but also a new concept of “convergence-control of series”. By means of extremely large freedom on constructing a homotopy of equations, we can construct rather generalized zeroth-order deformation equations. The more general a zeroth-order deformation equation, the more larger freedom it provides us to guarantee the convergence of homotopy series. However, this does not mean that more complicated zeroth-order deformation equations are always better than simple ones. In general, the simple types of the zeroth-order deformation equations, such as (4.38), (4.39) and (4.43), can often give rather accurate approximations, as shown later in this book. For a few complicated nonlinear problems, it might be necessary to apply more generalized type of the zeroth-order deformation equations.

In Sect. 4.3.2, different types of the zeroth-order deformation equations and the corresponding high-order deformation equations are given. In Sect. 4.3.3, some examples are given to illustrate how to gain the high-order deformation equation by means of these theorems.

4.3.2 High-order deformation equations

In this section, the high-order deformation equations for various types of zeroth-order deformation equations are given.

Lemma 4.1. *Let*

$$\phi = \sum_{m=0}^{+\infty} u_m q^m$$

denote a homotopy-Maclaurin series, where $q \in [0, 1]$ is the homotopy-parameter, and $\mathcal{L}$ denotes an auxiliary linear operator which has the property $\mathcal{L}[0] = 0$ and is independent of q . It holds

$$\mathcal{D}_m [(1 - q) \cdot \mathcal{L}(\phi - u_0)] = \mathcal{L}(u_m - \chi_m u_{m-1})$$

where

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \quad (4.47)$$

Proof. Since $\mathcal{L}$ is a linear operator independent of q , it holds

$$(1 - q) \mathcal{L}(\phi - u_0) = \mathcal{L}(\phi - q\phi + u_0 q - u_0).$$

According to Theorem 4.3, Theorem 4.2 and the statement (4.8) of Theorem 4.1, we have

$$\begin{aligned}
& \mathcal{D}_m [(1-q)\mathcal{L}(\phi - u_0)] \\
&= \mathcal{D}_m [\mathcal{L}(\phi - q\phi + u_0 q - u_0)] \\
&= \mathcal{L} [\mathcal{D}_m (\phi - q\phi + u_0 q - u_0)] \\
&= \mathcal{L} [\mathcal{D}_m (\phi) - \mathcal{D}_m (q\phi) + u_0 \mathcal{D}_m (q)] \\
&= \mathcal{L} [u_m - u_{m-1} + u_0 \mathcal{D}_m (q)],
\end{aligned}$$

which equals to $\mathcal{L}(u_m)$ when $m = 1$, and $\mathcal{L}(u_m - u_{m-1})$ when $m > 1$, respectively. Thus, according to the definition (4.47) of χ_m , it holds

$$\mathcal{D}_m [(1-q)\mathcal{L}(\phi - u_0)] = \mathcal{L}(u_m - \chi_m u_{m-1}).$$

This ends the proof. $\square$

Theorem 4.15. *Let $\mathcal{L}$ denote an auxiliary linear operator which has the property $\mathcal{L}(0) = 0$ and is independent of the homotopy-parameter $q \in [0, 1]$, $\mathcal{N}$ denote a nonlinear operator, $u_0(\mathbf{x}, t)$ an initial approximation of the original equation $\mathcal{N}(u) = 0$, c_0 the convergence-control parameter independent of q , and $H(\mathbf{x}, t)$ an auxiliary function independent of q , respectively, where $\mathbf{x}$ denotes a vector of the spatial independent variable, and t is temporal independent variable. If*

$$\phi = \sum_{m=0}^{+\infty} u_m(\mathbf{x}, t) q^m$$

is the homotopy-Maclaurin series of the zeroth-order deformation equation

$$(1-q)\mathcal{L}(\phi - u_0) = q c_0 H(\mathbf{x}, t) \mathcal{N}(\phi), \quad (4.48)$$

then the corresponding m th-order deformation equation reads

$$\mathcal{L}[u_m(\mathbf{x}, t) - \chi_m u_{m-1}(\mathbf{x}, t)] = c_0 H(\mathbf{x}, t) \mathcal{D}_{m-1}[\mathcal{N}(\phi)], \quad (4.49)$$

where $\mathcal{D}_{m-1}$ is defined by (4.7) and χ_m is defined by (4.47).

Proof. According to Theorem 4.5, we have

$$\mathcal{D}_m [(1-q)\mathcal{L}(\phi - u_0)] = \mathcal{D}_m [q c_0 H(\mathbf{x}, t) \mathcal{N}(\phi)].$$

According to Lemma 4.1, it holds

$$\mathcal{D}_m [(1-q)\mathcal{L}(\phi - u_0)] = \mathcal{L}(u_m - \chi_m u_{m-1}).$$

According to Theorem 4.2 and (4.9), it holds

$$\mathcal{D}_m [q c_0 H(\mathbf{x}, t) \mathcal{N}(\phi)] = c_0 H(\mathbf{x}, t) \mathcal{D}_{m-1}[\mathcal{N}(\phi)].$$

Thus, we have the m th-order deformation equation

$$\mathcal{L}(u_m - \chi_m u_{m-1}) = c_0 H(\mathbf{x}, t) \mathcal{D}_{m-1}[\mathcal{N}(\phi)].$$

This ends the proof. $\square$

Due to the extremely large freedom on constructing the zeroth-order deformation equation, we can introduce an infinite number of constant convergence-control parameters $c_0, c_1, c_2, \dots$ to construct a more general zeroth-order deformation equation, as proved in the following theorem.

Theorem 4.16. *Let $\mathcal{L}$ denote an auxiliary linear operator which has the property $\mathcal{L}(0) = 0$ and is independent of the homotopy-parameter $q \in [0, 1]$, $\mathcal{N}$ denote a nonlinear operator, $u_0(\mathbf{x}, t)$ denote an initial approximation of the original equation $\mathcal{N}(u) = 0$, and $H(\mathbf{x}, t)$ be an auxiliary function independent of q , respectively, where $\mathbf{x}$ denotes a vector of the spatial independent variable, and t is temporal independent variable. If the series*

$$\sum_{k=0}^{+\infty} c_k q^k = c_0 + c_1 q + c_2 q^2 + \dots$$

converges at $q = 1$, where $c_0, c_1, \dots$ are constant convergence-control parameters with $c_0 \neq 0$ and $\sum_{k=0}^{+\infty} c_k \neq 0$, besides if

$$\phi = \sum_{m=0}^{+\infty} u_m(\mathbf{x}, t) q^m$$

is the homotopy-Maclaurin series of the zeroth-order deformation equation

$$(1 - q)\mathcal{L}(\phi - u_0) = H(\mathbf{x}, t) q \left(\sum_{k=0}^{+\infty} c_k q^k \right) \mathcal{N}(\phi), \quad (4.50)$$

then the corresponding m th-order deformation equation reads

$$\mathcal{L}[u_m(\mathbf{x}, t) - \chi_m u_{m-1}(\mathbf{x}, t)] = H(\mathbf{x}, t) \sum_{k=1}^m c_{k-1} \mathcal{D}_{m-k}[\mathcal{N}(\phi)], \quad (4.51)$$

where $\mathcal{D}_{m-k}$ is defined by (4.7) and χ_m is defined by (4.47).

Proof. Writing

$$\psi = \sum_{k=0}^{+\infty} c_k q^{k+1},$$

we have according to (4.8) that

$$\mathcal{D}_0(\psi) = 0, \quad \mathcal{D}_k(\psi) = c_{k-1}$$

for $k \geq 1$. According to Theorem 4.5, we have

$$\mathcal{D}_m[(1 - q)\mathcal{L}(\phi - u_0)] = \mathcal{D}_m[H(\mathbf{x}, t) \psi \mathcal{N}(\phi)].$$

According to Lemma 4.1, it holds

$$\mathcal{D}_m[(1-q)\mathcal{L}(\phi - u_0)] = \mathcal{L}(u_m - \chi_m u_{m-1}).$$

Then, using Theorem 4.2 and (4.10) of Theorem 4.1, we have

$$\begin{aligned} \mathcal{D}_m[H(\mathbf{x}, t) \psi \mathcal{N}(\phi)] &= H(\mathbf{x}, t) \sum_{k=0}^m \mathcal{D}_k(\psi) \mathcal{D}_{m-k}[\mathcal{N}(\phi)] \\ &= H(\mathbf{x}, t) \sum_{k=1}^m \mathcal{D}_k(\psi) \mathcal{D}_{m-k}[\mathcal{N}(\phi)] \\ &= H(\mathbf{x}, t) \sum_{k=1}^m c_{k-1} \mathcal{D}_{m-k}[\mathcal{N}(\phi)]. \end{aligned}$$

Thus, the corresponding high-order deformation equation reads

$$\mathcal{L}[u_m(\mathbf{x}, t) - \chi_m u_{m-1}(\mathbf{x}, t)] = H(\mathbf{x}, t) \sum_{k=1}^m c_{k-1} \mathcal{D}_{m-k}[\mathcal{N}(\phi)].$$

This ends the proof. $\square$

Note that the zeroth-order deformation equation (4.48) is a special case of the zeroth-order deformation equation (4.50) in case of $c_k = 0$ for $k \geq 1$. Besides, the convergence-control parameters c_0, c_1, c_2 and so on are *unnecessary* to be a constant. So, combining the auxiliary function $H(\mathbf{x}, t)$ with the convergence-control parameter c_k , we have a more general zeroth-order deformation equation.

Theorem 4.17. *Let $\mathcal{L}$ denote an auxiliary linear operator which has the property $\mathcal{L}(0) = 0$ and is independent of the homotopy-parameter $q \in [0, 1]$, $\mathcal{N}$ a nonlinear operator, $u_0(\mathbf{x}, t)$ an initial approximation of the original equation $\mathcal{N}(u) = 0$, respectively, where $\mathbf{x}$ denotes a vector of the spatial independent variable, and t is temporal independent variable. If the series*

$$\sum_{k=0}^{+\infty} \beta_k(\mathbf{x}, t) q^k$$

converges at $q = 1$ to a non-zero function, where $\beta_0(\mathbf{x}, t), \beta_1(\mathbf{x}, t), \dots$ are called convergence-control functions, and besides if

$$\phi = \sum_{m=0}^{+\infty} u_m(\mathbf{x}, t) q^m$$

is the homotopy-Maclaurin series of the zeroth-order deformation equation

$$(1-q)\mathcal{L}(\phi - u_0) = q \left[\sum_{k=0}^{+\infty} \beta_k(\mathbf{x}, t) q^k \right] \mathcal{N}(\phi), \quad (4.52)$$

then the corresponding m th-order deformation equation reads

$$\mathcal{L}[u_m(\mathbf{x}, t) - \chi_m u_{m-1}(\mathbf{x}, t)] = \sum_{k=1}^m \beta_{k-1}(\mathbf{x}, t) \mathcal{D}_{m-k}[\mathcal{N}(\phi)], \quad (4.53)$$

where $\mathcal{D}_{m-k}$ is defined by (4.7) and χ_m is defined by (4.47).

Proof. Writing

$$\psi = \sum_{k=0}^{+\infty} \beta_k(\mathbf{x}, t) q^{k+1},$$

we have according to (4.8) that

$$\mathcal{D}_0(\psi) = 0, \quad \mathcal{D}_k(\psi) = \beta_{k-1}(\mathbf{x}, t)$$

for $k \geq 1$. According to Theorem 4.5, we have

$$\mathcal{D}_m[(1-q)\mathcal{L}(\phi - u_0)] = \mathcal{D}_m[\psi \mathcal{N}(\phi)].$$

According to Lemma 4.1, it holds

$$\mathcal{D}_m[(1-q)\mathcal{L}(\phi - u_0)] = \mathcal{L}(u_m - \chi_m u_{m-1}).$$

Then, using Theorem 4.2 and (4.10) of Theorem 4.1, we have

$$\begin{aligned} \mathcal{D}_m[\psi \mathcal{N}(\phi)] &= \sum_{k=0}^m \mathcal{D}_k(\psi) \mathcal{D}_{m-k}[\mathcal{N}(\phi)] \\ &= \sum_{k=1}^m \mathcal{D}_k(\psi) \mathcal{D}_{m-k}[\mathcal{N}(\phi)] \\ &= \sum_{k=1}^m \beta_{k-1}(\mathbf{x}, t) \mathcal{D}_{m-k}[\mathcal{N}(\phi)]. \end{aligned}$$

Thus, the corresponding high-order deformation equation reads

$$\mathcal{L}[u_m(\mathbf{x}, t) - \chi_m u_{m-1}(\mathbf{x}, t)] = \sum_{k=1}^m \beta_{k-1}(\mathbf{x}, t) \mathcal{D}_{m-k}[\mathcal{N}(\phi)].$$

This ends the proof. $\square$

Note that the zeroth-order deformation equation (4.50) is a special case of the zeroth-order deformation equation (4.52) in case of $\beta_k(\mathbf{x}, t) = c_k H(\mathbf{x}, t)$ for $k \geq 0$. So, the concept of the so-called convergence-control function is more general: it contains not only the constant convergence-control parameters but also the so-called auxiliary function $H(\mathbf{x}, t)$. This reveals the essence of the auxiliary function $H(\mathbf{x}, t)$.

Definition 4.2. An analytic function f of the homotopy-parameter $q \in [0, 1]$ is called a *deformation-function*, if $f = 0$ when $q = 0$, $f = 1$ when $q = 1$, and its Maclaurin series

$$f \sim \sum_{k=1}^{+\infty} \alpha_k q^k$$

absolutely converges at $q = 1$, i.e.

$$\sum_{k=1}^{+\infty} \alpha_k = 1,$$

where α_k can be a constant or a function of spatial and temporal independent variables.

Lemma 4.2. *Let*

$$\phi = \sum_{m=0}^{+\infty} u_m q^m$$

denote a homotopy-Maclaurin series, where $q \in [0, 1]$ is the homotopy-parameter, $\mathcal{L}$ an auxiliary linear operator which has the property $\mathcal{L}(0) = 0$ and is independent of q . If $\alpha(q)$ is a deformation-function, i.e. $\alpha(0) = 0, \alpha(1) = 1$ and its Maclaurin series

$$\alpha(q) = \sum_{k=1}^{+\infty} \alpha_k q^k$$

exists and absolutely converges at $q = 1$, where α_k is constant, then it holds

$$\mathcal{D}_m \{ [1 - \alpha(q)] \mathcal{L}(\phi - u_0) \} = \mathcal{L} \left(u_m - \sum_{n=1}^{m-1} \alpha_n u_{m-n} \right)$$

Proof. Write

$$\Phi = \phi - u_0 = \sum_{k=1}^{+\infty} u_k q^k, \quad \Psi = \alpha(q) = \sum_{k=1}^{+\infty} \alpha_k q^k.$$

According to (4.8) of Theorem 4.1, we have

$$\mathcal{D}_0(\Phi) = \mathcal{D}_0(\Psi) = 0, \quad \mathcal{D}_n(\Phi) = u_n, \quad \mathcal{D}_n(\Psi) = \alpha_n, \quad n \geq 1.$$

Then, using Theorems 4.1–4.3, we have

$$\begin{aligned} & \mathcal{D}_m \{ [1 - \alpha(q)] \mathcal{L}(\phi - u_0) \} \\ &= \mathcal{D}_m [(1 - \Psi) \mathcal{L}(\Phi)] = \mathcal{D}_m [\mathcal{L}(\Phi) - \Psi \mathcal{L}(\Phi)] \\ &= \mathcal{D}_m [\mathcal{L}(\Phi)] - \sum_{n=0}^m \mathcal{D}_n(\Psi) \mathcal{D}_{m-n} [\mathcal{L}(\Phi)] \\ &= \mathcal{L} [\mathcal{D}_m(\Phi)] - \sum_{n=0}^m \mathcal{D}_n(\Psi) \mathcal{L} [\mathcal{D}_{m-n}(\Phi)] \end{aligned}$$

$$\begin{aligned}
&= \mathcal{L} [\mathcal{D}_m(\Phi)] - \sum_{n=1}^{m-1} \mathcal{D}_n(\Psi) \mathcal{L} [\mathcal{D}_{m-n}(\Phi)] \\
&= \mathcal{L}(u_m) - \sum_{n=1}^{m-1} \alpha_n \mathcal{L}(u_{m-n})
\end{aligned}$$

which gives, since α_k is constant, that

$$\mathcal{D}_m \{[1 - \alpha(q)] \mathcal{L}(\phi - u_0)\} = \mathcal{L} \left(u_m - \sum_{n=1}^{m-1} \alpha_n u_{m-n} \right).$$

This ends the proof. $\square$

Theorem 4.18. Let $\mathcal{L}$ denote an auxiliary linear operator which has the property $\mathcal{L}(0) = 0$ and is independent of the homotopy-parameter $q \in [0, 1]$, $\mathcal{N}$ denote a nonlinear operator; $u_0(\mathbf{x}, t)$ denote an initial approximation of the original equation $\mathcal{N}(u) = 0$, respectively, where $\mathbf{x}$ denotes a vector of the spatial independent variable, and t is temporal independent variable. Let $\alpha(q)$ denote a deformation-function, i.e. $\alpha(0) = 0, \alpha(1) = 1$ and its Maclaurin series

$$\alpha(q) \sim \sum_{k=1}^{+\infty} \alpha_k q^k,$$

exists and absolutely converges at $q = 1$, where α_k is constant. If the series

$$\sum_{k=0}^{+\infty} \beta_k(\mathbf{x}, t) q^k$$

converges at $q = 1$ to a non-zero function, where $\beta_0(\mathbf{x}, t), \beta_1(\mathbf{x}, t), \dots$ are called convergence-control functions, and besides if

$$\phi = \sum_{m=0}^{+\infty} u_m(\mathbf{x}, t) q^m$$

is the homotopy-Maclaurin series of the zeroth-order deformation equation

$$[1 - \alpha(q)] \mathcal{L}(\phi - u_0) = q \left[\sum_{k=0}^{+\infty} \beta_k(\mathbf{x}, t) q^k \right] \mathcal{N}(\phi), \quad (4.54)$$

then the corresponding m th-order deformation equation reads

$$\mathcal{L} \left[u_m(\mathbf{x}, t) - \sum_{n=1}^{m-1} \alpha_n u_{m-n}(\mathbf{x}, t) \right] = \sum_{k=1}^m \beta_{k-1}(\mathbf{x}, t) \mathcal{D}_{m-k} [\mathcal{N}(\phi)], \quad (4.55)$$

where $\mathcal{D}_{m-k}$ is defined by (4.7).

Proof. Writing

$$\psi = \sum_{k=0}^{+\infty} \beta_k(\mathbf{x}, t) q^{k+1},$$

we have according to (4.8) that

$$\mathcal{D}_0(\psi) = 0, \quad \mathcal{D}_k(\psi) = \beta_{k-1}(\mathbf{x}, t)$$

for $k \geq 1$. According to Theorem 4.5, we have

$$\mathcal{D}_m \{ [1 - \alpha(q)] \mathcal{L}(\phi - u_0) \} = \mathcal{D}_m [\psi \mathcal{N}(\phi)].$$

According to Lemma 4.2, it holds

$$\mathcal{D}_m \{ [1 - \alpha(q)] \mathcal{L}(\phi - u_0) \} = \mathcal{L} \left(u_m - \sum_{n=1}^{m-1} \alpha_n u_{m-n} \right).$$

Then, using Theorem 4.2 and (4.10) of Theorem 4.1, we have

$$\begin{aligned} \mathcal{D}_m [\psi \mathcal{N}(\phi)] &= \sum_{k=0}^m \mathcal{D}_k(\psi) \mathcal{D}_{m-k} [\mathcal{N}(\phi)] \\ &= \sum_{k=1}^m \mathcal{D}_k(\psi) \mathcal{D}_{m-k} [\mathcal{N}(\phi)] \\ &= \sum_{k=1}^m \beta_{k-1}(\mathbf{x}, t) \mathcal{D}_{m-k} [\mathcal{N}(\phi)]. \end{aligned}$$

Thus, the corresponding high-order deformation equation reads

$$\mathcal{L} \left[u_m(\mathbf{x}, t) - \sum_{n=1}^{m-1} \alpha_n u_{m-n}(\mathbf{x}, t) \right] = \sum_{k=1}^m \beta_{k-1}(\mathbf{x}, t) \mathcal{D}_{m-k} [\mathcal{N}(\phi)].$$

This ends the proof. $\square$

Note that the zeroth-order deformation equation (4.52) and the corresponding high-order deformation equation (4.53) are special cases of (4.54) and (4.55) when $\alpha_1 = 1$ and $\alpha_k = 0$ for $k \geq 2$, respectively.

Definition 4.3. An operator $\mathcal{A}(\phi, \mathbf{x}, t, q)$ is called *deformation-operator*, if $\mathcal{A}(\phi, \mathbf{x}, t, q) = 0$ at $q = 0$ and $q = 1$, where $q \in [0, 1]$ is the homotopy-parameter, $\phi = \sum_{k=0}^{+\infty} u_k q^k$ is a homotopy-Maclaurin series, $\mathbf{x}$ denotes a vector of the spatial independent variable, and t is temporal independent variable, respectively.

Theorem 4.19. Let $\mathcal{L}$ denote an auxiliary linear operator which has the property $\mathcal{L}(0) = 0$ and is independent of the homotopy-parameter $q \in [0, 1]$, $\mathcal{N}$ denote a

nonlinear operator, $u_0(\mathbf{x}, t)$ denote an initial approximation of the original equation $\mathcal{N}(u) = 0$, respectively, where $\mathbf{x}$ denotes a vector of the spatial independent variable, and t is temporal independent variable. Let $\alpha(q)$ denote a deformation-function, i.e. $\alpha(0) = 0, \alpha(1) = 1$ and its Maclaurin series

$$\alpha(q) \sim \sum_{k=1}^{+\infty} \alpha_k q^k,$$

exists and converges at $q = 1$, where α_k is constant. Let $\mathcal{A}$ be a deformation-operator, i.e.

$$\mathcal{A}(\phi, \mathbf{x}, t, q) = 0, \quad \text{when } q = 0 \text{ and } q = 1.$$

If the series

$$\sum_{k=0}^{+\infty} \beta_k(\mathbf{x}, t) q^k$$

converges at $q = 1$ to a nonzero function, where $\beta_0(\mathbf{x}, t), \beta_1(\mathbf{x}, t), \dots$ are called convergence-control functions, and besides if

$$\phi = \sum_{m=0}^{+\infty} u_m(\mathbf{x}, t) q^m$$

is the homotopy-Maclaurin series of the zeroth-order deformation equation

$$[1 - \alpha(q)] \mathcal{L}(\phi - u_0) = q \left[\sum_{k=0}^{+\infty} \beta_k(\mathbf{x}, t) q^k \right] \mathcal{N}(\phi) + \mathcal{A}(\phi, \mathbf{x}, t, q), \quad (4.56)$$

then the corresponding m th-order deformation equation reads

$$\begin{aligned} & \mathcal{L} \left[u_m(\mathbf{x}, t) - \sum_{n=1}^{m-1} \alpha_n u_{m-n}(\mathbf{x}, t) \right] \\ &= \sum_{k=1}^m \beta_{k-1}(\mathbf{x}, t) \mathcal{D}_{m-k}[\mathcal{N}(\phi)] + \mathcal{D}_m[\mathcal{A}(\phi, \mathbf{x}, t, q)], \end{aligned} \quad (4.57)$$

where $\mathcal{D}_{m-k}$ is defined by (4.7).

Proof. Writing

$$\psi = \sum_{k=0}^{+\infty} \beta_k(\mathbf{x}, t) q^{k+1},$$

we have according to (4.8) that

$$\mathcal{D}_0(\psi) = 0, \quad \mathcal{D}_k(\psi) = \beta_{k-1}(\mathbf{x}, t)$$

for $k \geq 1$. According to Theorem 4.5, we have

$$\mathcal{D}_m \{ [1 - \alpha(q)] \mathcal{L}(\phi - u_0) \} = \mathcal{D}_m [\psi \mathcal{N}(\phi) + \mathcal{A}(\phi, \mathbf{x}, t, q)].$$

According to Lemma 4.2, it holds

$$\mathcal{D}_m \{ [1 - \alpha(q)] \mathcal{L}(\phi - u_0) \} = \mathcal{L} \left(u_m - \sum_{n=1}^{m-1} \alpha_n u_{m-n} \right).$$

Then, using Theorem 4.2 and (4.10) of Theorem 4.1, we have

$$\begin{aligned} \mathcal{D}_m [\psi \mathcal{N}(\phi) + \mathcal{A}(\phi, \mathbf{x}, t, q)] &= \sum_{k=0}^m \mathcal{D}_k(\psi) \mathcal{D}_{m-k}[\mathcal{N}(\phi)] + \mathcal{D}_m[\mathcal{A}(\phi, \mathbf{x}, t, q)] \\ &= \sum_{k=1}^m \mathcal{D}_k(\psi) \mathcal{D}_{m-k}[\mathcal{N}(\phi)] + \mathcal{D}_m[\mathcal{A}(\phi, \mathbf{x}, t, q)] \\ &= \sum_{k=1}^m \beta_{k-1}(\mathbf{x}, t) \mathcal{D}_{m-k}[\mathcal{N}(\phi)] + \mathcal{D}_m[\mathcal{A}(\phi, \mathbf{x}, t, q)]. \end{aligned}$$

Thus, the corresponding high-order deformation equation reads

$$\begin{aligned} &\mathcal{L} \left[u_m(\mathbf{x}, t) - \sum_{n=1}^{m-1} \alpha_n u_{m-n}(\mathbf{x}, t) \right] \\ &= \sum_{k=1}^m \beta_{k-1}(\mathbf{x}, t) \mathcal{D}_{m-k}[\mathcal{N}(\phi)] + \mathcal{D}_m[\mathcal{A}(\phi, \mathbf{x}, t, q)]. \end{aligned}$$

This ends the proof. $\square$

Note that the zeroth-order deformation equation (4.54) and its high-order deformation equation (4.55) are special cases of (4.56) and (4.57) in case of

$$\mathcal{A}(\phi, \mathbf{x}, t, q) = 0,$$

respectively. Although (4.56) and (4.57) are rather general, a even more generalized zeroth-order deformation equation can be given by the following theorem.

Lemma 4.3. *Let $\mathcal{L}$ be an auxiliary linear operator which has the property $\mathcal{L}(0) = 0$ and is independent of the homotopy-parameter $q \in [0, 1]$, and*

$$\phi = \sum_{m=0}^{+\infty} u_m q^m$$

denote a homotopy-Maclaurin series. If $\alpha(\mathbf{x}, t, q)$ is a deformation-function, i.e. $\alpha(\mathbf{x}, t, q) = 0$ when $q = 0$ and $\alpha(\mathbf{x}, t, q) = 1$ when $q = 1$, and its Maclaurin series

$$\alpha(\mathbf{x}, t, q) \sim \sum_{k=1}^{+\infty} \alpha_k(\mathbf{x}, t) q^k,$$

exists and converges at $q = 1$, where α_k is a function of the vector $\mathbf{x}$ of the spatial independent variables and the temporal independent variable t , then it holds

$$\mathcal{D}_m \{ [1 - \alpha(\mathbf{x}, t, q)] \mathcal{L}(\phi - u_0) \} = \mathcal{L}(u_m) - \sum_{n=1}^{m-1} \alpha_n(\mathbf{x}, t) \mathcal{L}(u_{m-n}).$$

Lemma 4.3 can be proved similarly as Lemma 4.2.

Theorem 4.20. Let $\mathcal{L}$ denote an auxiliary linear operator which has the property $\mathcal{L}(0) = 0$ and is independent of the homotopy-parameter $q \in [0, 1]$, $\mathcal{N}$ denote a nonlinear operator; $u_0(\mathbf{x}, t)$ denote an initial approximation of the original equation $\mathcal{N}(u) = 0$, respectively, where $\mathbf{x}$ denotes a vector of the spatial independent variable, and t is temporal independent variable. Let $\alpha(\mathbf{x}, t, q)$ denote a deformation-function, i.e. $\alpha(\mathbf{x}, t, q) = 0$ when $q = 0$ and $\alpha(\mathbf{x}, t, q) = 1$ when $q = 1$, and its Maclaurin series

$$\alpha(\mathbf{x}, t, q) \sim \sum_{k=1}^{+\infty} \alpha_k(\mathbf{x}, t) q^k,$$

exists and converges at $q = 1$, where $\alpha_k(\mathbf{x}, t)$ is dependent on $\mathbf{x}$ and t . Let $\mathcal{B}$ be an operator dependent of $\phi, \mathbf{x}, t$ and the homotopy-parameter $q \in [0, 1]$, which satisfies

$$\begin{aligned} \mathcal{B}(\phi, \mathbf{x}, t, q) &= 0, & \text{when } q = 0, \\ \mathcal{B}(\phi, \mathbf{x}, t, q) &= \gamma(\mathbf{x}, t) \mathcal{N}(\phi), & \text{when } q = 1, \end{aligned}$$

where $\gamma(\mathbf{x}, t) \neq 0$ is a non-zero function. If

$$\phi = \sum_{m=0}^{+\infty} u_m(\mathbf{x}, t) q^m$$

is the homotopy-Maclaurin series of the zeroth-order deformation equation

$$[1 - \alpha(\mathbf{x}, t, q)] \mathcal{L}(\phi - u_0) = \mathcal{B}(\phi, \mathbf{x}, t, q), \quad (4.58)$$

then the corresponding m th-order deformation equation reads

$$\mathcal{L}[u_m(\mathbf{x}, t)] - \sum_{n=1}^{m-1} \alpha_n(\mathbf{x}, t) \mathcal{L}[u_{m-n}(\mathbf{x}, t)] = \mathcal{D}_m[\mathcal{B}(\phi, \mathbf{x}, t, q)], \quad (4.59)$$

where $\mathcal{D}_{m-k}$ is defined by (4.7).

Proof. According to Theorem 4.5, it holds

$$\mathcal{D}_m \{ [1 - \alpha(\mathbf{x}, t, q)] \mathcal{L}(\phi - u_0) \} = \mathcal{D}_m[\mathcal{B}(\phi, \mathbf{x}, t, q)].$$

According to Lemma 4.3, it holds

$$\mathcal{D}_m \{ [1 - \alpha(\mathbf{x}, t, q)] \mathcal{L}(\phi - u_0) \} = \mathcal{L}(u_m) - \sum_{n=1}^{m-1} \alpha_n(\mathbf{x}, t) \mathcal{L}(u_{m-n}).$$

Thus, the corresponding m th-order deformation equation reads

$$\mathcal{L}[u_m(\mathbf{x}, t)] - \sum_{n=1}^{m-1} \alpha_n(\mathbf{x}, t) \mathcal{L}[u_{m-n}(\mathbf{x}, t)] = \mathcal{D}_m[\mathcal{B}(\phi, \mathbf{x}, t, q)].$$

This ends the proof. $\square$

Note that the zeroth-order deformation equations (4.48), (4.50), (4.52), (4.54), (4.56) and (4.58) are more and more general, so are the corresponding high-order deformation equations (4.49), (4.51), (4.53), (4.55), (4.57) and (4.59). This is mainly because we have extremely large freedom to construct the so-called zeroth-order deformation equation, whose general definition is given in Chapter 2. The high-order deformation equations have some common properties:

1. The high-order deformation equations are always linear with respect to the unknown u_m ;
2. The terms on the left-hand side of the high-order deformation equation are rather similar, i.e.

$$\mathcal{L}(u_m - \chi_m u_{m-1}), \quad \text{when } \alpha(q) = q,$$

or

$$\mathcal{L}\left(u_m - \sum_{n=1}^{m-1} \alpha_{m-n} u_n\right), \quad \text{when } \alpha(q) = \sum_{n=1}^{+\infty} \alpha_k q^k,$$

where α_k is constant, or

$$\mathcal{L}(u_m) - \sum_{n=1}^{m-1} \alpha_{m-n}(\mathbf{x}, t) \mathcal{L}(u_n), \quad \text{when } \alpha(\mathbf{x}, t, q) = \sum_{n=1}^{+\infty} \alpha_k(\mathbf{x}, t) q^k,$$

where $\alpha_k(\mathbf{x}, t)$ is dependent on the vector $\mathbf{x}$ of spatial independent variables and the temporal independent variable t , respectively. All of them are completely determined by the auxiliary linear operator $\mathcal{L}$ and the deformation-function α . Note that we have great freedom to choose the auxiliary linear operator $\mathcal{L}$ and the deformation-function $\alpha(q)$ or $\alpha(\mathbf{x}, t, q)$;

3. For the m th-order deformation equation, $u_0, u_1, u_2, \dots, u_{m-1}$ are known and thus the term on the right-hand side of the high-order deformation equation is often regarded as known in essence. So, it is often convenient to successively solve the linear high-order deformation equation, especially by means of computer algebra systems like Mathematica, Maple and so on.
4. Given the type of a zeroth-order deformation equation, the corresponding high-order deformation equation is obtained directly by means of the theorems proved above. Mostly, it is necessary to calculate the homotopy-derivative $\mathcal{D}_m[\mathcal{N}(\phi)]$. As pointed out in Sect. 4.2, for *any* given smooth function $f(\phi)$, where ϕ is a homotopy-Maclaurin series, we can always obtain its m th-order homotopy-derivative $\mathcal{D}_m[f(\phi)]$ by means of the theorems proved in Sect. 4.2, especially by means of the recursion formula (4.27). Similarly, for *any* a given original nonlinear equation $\mathcal{N}(u) = 0$, we can always obtain the term $\mathcal{D}_m[\mathcal{N}(\phi)]$, no matter

how complicated the nonlinear operator $\mathcal{N}$ is. This indicates the importance of the so-called homotopy-derivative operator $\mathcal{D}_m$ defined by (4.7), and reveals the reason why the theorems about $\mathcal{D}_m$ are proved in Sect. 4.2.

Note that, as shown in Chapter 2, using the extremely large freedom on constructing a homotopy of equations, we can express an unknown physical parameter as a homotopy-Maclaurin series in the zeroth-order deformation equation. Therefore, all above theorems hold when we replace $\mathcal{D}_k[\mathcal{N}(\phi)]$ by $\mathcal{D}_k[\mathcal{N}(\phi, \Omega, q)]$, where $q \in [0, 1]$ is the homotopy-parameter and

$$\Omega \sim \sum_{k=0}^{+\infty} \omega_k q^k$$

is a homotopy-Maclaurin series of a physical parameter ω . Besides, we can treat the initial/boundary conditions of a nonlinear equation in a similar way as mentioned above.

4.3.3 Examples

Here, we use some simple examples to show how to apply the above theorems to deduce high-order deformation equations of nonlinear problems.

Example 4.1. Let us consider a nonlinear heat transfer problem (Abbasbandy, 2006):

$$(1 + \varepsilon u)u' + u = 0, \quad u(0) = 1,$$

where the prime denotes the differentiation with respect to the time t . Choosing $\mathcal{L}(u) = u' + u$ as the auxiliary linear operator, and defining the nonlinear operator

$$\mathcal{N}[\phi(t; q)] = (1 + \varepsilon \phi)\phi' + \phi,$$

where

$$\phi = \sum_{k=0}^{+\infty} u_k(t) q^k$$

is a homotopy-Maclaurin series, we construct such a zeroth-order deformation equation

$$(1 - q)\mathcal{L}[\phi(t; q) - u_0(t)] = q c_0 \mathcal{N}[\phi(t; q)],$$

subject to the initial condition

$$\phi(t; q) = 1, \quad \text{when } t = 0,$$

where $u_0(t)$ is an initial guess satisfying the initial condition $u(0) = 1$. According to Theorem 4.15, the corresponding m th-order deformation equation reads

$$\mathcal{L}[u_m(t) - \chi_m u_{m-1}(t)] = c_0 \mathcal{D}_{m-1} \{ \mathcal{N}[\phi(t; q)] \},$$

subject to the initial condition

$$u_m(0) = 0.$$

According to Theorem 4.1, Theorem 4.2 and Theorem 4.3, one has

$$\begin{aligned} \mathcal{D}_{m-1} \{ \mathcal{N}[\phi(t; q)] \} &= \mathcal{D}_{m-1}(\phi') + \mathcal{D}_{m-1}(\phi) + \varepsilon \mathcal{D}_{m-1}(\phi \phi') \\ &= u'_{m-1} + u_{m-1} + \varepsilon \sum_{n=0}^{m-1} u_{m-1-n} u'_n. \end{aligned}$$

The corresponding homotopy-series solution is given by

$$u(t) = \sum_{k=0}^{+\infty} u_k(t),$$

which is convergent for *any* physical parameter $0 \leq \varepsilon < +\infty$ if one chooses the convergence-control parameter $c_0 = -(1 + \varepsilon)^{-1}$. For details, please refer to Abbasbandy (Abbasbandy, 2006).

Example 4.2. Let us consider a nonlinear oscillation equation (Liao and Tan, 2007):

$$u''(t) + \lambda u(t) + \varepsilon u^3(t) = 0, \quad u(0) = 1, u'(0) = 0,$$

where the prime denotes the differentiation with respect to the time t , λ and ε are physical parameters. Let ω denote the unknown frequency of the periodic solution. Writing $\tau = \omega t$, the above equation becomes

$$\gamma u''(\tau) + \lambda u(\tau) + \varepsilon u^3(\tau) = 0, \quad u(0) = 1, u'(0) = 0,$$

where $\gamma = \omega^2$ is an unknown physical parameter. Regard γ as a homotopy-Maclaurin series and define a nonlinear operator

$$\mathcal{N}[\phi(\tau; q), \Gamma(q)] = \Gamma(q) \phi''(\tau; q) + \lambda \phi(\tau; q) + \varepsilon \phi^3(\tau; q),$$

where the prime denotes the differentiation with respect to τ , and

$$\phi(\tau; q) = \sum_{k=0}^{+\infty} u_k(\tau) q^k, \quad \Gamma(q) = \sum_{k=0}^{+\infty} \gamma_k q^k$$

are two homotopy-Maclaurin series. Choosing the auxiliary linear operator

$$\mathcal{L}(u) = u'' + u,$$

we construct the following zeroth-order deformation equation

$$(1 - q) \mathcal{L}[\phi(\tau; q) - u_0(\tau)] = q c_0 \mathcal{N}[\phi(\tau; q), \Gamma(q)], \quad q \in [0, 1]$$

subject to the initial conditions

$$\phi(\tau; q) = 1, \quad \phi'(\tau; q) = 0, \quad \text{at } \tau = 0,$$

where $u_0(t)$ is an initial guess satisfying the initial conditions. According to Theorem 4.15, the corresponding high-order deformation equation reads

$$\mathcal{L}[u_m(\tau) - \chi_m u_{m-1}(\tau)] = c_0 \mathcal{D}_{m-1} \{ \mathcal{N}[\phi(\tau; q), \Gamma(q)] \},$$

subject to the initial conditions

$$u_m(0) = 0, \quad u'_m(0) = 0.$$

According to Theorems 4.1 – 4.3 and Theorem 4.6, we have

$$\begin{aligned} & \mathcal{D}_{m-1} \{ \mathcal{N}[\phi(\tau; q), \Gamma(q)] \} \\ &= \mathcal{D}_{m-1} (\Gamma \phi'') + \lambda \mathcal{D}_{m-1} (\phi) + \varepsilon \mathcal{D}_{m-1} (\phi^3) \\ &= \sum_{n=0}^{m-1} \gamma_{m-1-n} u''_n + \lambda u_{m-1} + \varepsilon \sum_{n=0}^{m-1} u_{m-1-n} \sum_{k=0}^n u_{n-k} u_k. \end{aligned}$$

The corresponding homotopy-series solutions are given by

$$u(t) = \sum_{m=0}^{+\infty} u_m(\omega t), \quad \omega = \left(\sum_{m=0}^{+\infty} \gamma_m \right)^{1/2},$$

which are convergent if the convergence-control parameter c_0 is chosen properly. For example, when $\lambda = 0$, the homotopy-series solutions are convergent for *any* a physical parameter $0 \leq \varepsilon < +\infty$ by using $c_0 = -(1 + \varepsilon)^{-1}$. For details, please refer to Chapter 2.

Example 4.3. Let us consider a nonlinear differential equation with variable coefficients:

$$A(x) u''(x) + A'(x) u'(x) + \gamma \sin u(x) = 0, \quad u'(0) = 0, \quad u'(\pi) = 0, \quad u(0) = a,$$

where $A(x)$ is a given function, γ is an unknown eigenvalue, a is a given constant. Choose $u_0 = a \cos x$ as the initial approximation, $\mathcal{L}(u) = u'' + u$ as the auxiliary linear operator, respectively, and define a nonlinear operator

$$\mathcal{N}(\phi, \Gamma, q) = A(x) \phi''(x; q) + A'(x) \phi'(x; q) + \Gamma(q) \frac{\sin(q\phi)}{q},$$

where the prime denotes the differentiation with respect to x . Construct the zeroth-order deformation equation

$$(1 - q) \mathcal{L}(\phi - u_0) = c_0 q \mathcal{N}(\phi, \Gamma, q), \quad \phi'(0; q) = 0, \quad \phi'(\pi; q) = 0, \quad \phi(0; q) = a.$$

According to Theorem 4.15, the corresponding high-order deformation equation reads

$$\mathcal{L}(u_m - \chi_m u_{m-1}) = c_0 \mathcal{D}_{m-1} [\mathcal{N}(\phi, \Gamma, q)], \quad u'_m(0) = 0, \quad u'_m(\pi) = 0, \quad u_m(0) = 0.$$

According to Theorem 4.1, Theorem 4.2 and Theorem 4.3, we have

$$\begin{aligned} & \mathcal{D}_{m-1} [\mathcal{N}(\phi, \Gamma, q)] \\ &= \mathcal{D}_{m-1} [A(x) \phi''(x; q)] + \mathcal{D}_{m-1} [A'(x) \phi'(x; q)] + \mathcal{D}_{m-1} \left[\Gamma(q) \frac{\sin(q\phi)}{q} \right] \\ &= A(x) u''_{m-1}(x) + A'(x) u'_{m-1}(x) + \sum_{n=0}^{m-1} \gamma_{m-1-n} \mathcal{D}_n \left[\frac{\sin(q\phi)}{q} \right], \end{aligned}$$

where the term $\mathcal{D}_n[\sin(q\phi)/q]$ is given by (4.36), which is obtained by means of Theorem 4.13. For details, please refer to Liao (2009b).

4.4 Convergence theorems

In Chapter 2, two theorems about the convergence of the homotopy series of a nonlinear oscillation equation are given. Qualitatively speaking, these theorems have general meanings. Here, two similar theorems are proved in general.

Theorem 4.21. *Let $\mathcal{L}$ denote an auxiliary linear operator which has the property $\mathcal{L}(0) = 0$ and is independent of the homotopy-parameter $q \in [0, 1]$, $\mathcal{N}$ denote a nonlinear operator; $u_0(\mathbf{x}, t)$ denote an initial approximation of the original equation $\mathcal{N}[u(\mathbf{x}, t), \gamma] = 0$, respectively, where $\mathbf{x}$ denotes a vector of the spatial independent variable, t is temporal independent variable, and γ is an unknown physical parameter, respectively. Let $\alpha(\mathbf{x}, t; q)$ denote a deformation-function, i.e. $\alpha = 0$ at $q = 0$ and $\alpha = 1$ at $q = 1$, and assume that its Maclaurin series*

$$\alpha(\mathbf{x}, t; q) = \sum_{k=1}^{+\infty} \alpha_k(\mathbf{x}, t) q^k,$$

absolutely converges at $q = 1$ so that $\sum_{k=1}^{+\infty} \alpha_k(\mathbf{x}, t; q) = 1$. Let

$$\sum_{k=0}^{+\infty} \beta_k(\mathbf{x}, t) q^{k+1}$$

be a series that is absolutely convergent at $q = 1$ to a non-zero function $\beta(\mathbf{x}, t)$, i.e.

$$\beta(\mathbf{x}, t) = \sum_{k=0}^{+\infty} \beta_k(\mathbf{x}, t) \neq 0,$$

where $\beta_0(\mathbf{x}, t), \beta_1(\mathbf{x}, t), \dots$ are the so-called convergence-control functions. Let

$$\Gamma \sim \sum_{m=0}^{+\infty} \gamma_m q^m$$

be a homotopy-Maclaurin series of an unknown physical parameter γ , and

$$\phi \sim \sum_{m=0}^{+\infty} u_m(\mathbf{x}, t) q^m$$

be the homotopy-Maclaurin series of the zeroth-order deformation equation

$$[1 - \alpha(\mathbf{x}, t; q)] \mathcal{L}(\phi - u_0) = \left[\sum_{k=0}^{+\infty} \beta_k(\mathbf{x}, t) q^{k+1} \right] \mathcal{N}(\phi, \Gamma), \quad (4.60)$$

where u_m is governed by the corresponding m th-order deformation equation

$$\mathcal{L}[u_m(\mathbf{x}, t)] - \sum_{n=1}^{m-1} \alpha_n(\mathbf{x}, t) \mathcal{L}[u_{m-n}(\mathbf{x}, t)] = \sum_{k=1}^m \beta_{k-1}(\mathbf{x}, t) \delta_{m-k}(\mathbf{x}, t), \quad (4.61)$$

with the definition

$$\delta_k(\mathbf{x}, t) = \mathcal{D}_k[\mathcal{N}(\phi, \Gamma)]$$

and the definition (4.7) of $\mathcal{D}_k$. If the homotopy-series

$$u(\mathbf{x}, t) \sim \sum_{m=0}^{+\infty} u_m(\mathbf{x}, t), \quad \gamma \sim \sum_{m=0}^{+\infty} \gamma_m$$

converge, and besides $\sum_{m=0}^{+\infty} \mathcal{L}[u_m(\mathbf{x}, t)]$ also converges, then

$$\sum_{k=0}^{+\infty} \delta_k(\mathbf{x}, t) = \sum_{k=0}^{+\infty} \mathcal{D}_k[\mathcal{N}(\phi, \Gamma)] = 0. \quad (4.62)$$

Proof. According to (4.61), we have

$$\sum_{m=1}^{+\infty} \left\{ \mathcal{L}[u_m(\mathbf{x}, t)] - \sum_{n=1}^{m-1} \alpha_n(\mathbf{x}, t) \mathcal{L}[u_{m-n}(\mathbf{x}, t)] \right\} = \sum_{m=1}^{+\infty} \sum_{k=1}^m \beta_{k-1}(\mathbf{x}, t) \delta_{m-k}(\mathbf{x}, t).$$

Since $\sum_{k=1}^{+\infty} \alpha_k(\mathbf{x}, t)$ is absolutely convergent and $\sum_{m=0}^{+\infty} \mathcal{L}[u_m(\mathbf{x}, t)]$ is convergent, we have due to the theorem of Cauchy product that

$$\sum_{m=1}^{+\infty} \left\{ \mathcal{L}[u_m(\mathbf{x}, t)] - \sum_{n=1}^{m-1} \alpha_n(\mathbf{x}, t) \mathcal{L}[u_{m-n}(\mathbf{x}, t)] \right\}$$

$$\begin{aligned}
&= \sum_{m=1}^{+\infty} \mathcal{L}[u_m(\mathbf{x}, t)] - \sum_{m=1}^{+\infty} \sum_{n=1}^{m-1} \alpha_n(\mathbf{x}, t) \mathcal{L}[u_{m-n}(\mathbf{x}, t)] \\
&= \sum_{m=1}^{+\infty} \mathcal{L}[u_m(\mathbf{x}, t)] - \sum_{n=1}^{+\infty} \sum_{m=n+1}^{+\infty} \alpha_n(\mathbf{x}, t) \mathcal{L}[u_{m-n}(\mathbf{x}, t)] \\
&= \sum_{m=1}^{+\infty} \mathcal{L}[u_m(\mathbf{x}, t)] - \sum_{n=1}^{+\infty} \sum_{k=1}^{+\infty} \alpha_n(\mathbf{x}, t) \mathcal{L}[u_k(\mathbf{x}, t)] \\
&= \sum_{m=1}^{+\infty} \mathcal{L}[u_m(\mathbf{x}, t)] - \left[\sum_{n=1}^{+\infty} \alpha_n(\mathbf{x}, t) \right] \left\{ \sum_{k=1}^{+\infty} \mathcal{L}[u_k(\mathbf{x}, t)] \right\} \\
&= \left[1 - \sum_{n=1}^{+\infty} \alpha_n(\mathbf{x}, t) \right] \left\{ \sum_{m=1}^{+\infty} \mathcal{L}[u_m(\mathbf{x}, t)] \right\},
\end{aligned}$$

which gives, due to $\sum_{n=1}^{+\infty} \alpha_n(\mathbf{x}, t) = 1$, that

$$\sum_{m=1}^{+\infty} \left\{ \mathcal{L}[u_m(\mathbf{x}, t)] - \sum_{n=1}^{m-1} \alpha_n(\mathbf{x}, t) \mathcal{L}[u_{m-n}(\mathbf{x}, t)] \right\} = 0.$$

Similarly, since $\sum_{k=1}^{+\infty} \beta_{k-1}(\mathbf{x}, t)$ is absolutely convergent, we have due to the theorem of Cauchy product that

$$\begin{aligned}
&\sum_{m=1}^{+\infty} \sum_{k=1}^m \beta_{k-1}(\mathbf{x}, t) \delta_{m-k}(\mathbf{x}, t) \\
&= \sum_{k=1}^{+\infty} \sum_{m=k}^{+\infty} \beta_{k-1}(\mathbf{x}, t) \delta_{m-k}(\mathbf{x}, t) \\
&= \sum_{k=1}^{+\infty} \sum_{n=0}^{+\infty} \beta_{k-1}(\mathbf{x}, t) \delta_n(\mathbf{x}, t) \\
&= \left[\sum_{k=1}^{+\infty} \beta_{k-1}(\mathbf{x}, t) \right] \left[\sum_{n=0}^{+\infty} \delta_n(\mathbf{x}, t) \right] \\
&= \left[\sum_{m=0}^{+\infty} \beta_m(\mathbf{x}, t) \right] \left[\sum_{n=0}^{+\infty} \delta_n(\mathbf{x}, t) \right].
\end{aligned}$$

Thus, it holds

$$\left[\sum_{m=0}^{+\infty} \beta_m(\mathbf{x}, t) \right] \left[\sum_{n=0}^{+\infty} \delta_n(\mathbf{x}, t) \right] = 0,$$

which gives, since $\sum_{m=0}^{+\infty} \beta_m(\mathbf{x}, t) \neq 0$, that

$$\sum_{n=0}^{+\infty} \delta_n(\mathbf{x}, t) = 0,$$

i.e.

$$\sum_{n=0}^{+\infty} \mathcal{D}_n[\mathcal{N}(\phi, \Gamma)] = 0.$$

This ends the proof. $\square$

Theorem 4.22. *Let $\phi(\mathbf{x}, t; q)$ and $\Gamma(q)$ denote the solution of the zeroth-order deformation equation (4.60), whose homotopy-Maclaurin series read*

$$\phi \sim \sum_{m=0}^{+\infty} u_m(\mathbf{x}, t) q^m, \quad \Gamma \sim \sum_{m=0}^{+\infty} \gamma_m q^m.$$

Assume that the homotopy-series

$$u(\mathbf{x}, t) \sim \sum_{m=0}^{+\infty} u_m(\mathbf{x}, t), \quad \gamma \sim \sum_{m=0}^{+\infty} \gamma_m$$

converge, and besides $\sum_{m=0}^{+\infty} \mathcal{L}[u_m(\mathbf{x}, t)]$ also converges, where $\mathcal{L}$ is an auxiliary linear operator, so that Theorem 4.21 holds. If $\mathcal{N}(\phi, \Gamma)$ is analytic about q in $q \in [0, 1]$, then $\sum_{m=0}^{+\infty} u_m(\mathbf{x}, t)$ and $\sum_{m=0}^{+\infty} \gamma_m$ satisfy the original equation $\mathcal{N}(u, \gamma) = 0$.

Proof. Since the two homotopy-series

$$u(\mathbf{x}, t) \sim \sum_{m=0}^{+\infty} u_m(\mathbf{x}, t), \quad \gamma \sim \sum_{m=0}^{+\infty} \gamma_m$$

converge and besides $\sum_{m=0}^{+\infty} \mathcal{L}[u_m(\mathbf{x}, t)]$ is also convergent, we have by Theorem 4.21 that

$$\sum_{n=0}^{+\infty} \delta_n(\mathbf{x}, t) = \sum_{n=0}^{+\infty} \mathcal{D}_n[\mathcal{N}(\phi, \Gamma)] = 0.$$

Note that

$$\mathcal{D}_0[\mathcal{N}(\phi, \Gamma)] = \mathcal{N}(u_0, \gamma_0)$$

denotes the residual of the original governing equation for the initial approximations u_0 and γ_0 . So, $\mathcal{N}(\phi, \Gamma)$ can be regarded as the residual of the governing equation in $q \in [0, 1]$. According to Theorem 4.10, its homotopy-Maclaurin series reads

$$\mathcal{N}(\phi, \Gamma) \sim \sum_{k=0}^{+\infty} \delta_k(\mathbf{x}, t) q^k,$$

say,

$$\mathcal{N} \left(\sum_{m=0}^{+\infty} u_m q^m, \sum_{n=0}^{+\infty} \gamma_n q^n \right) \sim \sum_{k=0}^{+\infty} \delta_k(\mathbf{x}, t) q^k.$$

Since $\mathcal{N}(\phi, \Gamma)$ is analytic about q in $q \in [0, 1]$, then the above series converges to $\mathcal{N}(\phi, \Gamma)$ in $q \in [0, 1]$. Thus, it holds

$$\mathcal{N} \left(\sum_{m=0}^{+\infty} u_m q^m, \sum_{n=0}^{+\infty} \gamma_n q^n \right) = \sum_{k=0}^{+\infty} \delta_k(\mathbf{x}, t) q^k.$$

Setting $q = 1$ and using $\sum_{n=0}^{+\infty} \delta_n(\mathbf{x}, t) = 0$, we have

$$\mathcal{N} \left(\sum_{m=0}^{+\infty} u_m, \sum_{n=0}^{+\infty} \gamma_n \right) = \sum_{k=0}^{+\infty} \delta_k(\mathbf{x}, t) = 0.$$

Therefore, the two convergent homotopy-series satisfy the original equation. This ends the proof. $\square$

Theorem 4.22 reveals the importance of the convergence of homotopy-series. Due to this theorem, it is enough for us to guarantee the convergence of every homotopy-series given by the HAM. Theorem 4.21 indicates a necessary condition of the convergence of homotopy-series. It is very interesting that the homotopy-Maclaurin series of the residual of the original governing equation reads

$$\mathcal{N}(\phi, \Gamma) \sim \sum_{k=0}^{+\infty} \delta_k(\mathbf{x}, t) q^k,$$

where $\delta_k(\mathbf{x}, t) = \mathcal{D}_k[\mathcal{N}(\phi, \Gamma, q)]$. Define

$$\Delta_m = \int_{\Omega} \left[\sum_{k=0}^m \delta_k(\mathbf{x}, t) \right]^2 d\Omega. \quad (4.63)$$

According to Theorem 4.22, if each homotopy-series converges, then it holds

$$\lim_{m \rightarrow +\infty} \Delta_m = 0.$$

Therefore, Δ_m defined above indicates the accuracy of the m th-order homotopy-approximation. Obviously, the smaller the value of Δ_m , the better the corresponding homotopy-approximation. When the homotopy-approximations contain one convergence-control parameter c_0 , Δ_m is a function of c_0 , so that the optimal value of c_0 is determined by the minimum of Δ_m . This provides us a simple way to determine the optimal convergence-control parameter. Note that the term

$$\delta_k(\mathbf{x}, t) = \mathcal{D}_k[\mathcal{N}(\phi, \Gamma)]$$

is on the right-hand side of the high-order deformation equation, thus we need not spend additional CPU time to calculate it. So, it is more efficient to calculate Δ_m than to directly calculate the squared residual of the original equation, i.e.

$$\bar{\Delta}_m = \int_{\Omega} \left[\mathcal{N} \left(\sum_{n=0}^m u_n, \sum_{k=0}^m \gamma_k \right) \right]^2 d\Omega. \quad (4.64)$$

4.5 Solution expression

For a given nonlinear equation, the key of the HAM is to construct a good enough zeroth-order deformation equation by means of choosing a proper initial approximation u_0 and a proper auxiliary linear operator $\mathcal{L}$. As shown above, we have extremely large freedom to choose the initial approximation u_0 and the auxiliary linear operator $\mathcal{L}$: it is such kind of fantastic freedom that differs the HAM from other analytic techniques.

However, anything has its bright and dark sides. The extremely large freedom of constructing a zeroth-order deformation equation makes it difficult for a beginner to apply the HAM. Thus, for multifarious applications of the HAM in science and engineering, we need some rules to guide the choice of the initial approximation u_0 and the auxiliary linear operator $\mathcal{L}$.

It is well-known that the starting-point of perturbation techniques is the so-called small/large physical parameter, i.e. perturbation quantity. However, the HAM has nothing to do with any small/large physical parameters (this is one of the advantages of the HAM). So, we need a new but different starting-point for the HAM.

In essence, to analytically approximate a function $f(t)$ in a domain $t \in \Omega$ is to express it by a complete set of base functions, say,

$$f(t) \sim \sum_{k=1}^{+\infty} a_k e_k(t),$$

where $e_k(t)$ denotes the base function and a_k is a coefficient. The choice of the proper base functions are determined not only by the property of $f(t)$ but also by the domain Ω . For example, if $f(t)$ is periodic, it is convenient to choose periodic base functions $e_k(t)$. Besides, according to Weierstrass's Theorem (Mason and Handscomb, 2003), for any given $f(t)$ in $C[a, b]$ and for any given $\varepsilon > 0$, there exists a polynomial p_n for some sufficiently large n such that $\|f(t) - p_n\| < \varepsilon$. Therefore, for a given function $u(\mathbf{x}, t)$, the key-point is to find a proper set of base functions $e_k(\mathbf{x}, t)$ to fit it, i.e.

$$u(\mathbf{x}, t) \sim \sum_{k=1}^{+\infty} a_k e_k(\mathbf{x}, t),$$

where $e_k(\mathbf{x}, t)$ denotes the base function.

Definition 4.4. Let $e_k(\mathbf{x}, t)$ denote the base function, $u(\mathbf{x}, t)$ be a solution of a nonlinear equation $\mathcal{N}(u) = 0$, respectively. Then,

$$u(\mathbf{x}, t) \sim \sum_{k=1}^{+\infty} a_k e_k(\mathbf{x}, t), \quad (4.65)$$

is called *the solution expression* of $u(\mathbf{x}, t)$, if

$$\lim_{m \rightarrow \infty} \left\| u(\mathbf{x}, t) - \sum_{k=1}^m a_k e_k(\mathbf{x}, t) \right\| \rightarrow 0. \quad (4.66)$$

Note that a function can be expressed by different base functions. For example, an arbitrary continuous function $f(t) \in C[-1, 1]$ has a best approximation by means of Chebyshev series, i.e.

$$f(t) \sim \sum_{n=0}^{+\infty} b_n T_n(t),$$

where Chebyshev polynomial $T_n(t)$ of the first kind is a polynomial in t of degree n , defined by the relation (Mason and Handscomb, 2003):

$$T_n(t) = \cos(n\theta)$$

with the recursion formula

$$T_0(t) = 1, \quad T_1(t) = t,$$

$$T_n(t) = 2t T_{n-1}(t) - T_{n-2}(t), \quad n = 2, 3, 4, \dots$$

Besides, the solution-expression of $f \in C[a, b]$ can be a polynomial, according to Weierstrass's Theorem, or a Fourier series (Mason and Handscomb, 2003).

In fact, finding the so-called solution-expression of a given equation $\mathcal{N}(u) = 0$ is the destination (or goal) of solving the equation. In other words, it is the end-point for us. However, such kind of end-point is used as the starting-point of the HAM. Given a nonlinear differential equation $\mathcal{N}(u) = 0$, one should first ask himself an important question: what kinds of base functions can be used to approximate the unknown solution u ? For some types of equations, such as equations with continuous solutions defined in a finite domain, it is easy to answer this question. However, the answer of this question is not always obvious, especially for some new types of equations whose physical backgrounds are not very clear. In this case, it is helpful to gain some asymptotic properties of solutions. The more, the better. These asymptotic properties often provide us valuable information about the solution expression of a nonlinear equation.

Some rules are given here for the choice of the solution-expression of a given equation $\mathcal{N}(u) = 0$:

1. Equations defined in a finite domain:

- Polynomials and Fourier series can be always used as the solution-expression;
 - Chebyshev series gives the best approximation for arbitrary continuous solutions of $\mathcal{N}(u) = 0$.
2. Equations defined in an infinite domain:
- Periodic base functions should be used, if the solution is periodic;
 - Non-periodic base functions should be used, if the solution is not periodic;
 - The solution-expression should satisfy as many asymptotic properties of solution as possible, if the solution is not periodic.

Note that the above rules are not absolute, especially when the unknown solution is non-periodic and defined in an infinite domain. Fortunately, as mentioned before, even a unique solution of $\mathcal{N}(u) = 0$ can be often expressed by different types of base functions. So, even if one has little information about the solution-expression of a given equation $\mathcal{N}(u) = 0$, one can always guess some forms of its solution-expression and then check whether these guesses are correct or not.

The concept of solution-expressions of nonlinear equations is easy to understand for applied mathematicians who have rich physical knowledge. For example, it is well-known that oscillations of a conservative dynamic system are mostly periodic, although its period and amplitude of oscillations are unknown. Besides, it is a well-known knowledge that laminar viscous flows vary greatly near a solid boundary (i.e. the boundary-layer flow) but tend to the uniform flow exponentially at infinity. So, solution-expressions of nonlinear differential equations related to laminar viscous flows in fluid mechanics contain the exponential functions which exponentially tend to zero at infinity. Furthermore, all periodic traveling waves can be expressed by periodic base functions. All of these physical knowledge, if possible, are rather helpful for the choice of solution-expression of a given nonlinear equation.

The concept of the so-called solution-expression mentioned above is important in the frame of the HAM, because it is the start-point for us to choose the initial approximation u_0 and the auxiliary linear operator $\mathcal{L}$, as shown below.

4.5.1 Choice of initial approximation

Assume that a solution expression (4.65) is chosen for a given equation $\mathcal{N}(u) = 0$. Our aim is to find a proper initial approximation u_0 and a proper auxiliary linear operator $\mathcal{L}$ such that the corresponding homotopy-series converge.

Obviously, the initial approximation u_0 must obey the so-called solution expression (4.65). Since we have freedom to choose the initial approximation, we could choose such a kind of initial approximation

$$u_0(\mathbf{x}, t) \sim \sum_{k=1}^{n_0} \bar{a}_k e_k(\mathbf{x}, t), \quad (4.67)$$

where $e_k(\mathbf{x}, t)$ is the base function, $\bar{a}_k$ is unknown constant, and n_0 is equal to or greater than the number of linear boundary/initial conditions¹, denoted by κ . Then, enforcing the above initial approximation to satisfy the κ boundary/initial linear conditions, we have $n_0 - \kappa$ unknown coefficients left. So, when $n_0 = \kappa$, then all coefficients of the initial approximation are known so that it is completely determined. However, when $n_1 = n_0 - \kappa > 0$, we have n_1 unknown coefficients, denoted by $b_1, b_2, \dots, b_{n_1}$. To gain an optimal initial approximation, we define the squared residual of the governing equation

$$E_0(b_1, b_2, \dots, b_{n_1}) = \int_{\Omega} [\mathcal{N}(u_0)]^2 d\Omega. \quad (4.68)$$

It is well-known that the minimum of the squared residual $E_0(b_1, b_2, \dots, b_{n_1})$ is determined by a set of nonlinear algebraic equations

$$\frac{\partial E_0}{\partial b_k} = 0, \quad 1 \leq k \leq n_1,$$

whose solution gives the optimal values $b_1^*, b_2^*, \dots, b_{n_1}^*$ of the unknown coefficients. In this way, we obtain an optimal approximation u_0 . Obviously, the larger the number n_1 is, i.e. there are more unknown coefficients, the better the optimal initial approximation, but it needs more CPU time to solve the set of more complicated nonlinear algebraic equations. In case of very large n_1 , the set of nonlinear algebraic equations become difficult to solve, and it becomes the method of least squares. So, a balance is needed. In practice, it is often suggested to have one unknown coefficient in the initial approximation (i.e. $n_1 = 1$), whose optimal value is then determined by the minimum of the squared residual defined by (4.68). Mostly, such kind of optimal initial approximations with one optimal coefficient are good enough, as illustrated in Chapter 2.

Therefore, as long as a solution-expression of a given equation $\mathcal{N}(u) = 0$ is known, it is straight-forward to gain an optimal initial approximation u_0 in the way mentioned above.

4.5.2 Choice of auxiliary linear operator

To obey the so-called solution-expression (4.65) of a given equation $\mathcal{N}(u) = 0$, the initial guess u_0 and the auxiliary linear operator $\mathcal{L}$ must be chosen so that the solution u_m of the high-order deformation equation exists and obeys (4.65), and besides the homotopy-series

$$u_0 + \sum_{m=1}^{+\infty} u_m$$

¹ For simplicity, we assume here that all boundary/initial conditions are linear. The HAM also works for nonlinear boundary/initial conditions, as shown in Chapter 15 and Chapter 16.

converges.

The choice of the auxiliary linear operator $\mathcal{L}$ is mainly determined by the solution-expression (4.65), but sometimes also by the boundary/initial conditions. For example, if the solution $u(t)$ is a periodic function with a known frequency ω , then the auxiliary linear operator should be

$$\mathcal{L}(u) = u'' + \omega^2 u,$$

where the prime denotes the differentiation with respect to t . If the solution $u(t)$ is a periodic function with unknown frequency ω , we first use the transform $\tau = \omega t$ and then² choose such an auxiliary linear operator

$$\mathcal{L}(u) = u'' + u,$$

where the prime denotes the differentiation with respect to τ . If the solution-expression of $u(t)$ is polynomial, then one can simply choose the auxiliary linear operator

$$\mathcal{L}(u) = \frac{d^\sigma u}{dt^\sigma},$$

where the positive integer $\sigma > 0$ denotes the highest order of derivative of $\mathcal{N}(u) = 0$.

In general, let the integer $\sigma > 0$ denote the highest-order of derivative of an ordinary differential equation $\mathcal{N}[u(t)] = 0$, $u_m^*(t)$ a special solution of the corresponding m th-order deformation equation, respectively. Let

$$\mathcal{L}(u) = u^{(\sigma')} + \sum_{k=1}^{\sigma'} \mu_k(t) u^{(\sigma'-k)} \quad (4.69)$$

denote the unknown auxiliary linear operator, where $u^{(k)}$ denotes the k th-order derivative of $u(t)$, σ' is the highest order of derivative, and the unknown coefficient $\mu_i(t)$ is determined later. Let

$$u_m(t) = u_m^*(t) + \sum_{k=1}^{\sigma'_1} A_k e_k(t) + \sum_{k=1}^{\sigma'_2} B_k \bar{e}_k(t), \quad (4.70)$$

denote the common solution of the m th-order deformation equation, where $\sigma'_1 + \sigma'_2 = \sigma'$, A_k and B_k are unknown coefficients, $e_k(t)$ is the base functions, but $\bar{e}_k(t)$ is not, i.e.

$$\bar{e}_k(t) \notin \{e_1(t), e_2(t), e_3(t), \dots\}.$$

Obviously, to obey the solution expression, it must hold

$$B_k = 0, \quad 1 \leq k \leq \sigma'_2. \quad (4.71)$$

² In this case, the unknown ω in the governing equation is often replaced by its homotopy-Maclaurin series $\Omega(q) = \omega_0 + \sum_{k=1}^{+\infty} \omega_k q^k$.

Thus, the common solution reads

$$u_m(t) = u_m^*(t) + \sum_{k=1}^{\sigma'_1} A_k e_k(t). \quad (4.72)$$

The unknown auxiliary linear operator $\mathcal{L}$ must be so chosen that the σ'_1 unknown coefficients A_k of the above expression are *uniquely* determined by means of *all* related boundary/initial conditions of the m th-order deformation equation, say, the solution $u_m(t)$ uniquely exists and besides obeys the solution expression. If this is not true, then we had to change the number σ'_1 until it is satisfied. If this comes true, then the unknown coefficients $\mu_k(t)$ ($1 \leq k \leq \sigma'$) of the corresponding auxiliary linear operator $\mathcal{L}$ is determined by solving the set of linear algebraic equations

$$\mathcal{L}[e_k(t)] = 0, \quad 1 \leq k \leq \sigma'_1$$

and

$$\mathcal{L}[\bar{e}_k(t)] = 0, \quad 1 \leq k \leq \sigma'_2.$$

In this way, we obtain the auxiliary linear operator $\mathcal{L}$ defined by (4.69). For example, please refer to Liao and Tan (2007). Note that, although one can choose $\sigma' = \sigma$ in most cases, this is however not absolutely necessary, as shown in Chapter 2, mainly because we have extremely large freedom to choose the auxiliary linear operator $\mathcal{L}$. Similarly, the above method can be used to find a proper auxiliary linear operator for nonlinear partial differential equations, too.

It is interesting that the zeroth-order deformation equation

$$(1 - q)\mathcal{L}(\phi - u_0) = c_0 q \mathcal{N}(\phi), \quad q \in [0, 1], \quad c_0 \neq 0$$

can be rewritten in the form

$$(1 - q)\bar{\mathcal{L}}(\phi - u_0) = q \mathcal{N}(\phi),$$

where $\bar{\mathcal{L}} = \mathcal{L}/c_0$, and c_0 is the convergence-control parameter. So, in essence, choosing the convergence-control parameter c_0 is a part of choosing the auxiliary linear operator. Therefore, the optimal convergence-control parameter corresponds to the optimal auxiliary linear operator! Let $\|\mathcal{L}^{-1}\|$ denote the norm of the inverse operator $\mathcal{L}^{-1}$. Then, we have $\|\bar{\mathcal{L}}^{-1}\| = |c_0| \|\mathcal{L}^{-1}\|$. Thus, we can adjust the norm of $\mathcal{L}^{-1}$, and this is the essential reason why we can guarantee the convergence of homotopy-series by means of choosing a proper convergence-control parameter c_0 .

Similarly, the more generalized zeroth-order deformation equation

$$(1 - q)\mathcal{L}(\phi - u_0) = q \left(\sum_{k=0}^{+\infty} c_k q^k \right) \mathcal{N}(\phi), \quad q \in [0, 1], \quad c_0 \neq 0$$

can be rewritten in the “basic” form

$$(1 - q)\tilde{\mathcal{L}}(\phi - u_0) = q \mathcal{N}(\phi),$$

where c_k is convergence-control parameter and

$$\tilde{\mathcal{L}} = \left(\sum_{k=0}^{+\infty} c_k q^k \right)^{-1} \mathcal{L}.$$

Therefore, choosing the convergence-control parameters c_k is essentially a part of choosing the auxiliary linear operator $\tilde{\mathcal{L}}$. Obviously, the norm $\|\tilde{\mathcal{L}}^{-1}\|$ is determined by the convergence-control parameters c_k , and the optimal convergence-control parameters c_k correspond to an optimal auxiliary linear operator $\tilde{\mathcal{L}}$. In other words, we choose the auxiliary linear operator in two steps: the “basic” auxiliary linear operator $\mathcal{L}$ is first chosen according to the so-called solution-expression, then the auxiliary linear operator as a whole is modified by choosing optimal convergence-control parameters. Therefore, even if the “basic” auxiliary linear operator $\mathcal{L}$ is not perfect, we can still guarantee the convergence of homotopy-series by choosing optimal convergence-control parameters c_k .

In summary, the *solution-expression* is an important concept of the HAM, which provides us a start-point to choose the initial approximation and the “basic” auxiliary linear operator $\mathcal{L}$.

4.6 Convergence control and acceleration

Another important concept of the HAM is the *convergence-control*: the convergence of homotopy-series is guaranteed by choosing optimal convergence-control parameters.

According to Theorem 4.22, it is very important to guarantee the convergence of homotopy-series. However, it is a pity that there do not exist any mathematical theorems which can guide us in details how to construct a good enough homotopy of equations for any a given nonlinear equation so as to gain its convergent series solution. This is mainly because nonlinear equations differ in thousands ways so that it seems rather difficult to give a common approach for all of them.

In theory, the convergence of homotopy-series is strongly dependent upon the initial approximation and the auxiliary linear operator as a whole. As shown in Sect. 4.5.1, it is easy to choose an optimal initial approximation that obeys the given solution-expression. Then, we use such a strategy to choose the auxiliary linear operator as a whole: a “basic” auxiliary linear operator $\mathcal{L}$ is chosen first by means of the given solution-expression, and then some unknown convergence-control parameters are introduced into the zeroth-order deformation equation, whose optimal values are determined by the minimum of the squared residual of the original equation $\mathcal{N}(u) = 0$. In this way, the convergence of the homotopy-series is “controlled” by the so-called convergence-control parameters.

The convergence-control of homotopy-series is similar to the control of a dynamic system (Levine, 1996). Here, “the desired output” is the minimum of the

squared residual of a given equation $\mathcal{N}(u) = 0$, “the inputs” are the unknown convergence-control parameters, which have no physical meanings at all, as mentioned in Chapter 2. It is the concept of convergence-control that differs the HAM from all other analytic techniques, such as perturbation techniques (Cole, 1992; Hilton, 1953; Hinch, 1991; Murdock, 1991; Nayfeh, 1973, 2000), Lyapunov’s artificial small parameter method (Lyapunov, 1992), Adomian decomposition method (Adomian, 1976, 1991, 1994), the δ -expansion method (Karmishin et al., 1990) and so on.

4.6.1 Optimal convergence-control parameter

Our aim is to construct a good enough zeroth-order deformation equation for a given nonlinear equation

$$\mathcal{N}(u) = 0$$

so that the corresponding homotopy-series converge, i.e. the squared residual of the m th-order homotopy-approximation

$$E_m(c_0, c_1, \dots, c_\kappa) = \int_{\Omega} \left[\mathcal{N} \left(\sum_{n=0}^m u_n \right) \right]^2 d\Omega \quad (4.73)$$

tends to zero as $m \rightarrow +\infty$, where c_i ($0 \leq i \leq \kappa$) is the convergence-control parameter. Obviously, at a given order m of approximation, the optimal homotopy-approximation is given by the minimum of the squared residual E_m , and the corresponding optimal convergence-control parameters c_n^* are determined by a set of $(\kappa + 1)$ nonlinear algebraic equations

$$\frac{\partial E_m}{\partial c_n} = 0, \quad 0 \leq n \leq \kappa. \quad (4.74)$$

In theory, the more convergence-control parameters we have, the better the optimal homotopy-approximation. Mostly, even one optimal convergence-control parameter can greatly accelerate the convergence of homotopy-series solution, as shown in Chapter 2. In general, one or two convergence-control parameters are enough to give accurate homotopy-approximations, as illustrated by Liao (2010b).

According to Theorem 4.21, the approximate squared residual

$$E_m^\delta(c_0, c_1, \dots, c_\kappa) = \int_{\Omega} \left(\sum_{k=0}^m \delta_k \right)^2 d\Omega, \quad (4.75)$$

tends to zero as $m \rightarrow +\infty$, if the homotopy-Maclaurin series $\phi = \sum_{n=0}^{+\infty} u_n q^n$ converges at $q = 1$, where $\delta_k = \mathcal{D}_k[\mathcal{N}(\phi)]$. Thus, as an alternative, the optimal convergence-

control parameters c_n^* ($0 \leq n \leq \kappa$) are determined, approximately, by the minimum of E_m^δ , i.e. a set of the $(\kappa + 1)$ nonlinear algebraic equations

$$\frac{\partial E_m^\delta}{\partial c_n} = 0, \quad 0 \leq n \leq \kappa. \quad (4.76)$$

Note that the term $\delta_k = \mathcal{D}_k[\mathcal{N}(\phi)]$ is on the right-hand side of the high-order deformation equation and thus can be regarded as known, i.e. we need not additional CPU time to calculate it. So, it is more efficient to use E_m^δ to gain the optimal convergence-control parameters c_n , where $0 \leq n \leq \kappa$. It is found that the optimal convergence-control parameters given by the above approach are rather close to those given by the exact squared residual E_m of governing equations.

4.6.2 Optimal initial approximation

As illustrated in Chapter 2, the optimal initial approximation can greatly accelerate the convergence of homotopy-series. It is relatively simple to get an optimal initial approximation, as shown in Sect. 4.5.1. First, we introduce n_1 unknown coefficient $b_1, b_2, \dots, b_{n_1}$ into the initial approximation u_0 , which satisfies as many boundary/initial conditions as possible. The optimal values of these unknown coefficients are determined by the minimum of E_0 defined by (4.68), which gives a set of the corresponding n_1 nonlinear algebraic equations

$$\frac{\partial E_0}{\partial b_k} = 0, \quad 1 \leq k \leq n_1.$$

Note that the optimal initial approximation has an influence on the choice of the optimal convergence-control parameters. In many cases, if both of the optimal initial approximation and the optimal convergence-control parameters are used, the homotopy-series converges rather fast, as shown in Chapter 2. So, the optimal initial approximation is strongly suggested to use in the frame of the HAM to accelerate the convergence.

4.6.3 Homotopy-iteration technique

Obviously, a better initial approximation gives a better homotopy-approximation. Since we have great freedom to choose the initial approximation in zeroth-order deformation equations, we can replace the initial approximation u_0 by means of the m th-order homotopy-approximation

$$u \sim u_0 + \sum_{n=1}^m u_n,$$

which is mostly better than the initial approximation u_0 . The above expression is called the m th-order homotopy-iteration formula. For a renewed initial approximation, we can choose the corresponding optimal convergence-control parameters in a similar way as mentioned above. As shown in Chapter 2, such kind of homotopy-iteration method can greatly accelerate the convergence of homotopy-series.

The key-point of the homotopy-iteration method is the truncation of the solution expression, i.e.

$$u_m(\mathbf{x}, t) \approx \sum_{k=1}^M a_{m,k} e_k(\mathbf{x}, t), \quad (4.77)$$

where $e_k(\mathbf{x}, t)$ is the base function, $M > 0$ is a large enough integer, respectively. In other words, the homotopy-approximation contains at most the first M base functions. For this reason, we often rewrite the right-hand side term of the high-order deformation equation in a truncated form

$$\mathcal{D}_{m-1}[\mathcal{N}(\phi)] \approx \sum_{k=1}^M c_{m,k} e_k(\mathbf{x}, t), \quad (4.78)$$

where $e_k(\mathbf{x}, t)$ is the base function. Note that the coefficient $c_{m,k}$ in above expression is easy to calculate when the base functions $e_k(\mathbf{x}, t)$ are orthogonal, i.e.

$$c_{m,k} = \frac{\langle \mathcal{D}_{m-1}[\mathcal{N}(\phi)], e_k(\mathbf{x}, t) \rangle}{\langle e_k(\mathbf{x}, t), e_k(\mathbf{x}, t) \rangle}, \quad (4.79)$$

where $\langle x, y \rangle$ is an inner product of x and y .

4.6.4 Homotopy-Padé technique

The so-called Padé approximant developed by French mathematician Henri Eugène Padé (1863 — 1953) is widely applied, which gives the “best” approximation of a given function by a rational function of given order. For a power series

$$\sum_{n=0}^{+\infty} c_n z^n,$$

the corresponding $[m, n]$ Padé approximant is expressed by

$$\frac{\sum_{k=0}^m a_{m,k} z^k}{\sum_{k=0}^n b_{n,k} z^k},$$

where $a_{m,k}, b_{n,k}$ are determined by the coefficients c_j ($j = 0, 1, 2, 3, \dots, m+n$).

The so-called homotopy-Padé technique is a combination of the traditional Padé technique with the homotopy analysis method. Regarding a homotopy-Maclaurin series

$$u(\mathbf{x}, t; q) \sim u_0(\mathbf{x}, t) + \sum_{n=1}^{+\infty} u_n(\mathbf{x}, t) q^n$$

of a given nonlinear equation $\mathcal{N}(u) = 0$ as a power series of q , we first employ the traditional $[m, n]$ Padé technique about the homotopy-parameter q to obtain the $[m, n]$ Padé approximant

$$\frac{\sum_{k=0}^m A_{m,k}(\mathbf{x}, t) q^k}{\sum_{k=0}^n B_{m,k}(\mathbf{x}, t) q^k}, \quad (4.80)$$

where the coefficients $A_{m,k}(\mathbf{x}, t)$ and $B_{m,k}(\mathbf{x}, t)$ are determined by the first $(m+n)$ terms

$$u_0(\mathbf{x}, t), u_1(\mathbf{x}, t), u_2(\mathbf{x}, t), \dots, u_{m+n}(\mathbf{x}, t)$$

of the homotopy-Maclaurin series. Since the homotopy-approximation is obtained at $q = 1$, setting $q = 1$ in (4.80), we have the so-called $[m, n]$ homotopy-Padé approximation

$$u(\mathbf{x}, t; q) \approx \frac{\sum_{k=0}^m A_{m,k}(\mathbf{x}, t)}{\sum_{k=0}^n B_{m,k}(\mathbf{x}, t)}. \quad (4.81)$$

In general, the homotopy-Padé technique can greatly accelerate the convergence of homotopy-series, as shown in Chapter 2.

4.7 Discussions and open questions

In this chapter, the HAM is systematically described in details as a whole. Mathematical theorems related to the so-called homotopy-derivative operator and deformation equations are proved, which are helpful to gain high-order approximations. Some theorems of convergence are proved, and the methods to control and accelerate convergence are generally described.

Note that, based on the homotopy of topology, the HAM provides us extremely large freedom to construct zeroth-order deformation equation. Such kind of fantastic freedom provides us great flexibility to choose initial approximation and auxiliary linear operator $\mathcal{L}$ so as to construct zeroth-order deformation equation in a quite general form, as shown in Sect. 4.3. Especially, it is due to this kind of freedom that we can introduce the so-called convergence-control parameters into the zeroth-order deformation equation, which provide us a convenient way to guarantee the convergence of homotopy-series solution.

The “solution-expression” and “convergence-control” are two important concepts in the frame of the HAM. The solution-expression provides a start-point and a guide to choose the initial approximation and the “basic” auxiliary linear operator $\mathcal{L}$. The so-called convergence-control parameters are used to control and accelerate the convergence of homotopy-series solution. In essence, it is the so-called convergence-control parameter that differs the HAM from all other analytic techniques. In the frame of the HAM, the convergence of homotopy-approximation can be controlled and greatly accelerated by means of optimal initial approximation, and/or optimal convergence-control parameters, and/or iteration approach. Therefore, unlike other analytic approximation methods, the HAM is valid even for highly nonlinear problems.

However, nonlinear problems are often hard to understand in essence. Naturally, there are some open questions for the HAM. First, for a given nonlinear equation $\mathcal{N}(u) = 0$ in general, there are no mathematical theorems up to now, which can clearly guide us in details how to construct a good enough homotopy of equations so as to gain a convergent series solution. It seems very difficult to give such a kind of general theorems, because nonlinear equations differ in thousand ways. But, such a theorem, if it indeed exists, could heighten the HAM in theory and greatly simplify its applications, especially for beginners of the HAM. If such a theorem indeed exists, it would be possible to develop a code, which can automatically solve most of nonlinear equations by computer algebra system like Mathematica, Maple and so on.

Secondly, the zeroth-order deformation equation

$$(1 - q)\mathcal{L}(\phi - u_0) = c_0 q \mathcal{N}(\phi)$$

can be rewritten in the “basic” form

$$(1 - q)\tilde{\mathcal{L}}(\phi - u_0) = q \mathcal{N}(\phi),$$

where $\mathcal{L}$ is the “basic” auxiliary linear operator and $\tilde{\mathcal{L}} = \mathcal{L}/c_0$ is the auxiliary linear operator as a whole, u_0 is the initial approximation and ϕ is a homotopy-Maclaurin series, respectively. Since

$$\|\tilde{\mathcal{L}}^{-1}\| = |c_0| \|\mathcal{L}^{-1}\|, \quad (4.82)$$

for any a given small $\varepsilon > 0$, we can always find such a convergence-control parameter c_0 that

$$\|\tilde{\mathcal{L}}^{-1}\| < \varepsilon, \quad (4.83)$$

if $\|\mathcal{L}^{-1}\|$ is bounded. So, it seems that, if $\|\mathcal{L}^{-1}\|$ is bounded, and if the initial approximation u_0 is close enough to the exact solution u^* so that $\|u_0 - u^*\|$ is bounded, then we could always find such a convergence-control parameter c_0 that the corresponding homotopy-series converges. Unfortunately, we can not prove this guess in general up to now.

In addition, although in theory we have extremely large freedom to construct a zeroth-order deformation equation for a given nonlinear equation $\mathcal{N}(u) = 0$, it is not very clear how to use such kind of freedom, especially for the rather general forms of the zeroth-order deformation equations, such as (4.52), (4.54), (4.56) and (4.58). Currently, these rather generalized zeroth-order deformation equations are hardly used in practice. There are many related open questions. For example, how to find the “best” auxiliary linear operator for a given nonlinear differential equation in general? How to find the “best” convergence-control function $\beta_k(\mathbf{x}, t)$? What happens if the convergence-control functions $\beta_k(\mathbf{x}, t)$ are orthogonal to the terms $\mathcal{D}_k[\mathcal{N}(\phi)]$? How to obtain the “best” deformation-operator $\mathcal{A}$ in (4.56) for a given nonlinear equation in general? Is the HAM valid for complicated nonlinear problems related to chaos and turbulence?

So, there is still a long way ahead for us, although the HAM has been successfully applied to so many nonlinear problems in science, finance and engineering. Indeed, nonlinear problems are often hard to understand in essence, especially those related to chaos and turbulence. That is the reason why our aim is to develop an analytic approximation method valid for as *many* nonlinear problems as *possible*.

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Chapter 5

Relationship to Euler Transform

Abstract The so-called generalized Taylor series and homotopy-transform are derived in the frame of the homotopy analysis method (HAM). Some related theorems are proved, which reveal in theory the reason why convergence-control parameter provides us a convenient way to guarantee the convergence of the homotopy-series solution. Especially, it is proved that the homotopy-transform logically contains the famous Euler transform that is often used to accelerate convergence of a series or to make a divergent series convergent. All of these provide us a conner-stone for the concept of convergence-control and the great generality of the HAM.

5.1 Introduction

As mentioned in Chapter 2 and Chapter 3, the convergence-control is an important concept in the frame of the homotopy analysis method (HAM) (Liao, 1992, 1997, 1999a,b; Liao and Campo, 2002; Liao, 2003a,b, 2004, 2005, 2006, 2009, 2010a,b; Liao and Tan, 2007; Li et al., 2010; Xu et al., 2010): the convergence of homotopy-series can be guaranteed by means of a non-zero auxiliary parameter c_0 , called today the convergence-control parameter, that was first introduced into the zeroth-order deformation equations by Liao (1997). In Chapter 2, a nonlinear oscillation equation is used to illustrate how powerful and efficient such kind of convergence-control parameter is to guarantee the convergence of homotopy-series. Especially, we prove that, by means of introducing a non-zero auxiliary parameter c_0 , we can give a power series of $(1+z)^{-1}$ convergent in the *whole* domain except the singular point $z = -1$, i.e. $(-\infty, -1) \cup (-1, +\infty)$ (see Theorem 2.3). Note that the traditional Taylor series of $(1+z)^{-1}$ converges only in a bounded domain $|1+z| < 1$. This shows the validity and great potential of the HAM from another view-point.

The function $(1+z)^{-1}$ considered in Theorem 2.3 is simple and special. In 2009, Liao (2010a) considered a smooth function $f(z)$ in general, and derived a more generalized power series of $f(z)$ by introducing the convergence-control parameter c_0 and two deformation-functions $A(q)$ and $B(q)$, satisfying $A(0) = B(0) = 0$ and

$A(1) = B(1) = 1$. Such kind of power series is called the generalized Taylor series, since it logically contains the traditional Taylor series of $f(z)$. It is found that the convergence-region of the generalized Taylor series can be fantastically enlarged by the so-called convergence-control parameter c_0 , and this explains in theory why the convergence-control parameter c_0 can guarantee the convergence of homotopy-series obtained by the HAM. Besides, based on this generalized Taylor series, a kind of transform, called the homotopy-transform, is defined, which is more general than the famous Euler transform, as proved by Liao (2010a). Especially, Liao (2010a) illustrated that the so-called homotopy-transform can be derived in the frame of the HAM. This not only further confirms the generality and great potential of the HAM for strongly nonlinear problems, but also provides the HAM a solid base in mathematics.

5.2 Generalized Taylor series

The traditional Taylor series of a function $f(z)$, i.e.

$$f(z) \sim f(z_0) + \sum_{k=1}^{+\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k,$$

was formally introduced by the English mathematician Brook Taylor FRS (1685 — 1731) in 1715, which is called Maclaurin series when $z_0 = 0$, named after the Scottish mathematician Colin Maclaurin (1698 — 1746). The traditional Taylor series converges to $f(z)$ in the domain

$$\left| \frac{z - z_0}{\zeta - z_0} \right| < 1,$$

where ζ is a pole of $f(z)$ closest to z_0 . It is well known that the traditional Taylor series of many functions converge in a bounded domain. For example, the traditional Taylor series of $(1 + z)^{-1}$ is convergent only in an unit circle $|z| < 1$.

In Chapter 2, we give a definition of the so-called deformation-function by means of the real homotopy-parameter $q \in [0, 1]$. Here, we give a more general definition of a deformation function in a complex number z .

Definition 5.1. Let q be a complex number. A complex function $A(q)$ is called a *deformation-function* if it satisfies

$$A(0) = 0, \quad A(1) = 1 \tag{5.1}$$

and is analytic in the region $|q| \leq 1$ so that its Maclaurin series $\sum_{k=1}^{+\infty} a_k q^k$ is convergent in the region $|q| \leq 1$, say,

$$A(q) = \sum_{k=1}^{+\infty} a_k q^k, \quad |q| \leq 1.$$

Lemma 5.1. *Let $A(q)$ be a deformation function, whose Maclaurin series $A(q) = \sum_{k=1}^{+\infty} a_k q^k$ converges at $|q| \leq 1$. Then, it holds*

$$\left(\sum_{k=1}^{+\infty} a_k q^k \right)^m = \sum_{n=m}^{+\infty} \bar{a}_{m,n} q^n, \quad (5.2)$$

where

$$\bar{a}_{1,n} = a_n, \quad n \geq 1, \quad (5.3)$$

$$\bar{a}_{m,n} = \sum_{k=m-1}^{n-1} a_{n-k} \bar{a}_{m-1,k}, \quad n \geq m \geq 2. \quad (5.4)$$

Proof. Write

$$[A(q)]^m = \left(\sum_{k=1}^{+\infty} a_k q^k \right)^m = \sum_{k=m}^{+\infty} \bar{a}_{m,k} q^k, \quad m \geq 1, \quad (5.5)$$

where

$$\bar{a}_{m,k} = a_k, \quad \text{when } m = 1.$$

Assume that $\bar{a}_{m-1,k}$ ($k \geq m \geq 2$) are known, it follows that

$$\begin{aligned} [A(q)]^m &= \sum_{k=m}^{+\infty} \bar{a}_{m,k} q^k = \left(\sum_{k=1}^{+\infty} a_k q^k \right)^{m-1} \left(\sum_{k=1}^{+\infty} a_k q^k \right) \\ &= \left(\sum_{n=m-1}^{+\infty} \bar{a}_{m-1,n} q^n \right) \left(\sum_{j=1}^{+\infty} a_j q^j \right) = \sum_{k=m}^{+\infty} q^k \left(\sum_{n=m-1}^{k-1} \bar{a}_{m-1,n} a_{k-n} \right), \end{aligned}$$

which gives the recurrence formula

$$\bar{a}_{m,k} = \sum_{n=m-1}^{k-1} a_{k-n} \bar{a}_{m-1,n}, \quad k \geq m \geq 2.$$

This ends the proof. $\square$

Theorem 5.1. *Let q, z, z_0 and $c_0 \neq 0$ be complex numbers, $A(q)$ and $B(q)$ denote two deformation-functions satisfying $A(0) = B(0) = 0$, $A(1) = B(1) = 1$, whose Maclaurin series $\sum_{k=1}^{+\infty} a_k q^k$ and $\sum_{k=1}^{+\infty} b_k q^k$ are absolutely convergent in the region $|q| \leq 1$, and besides $|(1+c_0)B(q)| < 1$. Define*

$$T_{m,k}(c_0, A, B) = (-c_0)^k \sum_{n=0}^{m-k} \sum_{r=0}^n \binom{k+r-1}{r} (1+c_0)^r \sum_{s=0}^{n-r} \bar{a}_{k,k+s} \bar{b}_{r,n-s}, \quad (5.6)$$

where $m \geq 1$ and $1 \leq k \leq m$, and

$$\bar{a}_{1,k} = a_k, \quad \bar{b}_{1,k} = b_k, \quad \bar{b}_{0,0} = 1, \quad \bar{b}_{0,k} = 0, \quad k \geq 1, \quad (5.7)$$

$$\bar{a}_{m,k} = \sum_{n=m-1}^{k-1} a_{k-n} \bar{a}_{m-1,n}, \quad k \geq m \geq 2, \quad (5.8)$$

$$\bar{b}_{m,k} = \sum_{n=m-1}^{k-1} b_{k-n} \bar{b}_{m-1,n}, \quad k \geq m \geq 2. \quad (5.9)$$

If a complex function $f(z)$ is analytic at z_0 but singular at ξ_k ($k = 1, 2, \dots, M_0$), where M_0 may be infinity, the series

$$f(z_0) + \lim_{m \rightarrow +\infty} \sum_{k=1}^m \left[\frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \right] T_{m,k}(c_0, A, B), \quad (5.10)$$

converges to $f(z)$ in the region $D = \bigcap_{k=0}^M S_k$, where $S_k = \{z : |\zeta_k| > 1\}$, and ζ_k is the solution of either

$$1 - (1 + c_0) B(\zeta_k) = 0,$$

or

$$1 - (1 + c_0) B(\zeta_k) + c_0 \left(\frac{z - z_0}{\xi_n - z_0} \right) A(\zeta_k) = 0, \quad 1 \leq n \leq M_0.$$

Proof. According to Lemma 5.1, we have

$$[A(q)]^m = \left(\sum_{k=1}^{+\infty} a_k q^k \right)^m = \sum_{k=m}^{+\infty} \bar{a}_{m,k} q^k, \quad m \geq 1,$$

and

$$[B(q)]^m = \left(\sum_{k=1}^{+\infty} b_k q^k \right)^m = \sum_{k=m}^{+\infty} \bar{b}_{m,k} q^k, \quad m \geq 1,$$

where $\bar{a}_{m,k}, \bar{b}_{m,k}$ are defined by (5.7) to (5.9), respectively. For simplicity, we define

$$\bar{b}_{0,0} = 1, \quad \bar{b}_{0,k} = 0 \quad (k \geq 1).$$

For simplicity, define

$$\begin{aligned} \Gamma_k(z) &= \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k, \quad k \geq 1, \\ \tau &= z_0 - \frac{c_0(z - z_0) A(q)}{1 - (1 + c_0) B(q)}, \quad c_0 \neq 0, \end{aligned} \quad (5.11)$$

and construct such a related complex function

$$F(q) = f(\tau) = f \left[z_0 - \frac{c_0 (z - z_0) A(q)}{1 - (1 + c_0) B(q)} \right]. \quad (5.12)$$

Since $A(0) = B(0) = 0$ and $A(1) = B(1) = 1$, it holds $\tau = z_0$ when $q = 0$ and $\tau = z$ when $q = 1$, respectively. Therefore, we have

$$F(0) = f(z_0), \quad F(1) = f(z). \quad (5.13)$$

In other words, $F(q)$ is a homotopy, i.e. $F(q) : f(z_0) \sim f(z)$. Writing

$$\delta\tau = -\frac{c_0 (z - z_0) A(q)}{1 - (1 + c_0) B(q)}, \quad (5.14)$$

we have $F(q) = f(\tau) = f(z_0 + \delta\tau)$. If $|\delta\tau|$ is sufficiently small and $|(1 + c_0)B(q)| < 1$ holds, then the Maclaurin series of $F(q)$ at $q = 0$ reads

$$\begin{aligned} & f(z_0) + \sum_{k=1}^{+\infty} \frac{f^{(k)}(z_0)}{k!} (\delta\tau)^k \\ &= f(z_0) + \sum_{k=1}^{+\infty} \frac{f^{(k)}(z_0)}{k!} \left[-\frac{c_0 (z - z_0) A(q)}{1 - (1 + c_0) B(q)} \right]^k \\ &= f(z_0) + \sum_{k=1}^{+\infty} \Gamma_k(z) (-c_0)^k [A(q)]^k [1 - (1 + c_0) B(q)]^{-k} \\ &= f(z_0) + \sum_{k=1}^{+\infty} \Gamma_k(z) (-c_0)^k [A(q)]^k \sum_{r=0}^{+\infty} \binom{k+r-1}{r} (1+c_0)^r [B(q)]^r \\ &= f(z_0) + \sum_{k=1}^{+\infty} \sum_{r=0}^{+\infty} \Gamma_k(z) \binom{k+r-1}{r} (-c_0)^k (1+c_0)^r \left(\sum_{i=k}^{+\infty} \bar{a}_{k,i} q^i \right) \left(\sum_{j=r}^{+\infty} \bar{b}_{r,j} q^j \right) \\ &= f(z_0) + \sum_{k=1}^{+\infty} \sum_{r=0}^{+\infty} \Gamma_k(z) \binom{k+r-1}{r} (-c_0)^k (1+c_0)^r \sum_{s=k+r}^{+\infty} q^s \left(\sum_{i=k}^{s-r} \bar{a}_{k,i} \bar{b}_{r,s-i} \right) \\ &= f(z_0) + \sum_{s=1}^{+\infty} q^s \left[\sum_{k=1}^s \Gamma_k(z) (-c_0)^k \sum_{r=0}^{s-k} \binom{k+r-1}{r} (1+c_0)^r \left(\sum_{i=k}^{s-r} \bar{a}_{k,i} \bar{b}_{r,s-i} \right) \right] \\ &= f(z_0) + \sum_{n=1}^{+\infty} \sigma_n(z) q^n, \end{aligned} \quad (5.15)$$

where

$$\sigma_n(z) = \sum_{k=1}^n \Gamma_k(z) (-c_0)^k \sum_{r=0}^{n-k} \binom{k+r-1}{r} (1+c_0)^r \left(\sum_{i=k}^{n-r} \bar{a}_{k,i} \bar{b}_{r,n-i} \right). \quad (5.16)$$

Let ζ denote a singular point of $F(q)$. According to the definition (5.12), all solutions of the equation

$$1 - (1 + c_0)B(\zeta) = 0,$$

are the singular points of $F(q)$. Besides, each original singular point ξ_k ($1 \leq k \leq M_0$) of $f(z)$ gives corresponding singular points of $F(q)$, governed by the equation

$$z_0 - \frac{c_0 (z - z_0) A(\zeta)}{1 - (1 + c_0) B(\zeta)} = \xi_k, \quad 1 \leq k \leq M_0,$$

i.e.

$$1 - (1 + c_0) B(\zeta) + c_0 \left(\frac{z - z_0}{\xi_k - z_0} \right) A(\zeta) = 0, \quad 1 \leq k \leq M_0.$$

Let ζ_k ($1 \leq k \leq M$) denote these singular points of $F(q)$. Note that these singular points are dependent upon z, z_0 and c_0 . The Maclaurin series (5.15) converges to $F(1) = f(z)$ at $q = 1$, if and only if all singular points ζ_k of $F(q)$ are out of the region $|q| \leq 1$, say,

$$|\zeta_k| > 1, \quad 1 \leq k \leq M.$$

In this case, according to (5.15) and (5.16), we have

$$\begin{aligned} f(z) &= F(1) = f(z_0) + \lim_{m \rightarrow +\infty} \sum_{n=1}^m \sigma_n \\ &= f(z_0) + \lim_{m \rightarrow +\infty} \sum_{n=1}^m \sum_{k=1}^n \Gamma_k(z) (-c_0)^k \sum_{r=0}^{n-k} \binom{k+r-1}{r} (1+c_0)^r \left(\sum_{i=k}^{n-r} \bar{a}_{k,i} \bar{b}_{r,n-i} \right) \\ &= f(z_0) + \lim_{m \rightarrow +\infty} \sum_{k=1}^m \Gamma_k(z) \left[(-c_0)^k \sum_{n=k}^m \sum_{r=0}^{n-k} \binom{k+r-1}{r} (1+c_0)^r \left(\sum_{i=k}^{n-r} \bar{a}_{k,i} \bar{b}_{r,n-i} \right) \right] \\ &= f(z_0) + \lim_{m \rightarrow +\infty} \sum_{k=1}^m \left[\frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \right] T_{m,k}(c_0, A, B) \end{aligned}$$

in the domain $D = \bigcap_{k=0}^M S_k$, where $S_k = \{z : |\zeta_k| > 1\}$ and

$$\begin{aligned} &T_{m,k}(c_0, A, B) \\ &= (-c_0)^k \sum_{n=k}^m \sum_{r=0}^{n-k} \binom{k+r-1}{r} (1+c_0)^r \left(\sum_{i=k}^{n-r} \bar{a}_{k,i} \bar{b}_{r,n-i} \right) \\ &= (-c_0)^k \sum_{n=0}^{m-k} \sum_{r=0}^n \binom{k+r-1}{r} (1+c_0)^r \left(\sum_{i=k}^{n+k-r} \bar{a}_{k,i} \bar{b}_{r,n+k-i} \right) \end{aligned}$$

$$= (-c_0)^k \sum_{n=0}^{m-k} \sum_{r=0}^n \binom{k+r-1}{r} (1+c_0)^r \left(\sum_{s=0}^{n-r} \bar{a}_{k,k+s} \bar{b}_{r,n-s} \right).$$

This ends the proof. $\square$

Definition 5.2. Let $c_0 \neq 0$ denote the convergence-control parameter, $A(q)$ and $B(q)$ be two deformation-functions whose Maclaurin series $A(q) = \sum_{k=1}^{+\infty} a_k q^k$ and $B(q) = \sum_{k=1}^{+\infty} b_k q^k$ converge at $q = 1$ so that $\sum_{k=1}^{+\infty} a_k = 1$ and $\sum_{k=1}^{+\infty} b_k = 1$. If a complex function $f(z)$ is analytic at $z = z_0$, then

$$f(z_0) + \lim_{m \rightarrow +\infty} \sum_{k=1}^m \left[\frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \right] T_{m,k}(c_0, A, B) \quad (5.17)$$

is called the generalized Taylor series of $f(z)$, where $T_{m,k}(c_0, A, B)$ is defined by (5.6) and (5.7) – (5.9).

Note that the so-called generalized Taylor series (5.17) of a complex $f(z)$ at $z = z_0$ is dependent upon one auxiliary parameter c_0 and the two complex analytic functions $A(q)$ and $B(q)$ with their Maclaurin series

$$A(q) = \sum_{k=1}^{+\infty} a_k q^k, \quad B(q) = \sum_{k=1}^{+\infty} b_k q^k$$

under the restriction

$$\sum_{k=1}^{+\infty} a_k = 1, \quad \sum_{k=1}^{+\infty} b_k = 1.$$

Here, the two convergent series $\sum_{k=1}^{+\infty} a_k$ and $\sum_{k=1}^{+\infty} b_k$ are derived from the two analytic functions $A(q)$ and $B(q)$, the so-called deformation-functions. Alternatively, any two convergent series $\sum_{k=1}^{+\infty} a_k = 1$ and $\sum_{k=1}^{+\infty} b_k = 1$ can be used. For example,

$$a_k = (1 - \gamma) \gamma^{k-1}, \quad |\gamma| < 1$$

$$b_k = \frac{6}{(k\pi)^2},$$

completely define two deformation-functions $A(q)$ and $B(q)$, and $T_{m,k}(c_0, A, B)$ by (5.6). There are many such kinds of convergent series, which can be used to define $T_{m,k}(c_0, A, B)$.

Theorem 5.2. Let c_0 denote a convergence-control parameter, $A(q) = q$ and $B(q) = q$ denote two deformation-functions, respectively. If a real function $f(x)$ is analytic

in the whole domain $-\infty < x < +\infty$ except at the unique singular point $x = \xi$, its generalized Taylor series at $x_0 \neq \xi$, i.e.

$$f(x_0) + \lim_{m \rightarrow +\infty} \sum_{k=1}^m \left[\frac{f^{(k)}(x_0)}{k!} (x - x_0)^k \right] T_{m,k}(c_0, A, B)$$

converges to $f(x)$ in the region

$$1 - \frac{2}{|c_0|} < \frac{x - x_0}{\xi - x_0} < 1, \quad -2 < c_0 < 1,$$

which becomes an infinite domain

$$-\infty < \frac{x - x_0}{\xi - x_0} < 1 \quad \text{as } c_0 \rightarrow 0.$$

Proof. Since $A(q) = q$, $B(q) = q$ and there exists only one singular point $x = \xi$, according to Theorem 5.1, we have two singular points ζ_0 and ζ_1 , governed by

$$1 - (1 + c_0) \zeta_0 = 0$$

and

$$1 - (1 + c_0) \zeta_1 + c_0 \left(\frac{x - x_0}{\xi - x_0} \right) \zeta_1 = 0,$$

respectively. Obviously,

$$\zeta_0 = (1 + c_0)^{-1}$$

and

$$\zeta_1 = \left[1 + c_0 - c_0 \left(\frac{x - x_0}{\xi - x_0} \right) \right]^{-1}.$$

According to Theorem 5.1, the generalized Taylor series

$$f(x_0) + \lim_{m \rightarrow +\infty} \sum_{k=1}^m \left[\frac{f^{(k)}(x_0)}{k!} (x - x_0)^k \right] T_{m,k}(c_0, A, B)$$

converges to $f(x)$ in the region $\{|\zeta_0| > 1\} \cap \{|\zeta_1| > 1\}$. From $|\zeta_0| > 1$, we have $|1 + c_0| < 1$, i.e.

$$-2 < c_0 < 0.$$

Besides, $|\zeta_1| > 1$ gives

$$\left| 1 + c_0 - c_0 \left(\frac{x - x_0}{\xi - x_0} \right) \right| < 1,$$

i.e.

$$-2 - c_0 < -c_0 \left(\frac{x - x_0}{\xi - x_0} \right) < -c_0.$$

Since $-2 < c_0 < 0$, we have the convergence domain

$$1 - \frac{2}{|c_0|} < \frac{x - x_0}{\xi - x_0} < 1, \quad -2 < c_0 < 0,$$

which becomes

$$-\infty < \frac{x - x_0}{\xi - x_0} < 1 \quad \text{as } c_0 \rightarrow 0.$$

This ends the proof. $\square$

Therefore, if a real function $f(x)$ has only one singular point at $x = \xi$, then its generalized Taylor series (5.17) by $A(q) = B(q) = q$ as $c_0 \rightarrow 0$ converges in the infinite domain $(-\infty, \xi)$ when $x_0 < \xi$, or in the infinite domain $(\xi, +\infty)$ when $x_0 > \xi$. This explains in theory why the convergence-control parameter c_0 can greatly enlarge the convergence-region of a series.

Theorem 5.3. Let $c_0 \in \Re$ denote a real convergence-control parameter; $A(q) = q$ and $B(q) = q$ denote two deformation-functions, respectively. If a complex function $f(z)$ is analytic in the whole plane except at the unique singular point $z = \xi$, its generalized Taylor series at $z_0 \neq \xi$, i.e.

$$f(z_0) + \lim_{m \rightarrow +\infty} \sum_{k=1}^m \left[\frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \right] T_{m,k}(c_0, A, B)$$

converges to $f(z)$ in the region

$$\rho = \left| \frac{z - z_0}{\xi - z_0} \right| < \frac{(1 + c_0) \cos \theta - \sqrt{1 - (1 + c_0)^2 \sin^2 \theta}}{c_0}, \quad -2 < c_0 < 0,$$

which becomes an infinite domain

$$\rho = \left| \frac{z - z_0}{\xi - z_0} \right| < \begin{cases} \frac{1}{\cos \theta}, & \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ +\infty, & \text{otherwise,} \end{cases}$$

as $c_0 \rightarrow 0$, where

$$\eta = \frac{z - z_0}{\xi - z_0} = \rho e^{i\theta}, \quad \theta \in [0, 2\pi], \quad i = \sqrt{-1}.$$

Proof. Since $A(q) = B(q) = q$ and there exists only one singular point $z = \xi$, according to Theorem 5.1, we have two singular points ζ_0 and ζ_1 , governed by

$$1 - (1 + c_0) \zeta_0 = 0$$

and

$$1 - (1 + c_0) \zeta_1 + c_0 \left(\frac{z - z_0}{\xi - z_0} \right) \zeta_1 = 0,$$

respectively. Obviously,

$$\zeta_0 = (1 + c_0)^{-1}$$

and

$$\zeta_1 = \left[1 + c_0 - c_0 \left(\frac{z - z_0}{\xi - z_0} \right) \right]^{-1}.$$

According to Theorem 5.1, the generalized Taylor series

$$f(z_0) + \lim_{m \rightarrow +\infty} \sum_{k=1}^m \left[\frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \right] T_{m,k}(c_0, A, B)$$

converges to $f(z)$ in the region $\{|\zeta_0| > 1\} \cap \{|\zeta_1| > 1\}$. From $|\zeta_0| > 1$, we have $|1 + c_0| < 1$, i.e.

$$-2 < c_0 < 0.$$

Besides, $|\zeta_1| > 1$ gives

$$\left| 1 + c_0 - c_0 \left(\frac{z - z_0}{\xi - z_0} \right) \right| < 1.$$

Writing

$$\eta = \frac{z - z_0}{\xi - z_0} = \rho e^{i\theta}, \quad \theta \in [0, 2\pi],$$

We have

$$c_0^2 \rho^2 - 2 c_0 (1 + c_0) \cos \theta \rho + (1 + c_0)^2 < 1,$$

whose solution reads

$$0 \leq \rho < \frac{(1 + c_0) \cos \theta - \sqrt{1 - (1 + c_0)^2 \sin^2 \theta}}{c_0}, \quad -2 < c_0 < 0.$$

Since $\sqrt{1 - (1 + c_0)^2 \sin^2 \theta} = |\cos \theta|$ as $c_0 \rightarrow 0$, we have when $\pi/2 \leq \theta \leq 3\pi/2$ that

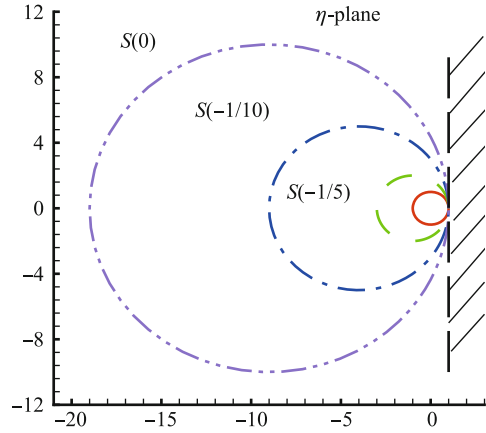
$$\frac{(1 + c_0) \cos \theta - \sqrt{1 - (1 + c_0)^2 \sin^2 \theta}}{c_0} = \frac{\cos \theta - |\cos \theta|}{c_0} = \frac{2 \cos \theta}{c_0} \rightarrow +\infty$$

as $c_0 \rightarrow 0$. Since $\rho = 0/0$ when $\theta \in (-\pi/2, \pi/2)$ and $c_0 \rightarrow 0$, we have by means of Bernoulli's rule in calculus that

$$\frac{(1 + c_0) \cos \theta - \sqrt{1 - (1 + c_0)^2 \sin^2 \theta}}{c_0} = \cos \theta + \frac{\sin^2 \theta}{|\cos \theta|} = \frac{1}{\cos \theta}.$$

This ends the proof. □

Fig. 5.1 Convergence-domain of the generalized Taylor series of a complex function with unique singular point at $z = \zeta$. Solid line: the boundary of convergence-domain $S(-1)$ of a traditional Taylor series; Dashed-line: the boundary of $S(-1/2)$; Dash-dotted line: the boundary of $S(-1/5)$; Dash-double-dotted line: the boundary of $S(-1/10)$; Long-dashed line: the boundary of $S(0)$.



In case of $A(q) = B(q) = q$ and there is an unique singular point, the convergence domain of the generalized Taylor series depends upon the convergence-control parameter c_0 , denoted by $S(c_0)$. According to Theorem 5.3, we have

$$\begin{aligned}
 S(-1) &= \{(\rho \cos \theta, \rho \sin \theta) \mid \rho < 1, \theta \in (0, 2\pi)\}, \\
 S\left(-\frac{1}{2}\right) &= \left\{(\rho \cos \theta, \rho \sin \theta) \mid \rho < 2\sqrt{1 - \frac{1}{4}\sin^2 \theta - \cos \theta}, \theta \in (0, 2\pi)\right\}, \\
 S\left(-\frac{1}{5}\right) &= \left\{(\rho \cos \theta, \rho \sin \theta) \mid \rho < 5\sqrt{1 - \frac{16}{25}\sin^2 \theta - 4\cos \theta}, \theta \in (0, 2\pi)\right\}, \\
 &\vdots \\
 S(0) &= \left\{(\rho \cos \theta, \rho \sin \theta) \mid \rho < \begin{cases} \frac{1}{\cos \theta}, & \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ +\infty, & \text{otherwise} \end{cases} \right\}.
 \end{aligned}$$

Note that $S(-1) \subset S(-1/2) \subset S(-1/5) \subset S(0)$, i.e. the convergence-region greatly enlarges as $c_0 \rightarrow 0$, as shown in Fig. 5.1. The convergence-domain $S(c_0)$ is in the shape of a circle: its centre is at $(1 + c_0^{-1})$, and its radius equals to $|c_0^{-1}|$. Note that $c_0 = -1$ corresponds to the traditional Taylor series which converges in the domain

$$\left| \frac{z - z_0}{\zeta - z_0} \right| < 1,$$

i.e. the traditional Taylor series converges in a circle with the centre at $z = z_0$ and the radius $\zeta - z_0$. However, the convergence-domain greatly enlarges as the convergence-control parameter tends to zero. Especially, when $c_0 \rightarrow 0$, the radius

$|c_0^{-1}|$ of the convergence-domain $S(c_0)$ tends to infinity so that the generalized Taylor-series converges in an infinite domain, i.e.

$$|\eta| = \left| \frac{z - z_0}{\xi - z_0} \right| < \begin{cases} \frac{1}{\cos \theta}, & \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ +\infty, & \text{otherwise,} \end{cases}$$

where $\eta = \rho \exp(i\theta)$. In the η -plane, it is a half-plane

$$S(0) = \left\{ (x', y') \mid x' < 1, -\infty < y' < +\infty \right\}$$

Note that the convergence-control parameter c_0 is a real number in Theorem 5.3. If a complex convergence-control parameter c_0 is used, we have a more general theorem.

Theorem 5.4. Let $i = \sqrt{-1}$ denote the imaginary unit,

$$c_0 = -1 + \varepsilon e^{i\gamma}, \quad \varepsilon \in [0, 1), \gamma \in [0, 2\pi),$$

be a complex convergence-control parameter, $A(q) = q$ and $B(q) = q$ be two deformation-functions, respectively. If a complex function $f(z)$ is analytic in the whole z -plane except at the unique singular point $z = \xi$, its generalized Taylor series at $z_0 \neq \xi$, i.e.

$$f(z_0) + \lim_{m \rightarrow +\infty} \sum_{k=1}^m \left[\frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \right] T_{m,k}(c_0, A, B)$$

converges to $f(z)$ in the region

$$\rho = \left| \frac{z - z_0}{\xi - z_0} \right| < \frac{\varepsilon [\varepsilon \cos \theta - \cos(\theta - \gamma)] + \sqrt{\Delta}}{1 - 2\varepsilon \cos \gamma + \varepsilon^2}, \quad \varepsilon \in [0, 1), \gamma \in [0, 2\pi), \quad (5.18)$$

where

$$\Delta = \varepsilon^2 [\varepsilon \cos \theta - \cos(\theta - \gamma)]^2 + (1 - \varepsilon^2) (1 - 2\varepsilon \cos \gamma + \varepsilon^2) \quad (5.19)$$

and

$$\eta = \frac{z - z_0}{\xi - z_0} = \rho e^{i\theta}, \quad \theta \in [0, 2\pi].$$

Epecially, in case of $\varepsilon = 1$, i.e. $c_0 = -1 + e^{i\gamma}$, it holds

$$0 \leq \rho < \max \left\{ \frac{\cos \theta - \cos(\theta - \gamma)}{1 - \cos \gamma}, 0 \right\}, \quad (5.20)$$

where $0 \leq \theta < 2\pi$, $0 \leq \gamma < 2\pi$.

Proof. Since $A(q) = B(q) = q$ and there exists only one singular point $z = \xi$, according to Theorem 5.1, we have two singular points ζ_0 and ζ_1 , governed by

$$1 - (1 + c_0) \zeta_0 = 0$$

and

$$1 - (1 + c_0) \zeta_1 + c_0 \left(\frac{z - z_0}{\xi - z_0} \right) \zeta_1 = 0,$$

respectively. Obviously,

$$\zeta_0 = (1 + c_0)^{-1}$$

and

$$\zeta_1 = \left[1 + c_0 - c_0 \left(\frac{z - z_0}{\xi - z_0} \right) \right]^{-1}.$$

According to Theorem 5.1, the generalized Taylor series

$$f(z_0) + \lim_{m \rightarrow +\infty} \sum_{k=1}^m \left[\frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \right] T_{m,k}(c_0, A, B)$$

converges to $f(z)$ in the region $\{|\zeta_0| > 1\} \cap \{|\zeta_1| > 1\}$. $|\zeta_0| > 1$ gives $|1 + c_0| < 1$, which is automatically satisfied since

$$c_0 = -1 + \varepsilon e^{i\gamma}, \quad 0 \leq \varepsilon < 1, \quad 0 \leq \gamma < 2\pi.$$

Besides, $|\zeta_1| > 1$ gives

$$\left| 1 + c_0 - c_0 \left(\frac{z - z_0}{\xi - z_0} \right) \right| < 1.$$

Writing

$$\eta = \frac{z - z_0}{\xi - z_0} = \rho e^{i\theta}, \quad \theta \in [0, 2\pi]$$

and using the definition of c_0 mentioned above, we have

$$\left| \varepsilon e^{i\gamma} - (\varepsilon e^{i\gamma} - 1) \rho e^{i\theta} \right| < 1,$$

i.e.

$$[1 - 2\varepsilon \cos \gamma + \varepsilon^2] \rho^2 - 2\varepsilon [\varepsilon \cos \theta - \cos(\theta - \gamma)] \rho + \varepsilon^2 < 1,$$

whose solution reads

$$\frac{\varepsilon [\varepsilon \cos \theta - \cos(\theta - \gamma)] - \sqrt{\Delta}}{1 - 2\varepsilon \cos \gamma + \varepsilon^2} < \rho < \frac{\varepsilon [\varepsilon \cos \theta - \cos(\theta - \gamma)] + \sqrt{\Delta}}{1 - 2\varepsilon \cos \gamma + \varepsilon^2},$$

where

$$\Delta = \varepsilon^2 [\varepsilon \cos \theta - \cos(\theta - \gamma)]^2 + (1 - \varepsilon^2) (1 - 2\varepsilon \cos \gamma + \varepsilon^2).$$

Obviously,

$$\frac{\varepsilon[\varepsilon \cos \theta - \cos(\theta - \gamma)] - \sqrt{\Delta}}{1 - 2\varepsilon \cos \gamma + \varepsilon^2} < 0.$$

Since $\rho \geq 0$, we have the solution

$$0 \leq \rho < \frac{\varepsilon[\varepsilon \cos \theta - \cos(\theta - \gamma)] + \sqrt{\Delta}}{1 - 2\varepsilon \cos \gamma + \varepsilon^2}.$$

Especially, when $\varepsilon = 1$, i.e. $c_0 = -1 + \exp(i\gamma)$, the above expression gives

$$0 \leq \rho < \max \left\{ \frac{\cos \theta - \cos(\theta - \gamma)}{1 - \cos \gamma}, 0 \right\}.$$

This ends the proof. $\square$

Obviously, Theorem 5.3 is a special case of Theorem 5.4 when $\gamma = 0$. Thus, as mentioned above, when $\gamma = 0$ and $\varepsilon \rightarrow 1$, i.e. $c_0 \rightarrow 0$, the generalized Taylor series converges to $f(z)$ in the half η -plane $\text{Re}(\eta) < 1$, as shown in Fig.5.1. Now, regarding c_0 as a complex number, there are an infinite number of different ways for c_0 to approach zero. For example, let us consider a special case: $\varepsilon = 1$ and $\gamma \rightarrow 0$, i.e. c_0 tends to zero along the circle $|c_0 + 1| = 1$. Note that there are two different ways to approach $c_0 = 0$ even along the circle: from above ($\gamma > 0$) or below ($\gamma < 0$). Let $S_0(\gamma)$ denote the convergence-domain given by (5.20), $S_0(0^+)$ and $S_0(0^-)$ denote the convergence-domain as γ tends to zero from above and below, respectively. It is found that, as γ tends to zero from above, the convergence-domain $S_0(\gamma)$ enlarges, for example,

$$S_0\left(\frac{\pi}{4}\right) \subset S_0\left(\frac{\pi}{9}\right) \subset S_0\left(\frac{\pi}{18}\right) \subset S_0\left(\frac{\pi}{36}\right) \subset S_0(0^+),$$

as shown in Fig.5.2. Especially, as $\gamma \rightarrow 0$ from above, the convergence-domain $S_0(0^+)$ becomes the below half-plane $\text{Im}(\eta) < 0$, i.e.

$$S_0(0^+) = \left\{ (x', y') \mid y' < 0, -\infty < x' < +\infty \right\}$$

in the plane $\eta = (x', y')$. Similarly, as γ tends to zero from below, the convergence-domain $S_0(\gamma)$ also enlarges, for example,

$$S_0\left(-\frac{\pi}{4}\right) \subset S_0\left(-\frac{\pi}{9}\right) \subset S_0\left(-\frac{\pi}{18}\right) \subset S_0\left(-\frac{\pi}{36}\right) \subset S_0(0^-),$$

as shown in Fig.5.3. Besides, as $\gamma \rightarrow 0$ from below, the convergence-domain $S_0(0^-)$ becomes the above half-plane $\text{Im}(\eta) > 0$, i.e.

$$S_0(0^-) = \left\{ (x', y') \mid y' > 0, -\infty < x' < +\infty \right\}$$

Fig. 5.2 Convergence-domain of the generalized Taylor series of a complex function $f(z)$ with unique singular point at $z = \zeta$ when $c_0 = 1 + e^{i\gamma}$ as $\gamma \rightarrow 0$ from above. Solid line: the boundary of convergence-domain $S_0(\pi/4)$; Dashed-line: the boundary of $S_0(\pi/9)$; Dash-dotted line: the boundary of $S_0(\pi/18)$; Dash-double-dotted line: the boundary of $S_0(\pi/36)$; Long-dashed line: the boundary of $S_0(0^+)$.

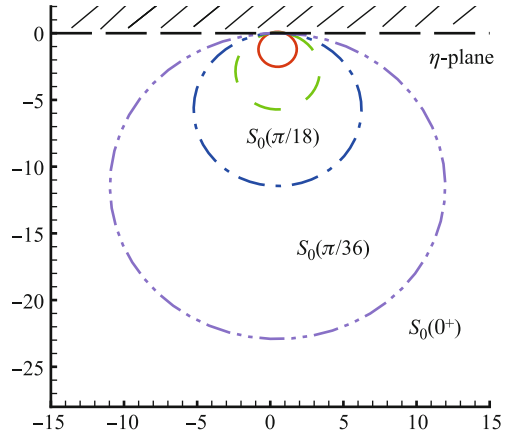
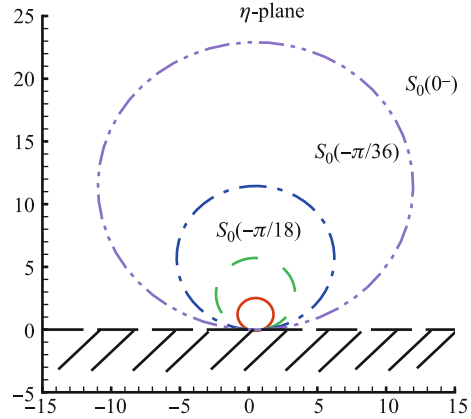


Fig. 5.3 Convergence-domain of the generalized Taylor series of a complex function $f(z)$ with unique singular point at $z = \zeta$ when $c_0 = 1 + e^{i\gamma}$ as $\gamma \rightarrow 0$ from below. Solid line: the boundary of convergence-domain $S_0(-\pi/4)$; Dashed-line: the boundary of $S_0(-\pi/9)$; Dash-dotted line: the boundary of $S_0(-\pi/18)$; Dash-double-dotted line: the boundary of $S_0(-\pi/36)$; Long-dashed line: the boundary of $S_0(0^-)$.



in the plane $\eta = (x', y')$. It should be emphasized that $S_0(0^+) \cup S_0(0^-)$ nearly covers the *whole* η -plane except the real axis $\text{Im}(\eta) = 0$.

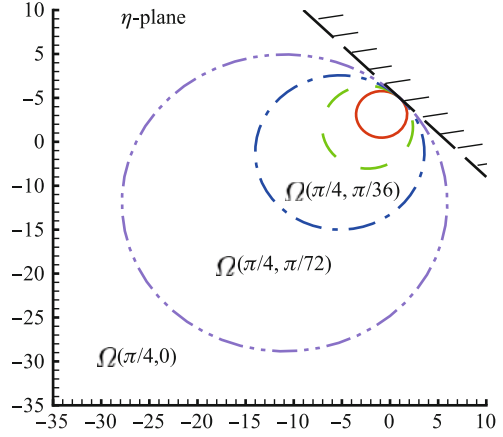
We can prove this by means of Theorem 5.4. According to (5.20), as $\gamma \rightarrow 0$, ρ has an expression of $0/0$. Then, using Bernoulli's rule in calculus, we have

$$\rho < \max \left\{ -\frac{\sin \theta}{\sin \gamma}, 0 \right\}.$$

Thus, as $\gamma > 0$ tends to zero, ρ tends to infinity when $\sin \theta < 0$, i.e. $-\pi < \theta < 0$, so that $S_0(0^+)$ is the below half-plane $\text{Im}(\eta) < 0$ in the η -plane. Similarly, as $\gamma < 0$ tends to zero, ρ tends to infinity when $\sin \theta > 0$, i.e. $0 < \theta < \pi$, so that $S_0(0^-)$ is the above half-plane $\text{Im}(\eta) > 0$ in the η -plane.

Since there are an infinite number of different ways to approach $c_0 = 0$ in the z -plane, we further consider such an approach

Fig. 5.4 Convergence-domain of the generalized Taylor series of a complex function $f(z)$ with unique singular point at $z = \zeta$ when $c_0 = 1 + \varepsilon e^{i\gamma}$, $\varepsilon = \sin \delta / \sin(\delta + \gamma)$ as $\gamma \rightarrow 0$. Solid line: the boundary of convergence-domain $\Omega(\pi/4, \pi/9)$; Dashed-line: the boundary of $\Omega(\pi/4, \pi/18)$; Dash-dotted line: the boundary of $\Omega(\pi/4, \pi/36)$; Dash-double-dotted line: the boundary of $\Omega(\pi/4, \pi/72)$; Long-dashed line: the boundary of $\Omega(\pi/4, 0)$.



$$\varepsilon = \frac{\sin \delta}{\sin(\delta + \gamma)}, \quad \gamma \rightarrow 0,$$

where $\delta > 0$, $\delta \in (-\pi/2, \pi/2)$, $\delta + \gamma \in (-\pi/2, \pi/2)$. In this case, the convergence-domain of the generalized Taylor series is dependent upon δ and γ , denoted by $\Omega(\delta, \gamma)$. Without loss of generality, let us consider two cases: $\delta = \pm\pi/4$. It is found that, in case of $\delta = \pi/4$, the convergence-domain enlarges as $\gamma \rightarrow 0$ from above, i.e.

$$\Omega\left(\frac{\pi}{4}, \frac{\pi}{9}\right) \subset \Omega\left(\frac{\pi}{4}, \frac{\pi}{18}\right) \subset \Omega\left(\frac{\pi}{4}, \frac{\pi}{36}\right) \subset \Omega\left(\frac{\pi}{4}, \frac{\pi}{72}\right) \subset \Omega\left(\frac{\pi}{4}, 0\right),$$

as shown in Fig. 5.4. Especially, as $\gamma \rightarrow 0$ from above, the convergence-domain becomes a half plane

$$\Omega\left(\frac{\pi}{4}, 0\right) = \left\{ (x', y') \mid y' < -x' + 1, -\infty < x' < +\infty \right\}$$

in the η -plane! Similarly, in case of $\delta = -\pi/4$, the convergence-domain enlarges as $\gamma \rightarrow 0$ from below, i.e.

$$\begin{aligned} \Omega\left(-\frac{\pi}{4}, -\frac{\pi}{9}\right) &\subset \Omega\left(-\frac{\pi}{4}, -\frac{\pi}{18}\right) \subset \Omega\left(-\frac{\pi}{4}, -\frac{\pi}{36}\right) \\ &\subset \Omega\left(-\frac{\pi}{4}, -\frac{\pi}{72}\right) \subset \Omega\left(-\frac{\pi}{4}, 0\right), \end{aligned}$$

as shown in Fig. 5.5. Especially, as $\gamma \rightarrow 0$ from below, the convergence-domain becomes a half plane

$$\Omega\left(-\frac{\pi}{4}, 0\right) = \left\{ (x', y') \mid y' < x' - 1, -\infty < x' < +\infty \right\}$$

in the η -plane!

Fig. 5.5 Convergence-domain of the generalized Taylor series of a complex function $f(z)$ with unique singular point at $z = \zeta$ when $c_0 = 1 + \varepsilon e^{i\gamma}$, $\varepsilon = \sin \delta / \sin(\delta + \gamma)$ as $\gamma \rightarrow 0$. Solid line: the boundary of convergence-domain $\Omega(-\pi/4, -\pi/9)$; Dashed-line: the boundary of $\Omega(-\pi/4, -\pi/18)$; Dash-dotted line: the boundary of $\Omega(-\pi/4, -\pi/36)$; Dash-double-dotted line: the boundary of $\Omega(-\pi/4, -\pi/72)$; Long-dashed line: the boundary of $\Omega(-\pi/4, 0)$.

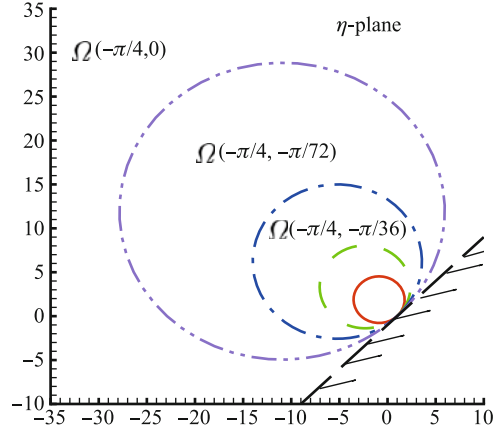
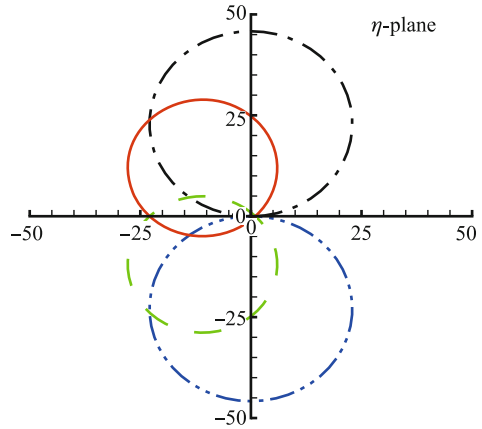


Fig. 5.6 Convergence-domain of the generalized Taylor series of a complex function $f(z)$ with unique singular point at $z = \zeta$ when $c_0 = 1 + \varepsilon e^{i\gamma}$, $\varepsilon = \sin \delta / \sin(\delta + \gamma)$. Solid line: the boundary of convergence-domain $\Omega(-\pi/4, -\pi/72)$; Dashed-line: the boundary of $\Omega(\pi/4, \pi/72)$; Dash-dotted line: the boundary of $\Omega(-35\pi/36, -\pi/72)$; Dash-double-dotted line: the boundary of $\Omega(35\pi/36, \pi/72)$.



Besides, the convergence-domains of the generalized Taylor series in cases of $\delta = \pm\pi/4$ and $\delta = \pm 35\pi/36$ are as shown in Fig. 5.6. It is easy to prove that, as $\delta \rightarrow \pi/2$ and $\gamma \rightarrow 0$ from above, the convergence-domain becomes the below half-plane $\text{Im}(\eta) < 0$, i.e.

$$\Omega\left(\frac{\pi}{2}, 0\right) = \left\{ (x', y') \mid y' < 0, -\infty < x' < +\infty \right\}$$

in the $\eta = (x', y')$ plane. Similarly, as $\delta \rightarrow -\pi/2$ and $\gamma \rightarrow 0$ from below, the convergence-domain becomes the above half-plane $\text{Im}(\eta) > 0$, i.e.

$$\Omega\left(\frac{\pi}{2}, 0\right) = \left\{ (x', y') \mid y' > 0, -\infty < x' < +\infty \right\}$$

in the plane $\eta = (x', y')$. All of these show that, as the convergence-control parameter $c_0 \rightarrow 0$, the convergence-domain Ω is dependent upon its approach-curve along which c_0 tends to zero: the convergence-domain is a different half-plane in the η -plane for a different way of $c_0 \rightarrow 0$.

Recall that $c_0 = -1 + \varepsilon \exp(i\gamma)$, where c_0 is a complex number. As shown above, as $\gamma = 0$ and $\varepsilon \rightarrow 1$, i.e. $c_0 \rightarrow 0$ along the parallel-axis, then the convergence-domain of the generalized Taylor series becomes a half-plane $\text{Re}(\eta) < 1$. In case of $\varepsilon = 1$, i.e. $c_0 \rightarrow 0$ along the circle $|1 + c_0| = 1$, the convergence domain is a below half-plane $\text{Im}(\eta) < 0$ when $\gamma \rightarrow 0$ from above, but is an above half-plane $\text{Im}(\eta) > 0$ when $\gamma \rightarrow 0$ from below. So, if a complex function $f(z)$ has an unique singular point at $z = \zeta$, we can always find a complex convergence-control parameter $|1 + c_0| < 1$ such that the corresponding generalized Taylor series converges in the whole η -plane except the half of the real axis $\text{Im}(\eta) = 0$ and $\text{Re}(\eta) \geq 1$, i.e.

$$\left\{ (x', y') \mid x' \geq 1, y' = 0 \right\},$$

where

$$\eta = (x', y') = \frac{z - z_0}{\zeta - z_0}.$$

Thus, we have the following theorem:

Theorem 5.5. *Let c_0 denote a complex convergence-control parameter, $A(q) = q$ and $B(q) = q$ be two deformation-functions, respectively. If a complex function $f(z)$ has a unique pole at $z = \zeta$ in the whole z -plane, its generalized Taylor series at $z_0 \neq \zeta$, i.e.*

$$f(z_0) + \lim_{m \rightarrow +\infty} \sum_{k=1}^m \left[\frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \right] T_{m,k}(c_0, A, B)$$

can converge to $f(z)$ in the whole z -plane except the points on a half-line

$$z = z_0 + \rho (\zeta - z_0), \quad \rho \geq 1.$$

So, if $f(z)$ has an unique pole at $z = \zeta$, we can find arbitrary two non-singular points $z_0 \neq \zeta$ and $z'_0 \neq \zeta$, which are not collinear with ζ , so that at any a given point in the whole z -plane except the pole $z = \zeta$, at least one of the two generalized Taylor series at $z = z_0$ and $z = z'_0$ converges to $f(z)$. So, we have the following theorem:

Theorem 5.6. *Let c_0 denote a complex convergence-control parameter, $A(q) = q$ and $B(q) = q$ be two deformation-functions, respectively. Let ζ denote an unique pole of a complex function $f(z)$, $z_0 \neq \zeta$ and $z'_0 \neq \zeta$ are two non-singular points. If ζ, z_0 and z'_0 are not collinear, then for any a given $z \neq \zeta$ in the whole plane, there exist such a convergence-control parameter c_0 that at least one of the generalized Taylor series*

$$f(z_0) + \lim_{m \rightarrow +\infty} \sum_{k=1}^m \left[\frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \right] T_{m,k}(c_0, A, B),$$

and/or

$$f(z'_0) + \lim_{m \rightarrow +\infty} \sum_{k=1}^m \left[\frac{f^{(k)}(z'_0)}{k!} (z - z'_0)^k \right] T_{m,k}(c_0, A, B),$$

converges to $f(z)$.

Proof. Choose two arbitrary non-singular points $z_0 \neq \zeta$ and $z'_0 \neq \zeta$, which are not collinear with the singular point. Let $z \neq \zeta$ be an arbitrary point in the z -plane. If z is not collinear with z_0 and ζ , then according to Theorem 5.5, there exists such a convergence-control parameter c_0 that the generalized Taylor series at z_0 converges at z . If z is collinear with z_0 and ζ , then z must be not collinear with z'_0 and ζ . Then, according to Theorem 5.5, there exists such a convergence-control parameter c'_0 that the generalized Taylor series at z'_0 converges at z . This ends the proof. $\square$

Theorem 5.7. Let c_0 be a convergence-control parameter, $A(q)$ and $B(q)$ are deformation functions whose Maclaurin series $A(q) = \sum_{k=1}^{+\infty} a_k q^k$ and $B(q) = \sum_{k=1}^{+\infty} b_k q^k$ are convergent at $q = 1$, i.e.

$$\sum_{k=1}^{+\infty} a_k = 1, \quad \sum_{k=1}^{+\infty} b_k = 1.$$

If $|1 + c_0| < 1$, then for any a finite integer $k \geq 1$, it holds

$$\lim_{m \rightarrow +\infty} T_{m,k}(c_0, A, B) = 1. \quad (5.21)$$

Proof. According to the definition of $T_{m,k}(c_0, A, B)$, it holds

$$\begin{aligned} & \lim_{m \rightarrow +\infty} T_{m,k}(c_0, A, B) \\ &= \lim_{m \rightarrow +\infty} (-c_0)^k \sum_{n=0}^{m-k} \sum_{r=0}^n \binom{k+r-1}{r} (1+c_0)^r \sum_{s=0}^{n-r} \bar{a}_{k,k+s} \bar{b}_{r,n-s} \\ &= (-c_0)^k \sum_{n=0}^{+\infty} \sum_{r=0}^n \binom{k+r-1}{r} (1+c_0)^r \sum_{s=0}^{n-r} \bar{a}_{k,k+s} \bar{b}_{r,n-s} \\ &= (-c_0)^k \sum_{r=0}^{+\infty} \sum_{n=r}^{+\infty} \binom{k+r-1}{r} (1+c_0)^r \sum_{s=0}^{n-r} \bar{a}_{k,k+s} \bar{b}_{r,n-s} \\ &= (-c_0)^k \sum_{r=0}^{+\infty} \binom{k+r-1}{r} (1+c_0)^r \sum_{n=r}^{+\infty} \sum_{s=0}^{n-r} \bar{a}_{k,k+s} \bar{b}_{r,n-s}. \end{aligned}$$

Since

$$\sum_{n=r}^{+\infty} \sum_{s=0}^{n-r} \bar{a}_{k,k+s} \bar{b}_{r,n-s} = \sum_{s=0}^{+\infty} \sum_{n=r+s}^{+\infty} \bar{a}_{k,k+s} \bar{b}_{r,n-s} = \sum_{m=k}^{+\infty} \sum_{j=r}^{+\infty} \bar{a}_{k,m} \bar{b}_{r,j}$$

$$= \left(\sum_{m=k}^{+\infty} \bar{a}_{k,m} \right) \left(\sum_{j=r}^{+\infty} \bar{b}_{r,j} \right) = \left(\sum_{m=1}^{+\infty} a_m \right)^k \left(\sum_{j=1}^{+\infty} b_j \right)^r = 1,$$

and

$$\sum_{r=0}^{+\infty} \binom{k+r-1}{r} (1+c_0)^r = [1 - (1+c_0)]^{-k} = (-c_0)^{-k}$$

due to $|1+c_0| < 1$, we have

$$\lim_{m \rightarrow +\infty} T_{m,k}(c_0, A, B) = (-c_0)^k \sum_{r=0}^{+\infty} \binom{k+r-1}{r} (1+c_0)^r = (-c_0)^k (-c_0)^{-k} = 1.$$

This ends the proof. $\square$

According to Theorem 5.7, it holds for any a *finite* positive number $k \geq 1$ that

$$\lim_{m \rightarrow +\infty} \frac{f^{(k)}(z_0)}{k!} (z-z_0)^k T_{m,k}(c_0, A, B) = \frac{f^{(k)}(z_0)}{k!} (z-z_0)^k.$$

Therefore, the first M terms of the generalized Taylor series, where M is any a *finite* positive number, are the same as the corresponding first M terms of the traditional Taylor series. However, as shown above, the generalized Taylor series converges in the much larger domain than the traditional one. Why? The key-point is that Theorem 5.7 holds only for a *finite* positive integer k . It is well-known that the convergence of a series

$$a_0 + a_1 t + a_2 t^2 + \cdots + a_m t^m + \cdots$$

depends on its property of the terms c_m as $m \rightarrow +\infty$, i.e. the radius of convergence of the above series is determined by

$$\lim_{m \rightarrow +\infty} \left| \frac{a_m}{a_{m+1}} \right|.$$

Therefore, the property of the generalized Taylor series at infinity is greatly modified by the convergence-control parameter c_0 and the deformation-functions so that its convergence-region is enlarged fantastically.

Finally, we explain, from the another points of view, why the convergence of the generalized Taylor series of $f(z)$, i.e.

$$\sum_{k=0}^{+\infty} \Gamma_k(z) T_{m,k}(c_0, A, B),$$

where

$$\Gamma_k(z) = \frac{f^{(k)}(z_0)}{k!} (z-z_0)^k,$$

can be fantastically modified by the convergence-control parameter c_0 and the deformation-functions $A(q)$ and $B(q)$. Regard

$$P^* = (\Gamma_0(z), \Gamma_1(z), \Gamma_2(z), \Gamma_3(z), \dots)$$

as a point of an infinite-dimension space $\mathbf{R}^\infty$ (for example, a Hilbert space). Then, the traditional Taylor series of $f(z)$, i.e.

$$\sum_{k=0}^{+\infty} \Gamma_k(z),$$

corresponds to a limit which tends to the point $P^* \in \mathbf{R}^\infty$ along such a traditional path:

$$(\Gamma_0(z), 0, 0, 0, \dots) \in \mathbf{R}^\infty,$$

$$(\Gamma_0(z), \Gamma_1(z), 0, 0, 0, \dots) \in \mathbf{R}^\infty,$$

$$(\Gamma_0(z), \Gamma_1(z), \Gamma_2(z), 0, 0, \dots) \in \mathbf{R}^\infty,$$

$$(\Gamma_0(z), \Gamma_1(z), \Gamma_2(z), \Gamma_3(z), 0, \dots) \in \mathbf{R}^\infty,$$

$\vdots$

According to Theorem 5.7, the generalized Taylor series

$$\sum_{k=0}^{+\infty} \Gamma_k(z) T_{m,k}(c_0, A, B),$$

corresponds to the same point $P^* \in \mathbf{R}^\infty$ as the traditional Taylor series

$$\sum_{k=0}^{+\infty} \Gamma_k(z).$$

However, the generalized Taylor series corresponds to a *family* of limits which tend to the *same* point $P^* \in \mathbf{R}^\infty$ along different paths that are dependent upon the convergence-control parameter c_0 and two deformation-functions $A(q)$ and $B(q)$, i.e.

$$(\Gamma_0(z) T_{0,0}(c_0, A, B), 0, 0, 0, 0, \dots) \in \mathbf{R}^\infty,$$

$$(\Gamma_0(z) T_{1,0}(c_0, A, B), \Gamma_1(z) T_{1,1}(c_0, A, B), 0, 0, 0, \dots) \in \mathbf{R}^\infty,$$

$$(\Gamma_0(z) T_{2,0}(c_0, A, B), \Gamma_1(z) T_{2,1}(c_0, A, B), \Gamma_2(z) T_{2,2}(c_0, A, B), 0, 0, \dots) \in \mathbf{R}^\infty,$$

$\vdots$

It is well-known that a limit of a function with multiple variables might be quite different for different paths. For example, the limit

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sqrt{x^2 + y^2}}{|x|} = \sqrt{1 + \alpha^2}$$

is dependent on an arbitrary real number α , if we gain the limit along the different approach paths $y = \alpha x$. This is exactly the reason why the convergence-domain of the generalized Taylor series of a complex function $f(z)$ with a unique pole at $z = \zeta$ converges to $f(z)$ in the whole half-plane $\text{Im}(\eta) < 0$ as c_0 tends to zero along the circle $|1 + c_0| = 1$ from above, but in the whole half-plane $\text{Im}(\eta) > 0$ as c_0 tends to zero from below, respectively.

All of these show in theory that the convergence of a series can be indeed modified fantastically by introducing a non-zero auxiliary parameter c_0 , called the convergence-control parameter. This well explains why the convergence-control parameter c_0 in the frame of the HAM can guarantee the convergence of homotopy-series. Note that the generalized Taylor series depends upon a convergence-control parameter c_0 and two deformation-functions $A(q)$ and $B(q)$, although most of the above theorems are given for the simplest deformation functions $A(q) = B(q) = q$. Note that, the convergence-control parameter can even be a complex number in the z -plane $|1 + c_0| < 1$, and the two deformation functions $A(q)$ and $B(q)$ can be also complex functions! Obviously, using better deformation-functions, it is quite possible for us to get even larger convergence-domain of a generalized Taylor series. Similarly, in the frame of the HAM, we have extremely large freedom to choose not only the convergence-control parameter c_0 , but also the initial approximation, the auxiliary linear operator, and the deformation-functions to construct different types of zeroth-order deformation equations. Therefore, the HAM provides us more degrees of freedom to control and adjust the convergence of homotopy-series.

Finally, we point out that the convergence-control is an important concept of the HAM. Here, we *prove* that the convergence of an infinite series can be indeed controlled and adjusted by means of a non-zero auxiliary parameter c_0 . This provides a corner-stone for the HAM in theory.

5.3 Homotopy transform

Write the sequence $s_n = \sum_{k=0}^n u_k$ for a series $\sum_{k=0}^{+\infty} u_k$. Due to Agnew's (Agnew, 1944) definition, the Euler transform, denoted by $\mathcal{E}(\lambda)$, of the sequence $\{s_n\}$ is the sequence $\{w_n\}$ defined by

$$w_n = \sum_{k=0}^n \binom{n}{k} \lambda^k (1 - \lambda)^{n-k} s_k. \quad (5.22)$$

The Euler transform was developed by Leonhard Euler (15 April 1707 — 18 September 1783), a pioneering Swiss mathematician. Today, it is widely applied to accelerate convergence of a sequence or even to make a divergent series convergent.

Using $T_{m,k}(c_0, A, B)$ defined by (5.6) and (5.7) – (5.9), Liao (2010a) defined a new transform of a sequence, namely the homotopy-transform, and proved that it logically contains the Euler-transform $\mathcal{E}(\lambda)$.

Definition 5.3. Let $c_0 \neq 0$ denote the convergence-control parameter, $A(q)$ and $B(q)$ be two deformation-functions whose Maclaurin series $A(q) = \sum_{k=1}^{+\infty} a_k q^k$ and $B(q) = \sum_{k=1}^{+\infty} b_k q^k$ converge at $q = 1$ so that $\sum_{k=1}^{+\infty} a_k = 1$ and $\sum_{k=1}^{+\infty} b_k = 1$. The so-called *homotopy-transform*, denoted by $\mathcal{H}(c_0, A, B)$, of a series $\sum_{k=0}^{+\infty} u_k$, is a sequence $\{\mu_n\}$ defined by

$$\mu_n = u_0 + \sum_{k=1}^n u_k T_{n,k}(c_0, A, B), \quad (5.23)$$

where $T_{n,k}(c_0, A, B)$ is given by (5.6) under the definitions (5.7) – (5.9). The series $\sum_{k=0}^{+\infty} u_k$ is called summable by the homotopy-transform $\mathcal{H}(c_0, A, B)$ if μ_n tends to a bounded value as $n \rightarrow +\infty$.

Lemma 5.2 Let c_0 denote a convergence-control parameter. If $A(q) = B(q) = q$, then

$$T_{m,k}(c_0, A, B) = \mu_0^{m,k}(c_0), \quad k \geq m, \quad (5.24)$$

where $T_{m,k}(c_0, A, B)$ is defined by (5.6) and (5.7) – (5.9), $\mu_0^{m,k}(c_0)$ is defined by

$$\mu_0^{m,k}(c_0) = (-c_0)^k \sum_{n=0}^{m-k} \binom{n+k-1}{n} (1+c_0)^n, \quad m \geq k \geq 1. \quad (5.25)$$

The same definition is given in (2.82).

Proof. When $A(q) = B(q) = q$, according to the definitions (5.7) – (5.9), we have

$$\bar{a}_{m,n} = \bar{b}_{m,n} = \begin{cases} 1, & \text{when } m = n, \\ 0, & \text{when } n > m. \end{cases}$$

Then, according to the definitions (5.6) and (5.25), it holds for $m \geq k \geq 1$ that

$$T_{m,k}(c_0, A, B) = (-c_0)^k \sum_{n=0}^{m-k} \sum_{r=0}^n \binom{k+r-1}{r} (1+c_0)^r \bar{a}_{k,k} \bar{b}_{r,n}$$

$$\begin{aligned}
&= (-c_0)^k \sum_{n=0}^{m-k} \binom{k+n-1}{n} (1+c_0)^r \bar{a}_{k,k} \bar{b}_{n,n} \\
&= (-c_0)^k \sum_{n=0}^{m-k} \binom{k+n-1}{n} (1+c_0)^r \\
&= \mu_0^{m,k}(c_0).
\end{aligned}$$

This completes the proof. $\square$

Theorem 5.8. Write the sequence $s_n = \sum_{k=0}^n u_k$ for a series $\sum_{k=0}^{+\infty} u_k$. Let λ be a complex number and c_0 a convergence-control parameter. The Euler transform $\mathcal{E}(\lambda)$ of the sequence $\{s_n\}$ is the same as the homotopy-transform $\mathcal{H}(c_0, A, B)$ of the series $\sum_{k=0}^{+\infty} u_k$ if $A(q) = B(q) = q$ and $c_0 = -\lambda$.

Proof. Due to Agnew's (Agnew, 1944) definition (5.22) of Euler transform $\mathcal{E}(\lambda)$, the sequence $\{\bar{\mu}_m\}$ given by the Euler transform of the sequence $\{s_n\}$ reads

$$\begin{aligned}
\bar{\mu}_m &= \sum_{k=0}^m \binom{m}{k} \lambda^k (1-\lambda)^{m-k} s_k \\
&= \sum_{k=0}^m \binom{m}{k} \lambda^k (1-\lambda)^{m-k} \sum_{n=0}^k u_n \\
&= \sum_{n=0}^m u_n \sum_{k=n}^m \binom{m}{k} \lambda^k (1-\lambda)^{m-k} \\
&= u_0 \sum_{k=0}^m \binom{m}{k} \lambda^k (1-\lambda)^{m-k} + \sum_{n=1}^m u_n \sum_{k=n}^m \binom{m}{k} \lambda^k (1-\lambda)^{m-k},
\end{aligned}$$

which gives, since

$$\sum_{k=0}^m \binom{m}{k} \lambda^k (1-\lambda)^{m-k} = [\lambda + (1-\lambda)]^m = 1,$$

that

$$\bar{\mu}_m = u_0 + \sum_{n=1}^m u_n \sum_{k=n}^m \binom{m}{k} \lambda^k (1-\lambda)^{m-k}. \quad (5.26)$$

According to Lemma 5.2, it holds $T_{m,n}(c_0, A, B) = \mu_0^{m,n}(c_0)$ when $A(q) = B(q) = q$. Therefore, by means of the definition (5.25), the sequence $\{\mu_m\}$ given by the homotopy-transform $\mathcal{H}(c_0, A, B)$ in case of $c_0 = -\lambda$ and $A(q) = B(q) = q$ is given by

$$\mu_m = u_0 + \sum_{n=1}^m u_n \mu_0^{m,n}(-\lambda) = u_0 + \sum_{n=1}^m u_n \lambda^n \sum_{k=0}^{m-n} \binom{n+k-1}{k} (1-\lambda)^k. \quad (5.27)$$

Enforcing $\bar{\mu}_m = \mu_m$ and comparing (5.26) with (5.27), it remains to show that

$$\lambda^n \sum_{k=0}^{m-n} \binom{n+k-1}{k} (1-\lambda)^k = \sum_{k=n}^m \binom{m}{k} \lambda^k (1-\lambda)^{m-k}, \quad 1 \leq n \leq m. \quad (5.28)$$

When $1 \leq n \leq m$, we have

$$\begin{aligned} & \lambda^n \sum_{k=0}^{m-n} \binom{n+k-1}{k} (1-\lambda)^k \\ &= \lambda^n \sum_{k=0}^{m-n} \binom{n+k-1}{k} \sum_{r=0}^k \binom{k}{r} (-\lambda)^r \\ &= \sum_{r=0}^{m-n} (-\lambda)^{n+r} (-1)^n \sum_{k=r}^{m-n} \binom{n+k-1}{k} \binom{k}{r} \end{aligned} \quad (5.29)$$

and

$$\begin{aligned} & \sum_{k=n}^m \binom{m}{k} \lambda^k (1-\lambda)^{m-k} \\ &= \sum_{k=n}^m \binom{m}{k} \lambda^k \sum_{r=0}^{m-k} \binom{m-k}{r} (-\lambda)^r \\ &= \sum_{k=n}^m \binom{m}{k} (-1)^k \sum_{r=0}^{m-k} \binom{m-k}{r} (-\lambda)^{k+r} \\ &= \sum_{k=0}^{m-n} \binom{m}{k+n} (-1)^{k+n} \sum_{r=0}^{m-n-k} \binom{m-n-k}{r} (-\lambda)^{n+k+r} \\ &= \sum_{s=0}^{m-n} (-\lambda)^{n+s} (-1)^n \sum_{k=0}^s (-1)^k \binom{m}{k+n} \binom{m-n-k}{s-k} \\ &= \sum_{r=0}^{m-n} (-\lambda)^{n+r} (-1)^n \sum_{k=0}^r (-1)^k \binom{m}{k+n} \binom{m-n-k}{r-k} \\ &= \sum_{r=0}^{m-n} (-\lambda)^{n+r} (-1)^n \sum_{k=0}^r (-1)^k \binom{m}{m-n-r} \binom{r+n}{k+n}, \end{aligned} \quad (5.30)$$

where we use such a formula

$$\binom{m}{k+n} \binom{m-k-n}{r-k} = \binom{m}{m-n-r} \binom{r+n}{k+n},$$

which holds in the relevant ranges. So, by (5.28), (5.29) and (5.30), we need to show

$$\sum_{k=r}^{m-n} \binom{n+k-1}{k} \binom{k}{r} = \sum_{k=0}^r (-1)^k \binom{m}{m-n-r} \binom{r+n}{k+n}, \quad (5.31)$$

where $1 \leq n \leq m$. Noticing that, for $n \geq 1$ and sufficiently small x ,

$$\sum_{r=0}^{+\infty} x^r \binom{n+r-1}{r} = (1-x)^{-n},$$

and that

$$\begin{aligned} & \sum_{r=0}^{+\infty} x^r \sum_{k=0}^r (-1)^k \binom{r+n}{k+n} \\ &= \sum_{k=0}^{+\infty} (-1)^k \sum_{r=k}^{+\infty} \binom{r+n}{k+n} x^r = \sum_{k=0}^{+\infty} (-1)^k x^k \sum_{r=0}^{+\infty} \binom{r+k+n}{k+n} x^r \\ &= \sum_{k=0}^{+\infty} (-1)^k x^k (1-x)^{-n-k-1} = (1-x)^{-n-1} \sum_{k=0}^{+\infty} (-1)^k x^k (1-x)^{-k} \\ &= (1-x)^{-n}, \end{aligned}$$

it holds

$$\binom{n+r-1}{r} = \sum_{k=0}^r (-1)^k \binom{r+n}{k+n}. \quad (5.32)$$

According to (5.31) and (5.32), we need to show

$$\sum_{k=r}^{m-n} \binom{n+k-1}{k} \binom{k}{r} = \binom{m}{m-n-r} \binom{n+r-1}{r}.$$

Noticing that

$$\begin{aligned} & \sum_{k=r}^{m-n} \binom{n+k-1}{k} \binom{k}{r} \\ &= \sum_{k=0}^{m-n-r} \binom{n+k+r-1}{k+r} \binom{k+r}{r} \\ &= \sum_{k=0}^{m-n-r} \frac{(k+r+n-1)!}{(n-1)! k! r!} \\ &= \binom{n+r-1}{r} \sum_{k=0}^{m-n-r} \binom{n+k+r-1}{k}, \end{aligned}$$

this reduces to show that

$$\binom{m}{m-n-r} = \sum_{k=0}^{m-n-r} \binom{n+k+r-1}{k}.$$

Writing $i = n+r$, the above expression reads

$$\binom{m}{m-i} = \sum_{k=0}^{m-i} \binom{k+i-1}{k} = \sum_{k=0}^{m-i} \binom{k+i-1}{i-1} = \sum_{j=i-1}^{m-1} \binom{j}{i-1}.$$

In order to prove this, we use the formula

$$\sum_{j=n}^N \binom{j}{n} = \binom{N+1}{n+1}$$

in a handbook of mathematics (Adams and Hippisley, 1922). Setting $N = m - 1$ and $n = i - 1$ in above formula gives

$$\sum_{j=i-1}^{m-1} \binom{j}{i-1} = \binom{m}{i} = \binom{m}{m-i}.$$

This completes the proof. $\square$

Remark 5.1. According to Theorem 5.8, the Euler transform $\mathcal{E}(\lambda)$ is only a special case of the so-called homotopy-transform $\mathcal{H}(c_0, A, B)$ defined by (5.23) when $c_0 = -\lambda$ and $A(q) = B(q) = q$, corresponding to $a_1 = b_1 = 1$ and $a_k = b_k = 0$ for $k > 1$.

Here, we would like to emphasize two points. First, Euler transform $\mathcal{E}(\lambda)$ is only a special case of the so-called homotopy-transformation $\mathcal{H}(c_0, A, B)$. Thus, the homotopy-transform is more general. Secondly, Euler transform is widely used to accelerate convergence of a series or to make a divergent series convergent. Thus, the homotopy-transform $\mathcal{H}(c_0, A, B)$ provides us with a new but more general way to accelerate convergence of a series or to make a divergent series convergent.

5.4 Relation between homotopy analysis method and Euler transform

It is very interesting that the so-called homotopy-transform $\mathcal{H}(c_0, A, B)$ defined by (5.23) can be obtained in the frame of the HAM, as shown by Liao (2010a).

To illustrate this, let us consider here a nonlinear ordinary differential equation

$$u'(z) + u(z) \left[1 - \frac{1}{2}u(z) \right] = 0, \quad u(0) = 1, \quad (5.33)$$

where the prime denotes the differentiation with respect to z . This equation has a closed-form solution $u(z) = 2/(1 + e^z)$.

Let $c_0 \neq 0$ denote a convergence-control parameter, $q \in [0, 1]$ the homotopy-parameter, $\alpha(q)$ and $\beta(q)$ be two deformation-functions satisfying

$$\alpha(0) = \beta(0) = 0, \quad \alpha(1) = \beta(1) = 1, \quad (5.34)$$

and their Maclaurin series $\alpha(q) = \sum_{k=1}^{+\infty} \alpha_k q^k$ and $\beta(q) = \sum_{k=1}^{+\infty} \beta_k q^k$ are convergent at $q = 1$, respectively. Choose $u_0(z) = 1$ as the initial approximation and $\mathcal{L}(u) = u'$ as the auxiliary linear operator, respectively, and define the nonlinear operator

$$\mathcal{N}(u) = \frac{du}{dz} + u \left(1 - \frac{1}{2}u \right). \quad (5.35)$$

Then, we construct the zeroth-order deformation equation

$$[1 - \alpha(q)] \mathcal{L}[\tilde{u}(z; q) - u_0(z)] = c_0 \beta(q) \mathcal{N}[\tilde{u}(z; q)], \quad \tilde{u}(0; q) = 1. \quad (5.36)$$

The m th-order homotopy-approximation is given by

$$u(z) \approx u_0(z) + \sum_{m=1}^m u_m(z), \quad (5.37)$$

where $u_m(z)$ is governed by the m th-order deformation equation

$$\mathcal{L} \left[u_m(z) - \sum_{k=1}^{m-1} \alpha_k u_{m-k}(z) \right] = c_0 \sum_{k=1}^{m-1} \beta_{m-k} \delta_{k-1}(z), \quad u_m(0) = 0, \quad (5.38)$$

with the definition

$$\delta_n(z) = \mathcal{D}_n \{ \mathcal{N}[\tilde{u}(z; q)] \} = u'_n(z) + u_n(z) - \frac{1}{2} \sum_{k=0}^n u_k(z) u_{n-k}(z). \quad (5.39)$$

Note that we have great freedom to choose the deformation functions $\alpha(q)$ and $\beta(q)$ in the zeroth-order deformation equation (5.36). Let $A(q)$ and $B(q)$ denote two deformation-functions satisfying $A(0) = B(0) = 0$ and $A(1) = B(1) = 1$, and their Maclaurin series $A(q) = \sum_{k=1}^{+\infty} a_k q^k$ and $B(q) = \sum_{k=1}^{+\infty} b_k q^k$ converge at $q = 1$, i.e.

$$\sum_{k=1}^{+\infty} a_k = 1, \quad \sum_{k=1}^{+\infty} b_k = 1.$$

To derive the so-called homotopy-transform, we define

$$\begin{aligned} \alpha(q) &= B(q) + c_0[B(q) - A(q)] = \sum_{k=1}^{+\infty} [(1 + c_0) b_k - c_0 a_k] q^k, \\ \beta(q) &= A(q) = \sum_{k=1}^{+\infty} a_k q^k, \end{aligned}$$

i.e.

$$\alpha_k = (1 + c_0) b_k - c_0 a_k, \quad \beta_k = a_k.$$

Then, since $u'_0(z) = 0$, the zeroth-order deformation equation (5.36) becomes

$$\begin{aligned}
& [1 - (1 + c_0) B(q) + c_0 A(q)] \frac{\partial \tilde{u}(z; q)}{\partial z} \\
& = c_0 A(q) \left\{ \frac{\partial \tilde{u}(z; q)}{\partial z} + \tilde{u}(z; q) \left[1 - \frac{1}{2} \tilde{u}(z; q) \right] \right\}
\end{aligned} \quad (5.40)$$

and the corresponding m th-order ($m \geq 1$) deformation equation reads

$$\frac{d}{dz} \left\{ u_m(z) - \sum_{k=1}^{m-1} [(1 + c_0) b_{m-k} - c_0 a_{m-k}] u_k(z) \right\} = c_0 \sum_{k=1}^{m-1} a_{m-k} \delta_{k-1}(z), \quad (5.41)$$

subject to the initial condition

$$u_m(0) = 0. \quad (5.42)$$

The solution of the above high-order deformation equation reads

$$\begin{aligned}
u_m(z) &= \sum_{k=1}^{m-1} [(1 + c_0) b_{m-k} - c_0 a_{m-k}] u_k(z) \\
&+ c_0 \sum_{k=1}^{m-1} a_{m-k} \int_0^z \left[u'_k(z) + u_k(z) - \frac{1}{2} \sum_{i=0}^k u_i(z) u_{k-i}(z) \right] dx.
\end{aligned} \quad (5.43)$$

Thus, using the initial approximation $u_0(z) = 1$ and above recurrence formula, we can get $u_1(z)$, $u_2(z)$ and so on. It is found that the corresponding m th-order approximation reads

$$u_0(z) + \sum_{k=1}^m u_k(z) = 1 + \sum_{k=1}^m \left(\gamma_k z^k \right) T_{m,k}(c_0, A, B), \quad (5.44)$$

where $T_{m,k}(c_0, A, B)$ is exactly defined by (5.6) and (5.7) – (5.9), and $1 + \sum_{k=1}^{+\infty} \gamma_k z^k$ is the Taylor series of the closed-form solution $u(z) = 2/(1 + e^z)$ of (5.33). This is indeed a surprise!

We can prove the correctness of (5.44) in another way. Notice that the zeroth-order deformation equation (5.40) can be rewritten as

$$\frac{1 - (1 + c_0) B(q)}{-c_0 A(q)} \frac{\partial \tilde{u}(z; q)}{\partial z} + \tilde{u}(z; q) \left[1 - \frac{1}{2} \tilde{u}(z; q) \right] = 0, \quad (5.45)$$

i.e.

$$\frac{d\tilde{u}}{d\tau} + \tilde{u} \left(1 - \frac{1}{2} \tilde{u} \right) = 0, \quad (5.46)$$

whose solution, satisfying the initial condition $\tilde{u} = 1$ at $z = 0$, is exactly

$$\tilde{u}(z; q) = \frac{2}{1 + \exp(\tau)} \quad (5.47)$$

where

$$\bar{\tau} = \frac{-c_0 A(q) z}{1 - (1 + c_0) B(q)}$$

is a special case of τ when $z_0 = 0$, as defined by (5.11). Similarly, expanding $\tilde{u}(z; q)$ in Maclaurin series of q and then setting $q = 1$, we get the homotopy-series

$$u(z) = \tilde{u}(z; 1) = 1 + \lim_{m \rightarrow +\infty} \sum_{k=1}^m \left(\gamma_k z^k \right) T_{m,k}(c_0, A, B), \quad (5.48)$$

where $T_{m,k}(c_0, A, B)$ is exactly defined by (5.6) and (5.7) – (5.9), and γ_k is the coefficient of the Taylor series of the exact solution $u(z) = 2/(1 + e^z)$.

Therefore, the so-called homotopy transform described in Sect. 5.3 can be indeed derived in the frame of the HAM in some special cases. As proved in Sect. 5.3, the famous Euler transform $\mathcal{E}(\lambda)$ is only a special case of the homotopy transform $\mathcal{H}(c_0, A, B)$ in case of $c_0 = -\lambda$ and $A(q) = B(q) = q$. Thus, for some special choices of the initial approximation and the auxiliary linear operator, the HAM in case of $A(q) = B(q) = q$ are sometimes equivalent to the famous Euler transform.

In theory, this fact explains why the convergence of homotopy-series given by the HAM can be guaranteed, because the Euler transform is widely applied to accelerate the convergence of a series or to make a divergent series convergent. On the other hand, it should be emphasized that the homotopy analysis method is more general than Euler transform, because we have extremely large freedom to choose not only different types of deformation functions $A(q)$ and $B(q)$, but also the auxiliary linear operator $\mathcal{L}$ and the initial approximation. Note that the homotopy-transform $\mathcal{H}(c_0, A, B)$ defined by (5.23) is obtained in the frame of the HAM by using a special initial approximation $u_0(z) = 1$ and the special auxiliary linear operator $\mathcal{L}(u) = u'$ for the considered example. However, by means of the HAM, we have great freedom to choose other initial approximations and other auxiliary linear operators. For example, if the auxiliary linear operator $\mathcal{L}(u) = u' + \kappa u$ and the initial approximation $u_0(x) = \exp(-\kappa x)$ are chosen for the considered simple example (5.33), where $\kappa > 0$ is the second auxiliary parameter, we can obtain approximations expressed by exponential base functions

$$\{\exp(-\kappa x), \exp(-2\kappa x), \exp(-3\kappa x), \dots\}.$$

Obviously, such kind of homotopy-approximations contain two non-zero auxiliary parameters c_0 and κ , and thus is more general than Euler transform $\mathcal{E}(\lambda)$ that has only one auxiliary parameter λ .

Euler transform is widely used to accelerate the convergence of a sequence or even to make a divergent series convergent. In this chapter, it is proved that the famous Euler transform is a special case of the so-called homotopy-transform, which can be derived in the frame of the HAM by means of special initial approximation and auxiliary linear operator. This further explains in theory why the HAM can guarantee the convergence of homotopy-series, and why it is generally valid for so many highly nonlinear equations.

5.5 Concluding remarks

The convergence-control is a key concept of the HAM: it is the convergence-control parameter c_0 that provides us a convenient way to control and adjust the convergence of homotopy-series, so that the HAM is valid even for strongly nonlinear problems. In Chapter 2, we proved that, by introducing a non-zero auxiliary parameter c_0 , the convergence-domain of the power series of a real function $(1+z)^{-1}$ can be fantastically enlarged to the whole real axis except the singular point $z = -1$ only. In this chapter, we further *prove* this point in general. Note that the traditional Taylor series of a complex function $f(z)$, i.e.

$$f(z_0) + \sum_{k=1}^{+\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k,$$

converges only within a circle

$$\left| \frac{z - z_0}{\zeta - z_0} \right| < 1,$$

where ζ is a unique pole of $f(z)$. In the z -plane, the convergence-domain of the traditional Taylor series is within a circle with the radius $r = |\zeta - z_0|$. However, according to Theorem 5.5, by means of the so-called convergence-control parameter c_0 , the generalized Taylor series

$$f(z_0) + \lim_{m \rightarrow +\infty} \sum_{k=1}^m \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k T_{m,k}(c_0, A, B),$$

can converge to $f(z)$ in the *whole* z -plane except a half-line

$$\eta = \frac{z - z_0}{\zeta - z_0} > 1.$$

The comparison of the convergence-domains of the traditional and the generalized Taylor series is as shown in Fig.5.7, which clearly illustrates that the convergence-domain of a series can be indeed fantastically enlarged in general by introducing a non-zero auxiliary parameter c_0 , called the convergence-control parameter. This well explains why the so-called convergence-control parameter c_0 can guarantee the convergence of homotopy-series in the frame of the HAM in general. More importantly, it provides a corner-stone for the concept convergence-control of the HAM in theory.

Such kind of generalized Taylor series is further used to define the so-called homotopy-transform for a given sequence, which is unnecessary to be a power series. It is well known that the Euler transform is widely applied to accelerate a sequence or even to make a divergent sequence convergent. However, we prove that the famous Euler transform is only a special case of the homotopy transform. Besides, we illustrate that the homotopy-transform can be derived in the frame of the

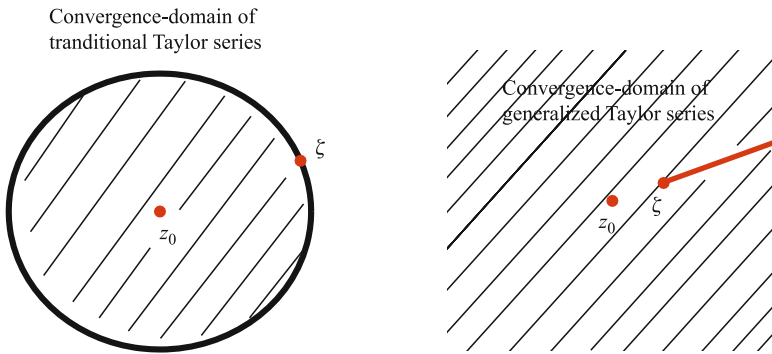


Fig. 5.7 Comparison of the convergence domains of the traditional and the generalized Taylor series of a complex function $f(z)$ with unique singular point at $z = \zeta$.

HAM by means of the simplest deformation-functions $A(q) = B(q) = q$. This explains from the another view point why the HAM can guarantee the convergence of series solution. In addition, this fact also shows the generality and great potential of the HAM.

Note that, unlike the Taylor series for a given function and Euler transform for a given sequence, the HAM is for nonlinear differential equations in general. Therefore, it is the HAM that introduces the convergence-control parameter c_0 and deformation-functions into the nonlinear differential equations in general so as to guarantee the convergence of series solutions. Note that, the HAM provides us extremely large freedom to choose not only the convergence-control parameter c_0 and deformation-functions $A(q)$ and $B(q)$, but also the auxiliary linear operator $\mathcal{L}$ and the initial approximation. Therefore, the HAM is much more general than the Euler transform, and thus should be valid for more complicated nonlinear problems in science and engineering.

In summary, the mathematical proofs given in this chapter provide us a cornerstone for the important concept *convergence-control* in the frame of the HAM, and well explain why the HAM is generally valid for so many highly nonlinear problems in science and engineering.

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Chapter 6

Some Methods Based on the HAM

Abstract In this chapter, some analytic and semi-analytic techniques based on the homotopy analysis method (HAM) are briefly described, including the so-called “homotopy perturbation method”, the optimal homotopy asymptotic method, the spectral homotopy analysis method, the generalized boundary element method, and the generalized scaled boundary finite element method. The relationships between these methods with the HAM are also revealed.

6.1 A brief history of the homotopy analysis method

To reveal the relationship between the homotopy analysis method (HAM) (Liao, 1992, 1997a, 1999, 2003, 2004; Liao and Tan, 2007; Liao, 2010) with other analytic approximation methods, we first briefly describe the basic ideas of the HAM and its history of development and modification.

The early HAM was first described by Shijun Liao (1992) in his PhD dissertation. For a given nonlinear differential equation

$$\mathcal{N}[u(x)] = 0, \quad x \in \Omega,$$

where $\mathcal{N}$ is a nonlinear operator and $u(x)$ is an unknown function, Liao (1992) used the concept of homotopy (Hilton, 1953) in topology (Sen, 1983) to construct a *one-parameter* family of equations in the embedding parameter $q \in [0, 1]$, called the zeroth-order deformation equation

$$(1 - q)\mathcal{L}[\phi(x; q) - u_0(x)] + q\mathcal{N}[\phi(x; q)] = 0, \quad x \in \Omega, \quad q \in [0, 1], \quad (6.1)$$

where $\mathcal{L}$ is an auxiliary linear operator and $u_0(x)$ is an initial guess. In theory, the concept of homotopy (Hilton, 1953) in topology (Sen, 1983) provides us extremely large freedom to choose the auxiliary linear operator $\mathcal{L}$ and the initial guess $u_0(x)$. At $q = 0$ and $q = 1$, we have $\phi(x; 0) = u_0(x)$ and $\phi(x; 1) = u(x)$, respectively. So, as the embedding parameter $q \in [0, 1]$ increases from 0 to 1, the solution $\phi(x; q)$ of the

zeroth-order deformation equation (6.1) varies (or deforms) from the initial guess $u_0(x)$ to the exact solution $u(x)$ of the original nonlinear equation $\mathcal{N}[u(x)] = 0$. Such kind of continuous variation is called deformation in topology, so that (6.1) is called the zeroth-order deformation equation. Since $\phi(x; q)$ is also dependent upon the embedding parameter $q \in [0, 1]$, we can expand it into the Maclaurin series with respect to q :

$$\phi(x; q) = u_0(x) + \sum_{n=1}^{+\infty} u_n(x) q^n, \quad (6.2)$$

called the homotopy-Maclaurin series. Assuming that, the auxiliary linear operator $\mathcal{L}$ and the initial guess $u_0(x)$ are so properly chosen that the above homotopy-Maclaurin series converges at $q = 1$, we have the so-called homotopy-series solution

$$u(x) = u_0(x) + \sum_{n=1}^{+\infty} u_n(x), \quad (6.3)$$

which satisfies the original equation $\mathcal{N}[u(x)] = 0$, as proved by Liao (1999, 2003) in general.

The governing equation of $u_n(x)$ is completely determined by the zeroth-order deformation equation (6.1). Differentiating (6.1) n times with respect to the embedding parameter q , then dividing by $n!$ and finally setting $q = 0$, we have the so-called high-order deformation equation

$$\mathcal{L}[u_n(x) - \chi_n u_{n-1}(x)] = -\mathcal{D}_{n-1} \{ \mathcal{N}[\phi(x; q)] \}, \quad (6.4)$$

where $\chi_1 = 0$ and $\chi_k = 1$ when $k \geq 2$, $\mathcal{D}_k$ is the so-called k th-order homotopy-derivative operator defined by

$$\mathcal{D}_k = \frac{1}{k!} \frac{\partial^k}{\partial q^k} \bigg|_{q=0}. \quad (6.5)$$

Note that the high-order deformation equation (6.4) is always linear with the known term on the right-hand side, therefore is easy to solve, as long as we choose the auxiliary linear operator $\mathcal{L}$ properly.

Unfortunately, the early HAM mentioned above can not guarantee the convergence of approximation series of nonlinear equations in general. To overcome this restriction, Liao (1997a) generalized the concept of homotopy and introduced such a non-zero auxiliary parameter c_0 to construct a *two-parameter* family of equations, i.e. the zeroth-order deformation equation

$$(1 - q) \mathcal{L}[\phi(x; q) - u_0(x)] = c_0 q \mathcal{N}[\phi(x; q)], \quad x \in \Omega, \quad q \in [0, 1]. \quad (6.6)$$

In this way, the homotopy-series solution (6.3) is dependent upon not only the physical variable x but also the auxiliary parameter c_0 . It has been proved (for details, please refer to Chapter 5) that the auxiliary parameter c_0 can adjust and control the convergence region of homotopy-series solutions, although c_0 has no physi-

cal meanings at all. In essence, the use of the auxiliary parameter c_0 introduces us one more “artificial” degree of freedom, which greatly improves the early HAM: it is the auxiliary parameter c_0 which provides us a convenient way to guarantee the convergence of homotopy-series solution. This is the reason why we call c_0 the *convergence-control parameter*. Differentiating (6.6) n times with respect to the embedding parameter q , then dividing by $n!$ and finally setting $q = 0$, we have the so-called high-order deformation equation

$$\mathcal{L}[u_n(x) - \chi_n u_{n-1}(x)] = c_0 \mathcal{D}_{n-1} \{ \mathcal{N}[\phi(x; q)] \}, \quad (6.7)$$

Note that (6.1) and (6.4) are special cases of (6.6) and (6.7) when $c_0 = -1$, respectively.

The use of the convergence-control parameter c_0 is a milestone in the development of the HAM. Realizing that more degrees of freedom imply larger possibility to gain better approximations, Liao (1999) further introduced more “artificial” degrees of freedom by using the zeroth-order deformation equation in a more general form:

$$[1 - \alpha(q)] \mathcal{L}[\phi(x; q) - u_0(x)] = c_0 \beta(q) \mathcal{N}[\phi(x; q)], \quad x \in \Omega, \quad q \in [0, 1], \quad (6.8)$$

where $\alpha(q)$ and $\beta(q)$ are the so-called *deformation functions* satisfying

$$\alpha(0) = \beta(0) = 0, \quad \alpha(1) = \beta(1) = 1, \quad (6.9)$$

whose Taylor series

$$\alpha(q) = \sum_{m=1}^{+\infty} \alpha_m q^m, \quad \beta(q) = \sum_{m=1}^{+\infty} \beta_m q^m, \quad (6.10)$$

are convergent for $|q| \leq 1$.

In fact, the zeroth-order deformation equation (6.8) can be further generalized, as shown by Liao (2003, 2004). Thus, the approximation series given by the HAM can contain many unknown convergence-control parameters, which provide us great possibility to guarantee the convergence of homotopy-series solution.

In addition, using these generalized zeroth-order deformation equation, Liao (2003) proved that the HAM logically contains other non-perturbation techniques, such as Lyapunov’s artificial small parameter method (Lyapunov, 1992), Adomian’s decomposition method (Adomian, 1976, 1994), the δ -expansion method (Karmishin et al., 1990), and so on, and thus is rather general.

6.2 Homotopy perturbation method

In 1998, six years later after Liao (1992) proposed the early HAM in his PhD dissertation, Jihuan He (1998, 1999) published the so-called “homotopy perturbation method”. Like the early HAM, the “homotopy perturbation method” is based on

constructing a homotopy equation

$$(1 - q)\mathcal{L}[\phi(x; q) - u_0(x)] + q\mathcal{N}[\phi(x; q)] = 0, \quad x \in \Omega, \quad q \in [0, 1], \quad (6.11)$$

which is exactly the same as the zeroth-order deformation equation (6.1). Like the HAM, the solution $\phi(x; q)$ is also expanded into Maclaurin series

$$\phi(x; q) = u_0(x) + \sum_{n=1}^{+\infty} u_n(x) q^n, \quad (6.12)$$

and the approximation is gained by setting $q = 1$, say,

$$u(x) = u_0(x) + \sum_{n=1}^{+\infty} u_n(x). \quad (6.13)$$

Obviously, (6.12) and (6.13) are exactly the same as (6.2) and (6.3), respectively. The only difference between the “homotopy perturbation method” and the early HAM is that the embedding parameter $q \in [0, 1]$ is regarded as a “small parameter” so that the governing equation of $u_n(x)$ is gained by substituting the series (6.12) into (6.11) and equating the coefficients of the like-power of q .

However, Hayat and Sajid (2007) proved that, substituting the Maclaurin series

$$\mathcal{N}[\phi(x; q)] = \sum_{n=0}^{+\infty} \mathcal{D}_n \{ \mathcal{N}[\phi(x; q)] \} q^n,$$

where $\mathcal{D}_n$ is defined by (6.5), and the series (6.12) into (6.11), then equating the coefficients of the like-power of q , one obtains

$$\mathcal{L}[u_n(x) - \chi_n u_{n-1}(x)] = -\mathcal{D}_{n-1} \{ \mathcal{N}[\phi(x; q)] \} \quad (6.14)$$

for $u_n(x)$, which is the *same* as the high-order deformation equation (6.4) exactly! So, no matter whether or not one regards the embedding parameter $q \in [0, 1]$ as a small parameter, one obtains the exactly same approximations as the early HAM. Therefore, Sajid and Hayat (2008) pointed out that “nothing is new in Dr. He’s approach, except the new name the homotopy perturbation method”.

This is easy to understand from the view points of mathematics: the so-called “homotopy perturbation method” is based on (6.11), which is exactly the same as the zeroth-order deformation equation (6.1) of the early HAM. Therefore, these two equations have the same solution $\phi(x; q)$. According to the fundamental theorem in calculus, the Maclaurin series of a function is unique. Therefore, as a coefficient of the Maclaurin series of $\phi(x; q)$, $u_n(x)$ is unique. Thus, $u_n(x)$ must be determined by the unique equation, say, (6.14) is exactly the same as (6.4).

Unfortunately, like the early HAM, the so-called “homotopy perturbation method” can not guarantee the convergence of approximations, so that it is valid only for weakly nonlinear problems with small physical parameters, as reported by many researchers. For example, Abbasbandy (2006) compared the modified HAM based on

(6.6) and the “homotopy perturbation method” by solving a nonlinear heat transfer equation

$$(1 + \varepsilon u)u' + u = 0, \quad u(0) = 1,$$

where the prime denotes the differentiation with respect to t , and $\varepsilon \geq 0$ is a physical parameter. By means of the “homotopy perturbation method” with the auxiliary linear operator $\mathcal{L}(u) = u' + u$ and the initial guess $u_0(t) = \exp(-t)$, one obtains the following approximations

$$u'(0) = -1 + \varepsilon - \varepsilon^2 + \varepsilon^3 - \varepsilon^4 + \dots,$$

which is convergent only for $0 \leq \varepsilon < 1$. Thus, “the HPM and perturbation method are valid only for small parameter ε ”, as concluded by Abbasbandy (2006). This illustrates that, like perturbation techniques, the “homotopy perturbation method” is *not* independent of small physical parameters in essence. “In fact, for many cases the solutions obtained by perturbation method and HPM are identical”, as pointed by Abbasbandy (2006). However, using the modified HAM based on (6.6) with the *same* auxiliary linear operator and the *same* initial guess but such a *different*¹ convergence-control parameter $c_0 = -(1 + \varepsilon)^{-1}$, Abbasbandy (2006) gained the first-order homotopy approximation of $u'(0)$ which is accurate for *all* physical parameter $0 \leq \varepsilon < +\infty$. Thus, Abbasbandy (2006) came to the conclusion that “HAM provides us with a convenient way to control the convergence of approximation series, which is a fundamental qualitative difference in analysis between HAM and other methods”.

Ganji et al. (2007) also applied the so-called “homotopy perturbation method” to solve the following differential equation

$$u_t + u_x = 2u_{xt}, \quad t > 0, \quad -\infty < x < +\infty,$$

subject to the initial condition

$$u(x, 0) = \exp(-x),$$

where the subscript denotes the differentiation. This problem has the closed-form solution

$$u(x, t) = \exp(-x - t).$$

Using the auxiliary linear operator $\mathcal{L}(u) = u_t$ and the initial guess $u_0(x, t) = \exp(-x)$, Ganji et al. (2007) reported “an analytic approximation” given by means of the so-called “homotopy perturbation method”. However, Liang and Jeffrey (2009) repeated their work and found that the approximations given by the so-called “homotopy perturbation method” are “divergent for *all* x and t except $t = 0$ ”. In other words, results given by the so-called “homotopy perturbation method” are divergent in the whole interval

$$-\infty < x < +\infty, \quad t > 0$$

¹ The “homotopy perturbation method” is in fact a special case of the HAM when $c_0 = -1$.

except $t = 0$ that however corresponds to the known initial condition! Liang and Jeffrey's work (Liang and Jeffrey, 2009) confirms Abbansbandy's conclusion (Abbansbandy, 2006) that the so-called "homotopy perturbation method" can not guarantee the convergence of the approximation and is valid for weakly nonlinear problems only. Besides, Liang and Jeffrey (2009) also applied the modified HAM based on (6.6) to solve the same problem: using the same auxiliary linear operator and the same initial guess, they gained the divergent series by means of the convergence-control parameter $c_0 = -1$, which is exactly the same as Ganji's approximation; however, using a different convergence-control parameter $c_0 = 1$, they obtained the accurate approximation which converges to the exact solution $\exp(-x - t)$ in the *whole* interval $-\infty < x < +\infty$ and $t \geq 0$. Liang and Jeffrey (2009) also came to the conclusion that "the HPM is a special case of the HAM" when $c_0 = -1$, and that "it is very important to investigate the convergence of approximation series, otherwise one might get useless results."

Turkyilmazoglu (2011) compared the HAM and the so-called "homotopy perturbation method" (HPM) from the view point of convergence, and came to the conclusion that "blindly using the HPM yields a non-convergence series to the sought solution. In addition to this, HPM is shown not always to generate a continuous family of solutions in terms of the homotopy parameter. By the convergence-control parameter this can however be prevented to occur in the HAM".

These examples illustrate that the so-called "homotopy perturbation method" is exactly the same as the early HAM. Thus, as a special case of the modified HAM when $c_0 = -1$, the so-called "homotopy perturbation method" can not give anything new indeed. Besides, they also reveal the importance of the convergence-control parameter c_0 in theory. The use of the convergence-control parameter c_0 is a milestone of the HAM: it is the convergence-control parameter c_0 which provides us a convenient way to guarantee the convergence of series solution so that the HAM becomes independent of small/large physical parameters in essence. In fact, it is the convergence-control parameter c_0 which differs the HAM from all other analytic approximation methods.

It is a pity that the basic ideas of the HAM were not mentioned at all by Dr. Jihuan He in his paper about the so-called "homotopy perturbation method (HPM)" in 1998: Liao's work related to the HAM was only simply cited (without saying anything about it) when mentioning the fundamental concept of the homotopy in topology. This is not strictly ethical.

6.3 Optimal homotopy asymptotic method

In 2007, Yabushita et al. (2007) first used the minimum of squared residual of governing equations to determine optimal convergence-control parameters in the frame of the HAM. In 2008, Akyildiz and Vajravelu (2008) suggested to use the optimal convergence-control parameter determined by the minimum of the squared residual of governing equation.

In 2008, Marinca and Heriřanu (2008, 2009) suggested the so-called “optimal homotopy asymptotic method” based on the homotopy equation

$$(1 - q)\mathcal{L}[\phi(x; q) - u_0(x)] = \left(\sum_{i=1}^{+\infty} c_i q^i \right) \mathcal{N}[\phi(x; q)], \quad x \in \Omega, \quad q \in [0, 1], \quad (6.15)$$

where the optimal value of c_i ($i = 1, 2, 3, \dots$) is determined by the minimum of squared residual of governing equations. Let

$$E_m = \int \left\{ \mathcal{N} \left[\sum_{n=0}^m u_n(x) \right] \right\}^2 dx$$

denote the squared residual of governing equation $\mathcal{N}[u(x)] = 0$ at the m th-order of approximation. Then, one had to solve a set of nonlinear algebraic equations

$$\frac{\partial E_m}{\partial c_i} = 0, \quad 1 \leq i \leq m \quad (6.16)$$

so as to gain the optimal m th-order approximation.

Note that, substituting

$$\alpha(q) = q, \quad \beta(q) = \frac{1}{c_0} \sum_{i=1}^{+\infty} c_i q^i, \quad c_0 = \sum_{i=1}^{+\infty} c_i$$

into the zeroth-order deformation equation (6.8), one obtains exactly the same equation as (6.15). So, Marinca and Heriřanu’s approach (Marinca and Heriřanu, 2008, 2009) is still in the frame of the HAM. However, Marinca and Heriřanu’s approach (Marinca and Heriřanu, 2008, 2009) is interesting: one can regard each c_i as a convergence-control parameter, as long as $\sum_{i=1}^{+\infty} c_i$ is convergent to a bounded value.

The optimal approach suggested by Marinca and Heriřanu (2008, 2009) is rigorous in theory. However, the number of unknown convergence-control parameters linearly increases as the order of approximation increases, so that it becomes time-consuming in practice to solve the set of corresponding nonlinear algebraic equations related to the high-order optimal approximations, as illustrated by Niu and Wang (2010). In 2010, Liao (2010) suggested an optimal HAM with only three convergence-control parameters at the most, and illustrated that the optimal HAM with one or two convergence-control parameters seems to be most efficient computationally.

In Chapter 3, an optimal HAM is proposed based on the zeroth-order deformation equation

$$(1 - q)\mathcal{L}[\phi(x; q) - u_0(x)] = \left(\sum_{i=0}^{\kappa} c_i q^{i+1} \right) \mathcal{N}[\phi(x; q)], \quad x \in \Omega, \quad q \in [0, 1], \quad (6.17)$$

where the optimal value of c_i is determined by the minimum of squared residual E_m of the m th-order approximation, i.e.

$$\frac{\partial E_m}{\partial c_i} = 0, \quad 0 \leq i \leq \min\{m, \kappa\}. \quad (6.18)$$

The above optimal HAM contains the basic convergence-control parameter c_0 when $\kappa = 0$, and becomes Marinca and Herișanu's approach (Marinca and Herișanu, 2008, 2009) when $\kappa \rightarrow +\infty$.

6.4 Spectral homotopy analysis method

In 2010, Motsa et al. (2010a) suggested the so-called “spectral homotopy analysis method” (SHAM) by using the Chebyshev pseudospectral method to solve the linear high-order deformation equations and choosing the auxiliary linear operator $\mathcal{L}$ in terms of the Chebyshev spectral collocation differentiation matrix described by Don and Solomonoff (1995).

For example, to solve the high-order deformation equation (6.7) in a finite interval $x \in [a, b]$ by means of the SHAM (Motsa et al., 2010a,b), one approximates $u_n(x)$ by means of the truncated Chebyshev polynomial

$$u_n(x) = \sum_{k=0}^M a_k T_k(x),$$

where $T_k(x)$ is the k th Chebyshev polynomial of the first kind, and the unknown coefficient a_k is determined by the discrete collocation points and related boundary conditions.

In theory, any a continuous function in a bounded interval can be best approximated by Chebyshev polynomial. So, the SHAM provides larger freedom to choose the auxiliary linear operator $\mathcal{L}$ and initial guess. The basic idea of the SHAM might be expanded to solve nonlinear partial differential equations. Besides, it is easy to employ the optimal convergence-control parameter in the frame of the SHAM. Thus, the SHAM has great potential to solve more complicated nonlinear problems in science and engineering, although further modifications in theory and more applications are needed.

Chebyshev polynomial is a kind of special function. There are many other special functions such as Hermite polynomial, Legendre polynomial, Airy function, Bessel function, Riemann zeta function, hypergeometric functions and so on. Since the HAM provides us extremely large freedom to choose the auxiliary linear operator $\mathcal{L}$ and the initial guess, it should be possible to develop a “generalized spectral HAM” which can use a proper special function for a given nonlinear problem.

6.5 Generalized boundary element method

In essence, the HAM replaces a nonlinear problem by means of an infinite number of linear sub-problems, since the high-order deformation equation is always linear

and governed by the auxiliary linear operator $\mathcal{L}$. If the initial guess and the auxiliary linear operator $\mathcal{L}$ are so properly chosen that the analytic solution of the high-order deformation equation can be gained, then we obtain the analytic homotopy-approximation exactly, whose convergence is guaranteed by choosing proper value of the convergence-control parameter. However, obviously, the linear high-order deformation equation can be solved by means of different numerical techniques, such as the finite difference method (FDM), the finite element method (FEM), the finite volume method (FVM), the boundary element method (BEM), and so on. So, in theory, it is very easy to combine the HAM with advanced numerical techniques. Since the numerical techniques are valid for differential equations defined in rather complicated domain, the combination of the HAM with numerical techniques can greatly enlarge the application fields of the HAM.

For example, based on the HAM, Liao (1997b,c,d) proposed the so-called “generalized boundary element method”. The traditional BEM is often valid for a linear differential equation $\mathcal{L}_0(u) = 0$, whose solution can be expressed by integration of a fundamental solution on the boundary. When the traditional BEM is applied to solve a nonlinear differential equation

$$\mathcal{L}_0(u) + \mathcal{N}_0(u) = 0,$$

where $\mathcal{L}_0(u)$ and $\mathcal{N}_0(u)$ denote the linear and nonlinear parts of the governing equation, one often rewrites $\mathcal{L}_0(u) = -\mathcal{N}_0(u)$ and uses iteration approach by regarding the right-hand side term as the known ones. Unfortunately, this approach has strong restrictions on the linear operator $\mathcal{L}_0$, and thus does not work if the fundamental solution of $\mathcal{L}_0$ is unknown, or if the highest order of derivative of $\mathcal{L}_0$ is lower than that of the governing equation, or if the linear operator $\mathcal{L}_0$ does not exist at all, and so on. However, the HAM provides us extremely large freedom to choose the auxiliary linear operator $\mathcal{L}$. So, in the frame of the HAM, we can always choose such a proper auxiliary linear operator $\mathcal{L}$ that the linear high-order deformation equation can be solved by means of the traditional BEM.

Combining the HAM with the traditional BEM in this way, many nonlinear problems can be solved by means of the so-called generalized BEM. For example, by means of the generalized BEM, Wu and Liao (2005) successfully obtained the convergent results of driven cavity viscous flows at Reynolds number up to $R_e = 10000$, governed by the exact Navier-Stokes equation. Note that, one often obtains convergent numerical result of driven cavity flow with only $R_e = 1000$ by means of traditional BEM. This illustrates the great potential of the generalized BEM.

6.6 Generalized scaled boundary finite element method

The scaled boundary finite-element method (SBFEM), developed by Song and Wolf (1997), is a novel semi-analytical method to solve linear partial differential equations in complicated domain. Briefly speaking, in the frame of the SBFEM (Song

and Wolf, 1997; Wolf and Song, 2000, 2001; Wolf, 2003), a scaled boundary coordinate system is first introduced, then the weighted residual approximation of finite elements is applied in the circumferential direction, and the governing partial differential equations are transformed to ordinary differential equations in the radial direction, which can be solved analytically in the radial direction. Like the BEM, only the boundary of the domain is discretized, but no fundamental solution is required. Thus, this semi-analytical method combines the advantages of the finite element method and boundary element methods.

The traditional SBFEM is widely applied to the problems related to elastostatics and elasto-dynamics, especially for soil-structure interaction problems in unbounded domains (Song and Wolf, 1997; Wolf, 2003). Recently, the SBFEM has been extended to the fluid flow problems (Tao et al., 2007). By coupling the finite element method and the SBFEM, Doherty and Deeks (2005) captured the nonlinearity of problems in the near field. However, the SBFEM only accurately modeled the linear elastic far field response. Up to now, all problems solved by the traditional SBFEM are governed by linear partial differential equations.

In essence, the HAM replaces a nonlinear problem by means of an infinite number of linear sub-problems, since the high-order deformation equations are always linear. Especially, the HAM provides us extremely large freedom to choose the auxiliary linear operator $\mathcal{L}$, and besides can guarantee the convergence of approximations by means of proper convergence-control parameter c_0 . Thus, using the traditional scaled boundary finite element method to solve the linear high-order deformation equations, Lin and Liao (2011) proposed the so-called “generalized scaled boundary finite element method” by combining the HAM with the traditional scaled boundary finite element method. Using a nonlinear heat transfer problem as an example, Lin and Liao (2011) illustrated the validity of the generalized SBFEM for nonlinear partial differential equations.

Since electronic computers appear, hundreds of numerical techniques and analytic methods have been developed independently. However, methods based on the combination of analytic and numerical techniques are much less. Such kind of semi-analytic methods may combine the advantages of the high accuracy of analytic methods and the flexibility of numerical methods for complicated domains. So, the generalized SBFEM has great potential in future, although it needs further modifications in theory and more applications in practice.

6.7 Predictor homotopy analysis method

Abbasbandy et al. (2009), Abbasbandy and Shivanian (2010, 2011) proposed the so-called “the predictor homotopy analysis method” (PHAM) to predict the multiplicity of solutions of nonlinear equations. Using the PHAM, they obtained multiple solutions of some nonlinear differential equations by means of different values of the convergence-control parameter c_0 with the same auxiliary linear operator $\mathcal{L}$ and even the same initial guess. As pointed out by Abbasbandy and Shivanian (2010),

this trait makes HAM to be different from the other analytical techniques which are used to approach one solution but possibly lose the others.

For details, please refer to Abbasbandy et al. (2009), Abbasbandy and Shivanian (2010, 2011).

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Part II

Mathematica Package BVPh and Its Applications

“If you shut your door to all errors, truth will be shut out.”

by Rabindranth Tagore (1861 — 1941)

Chapter 7

Mathematica Package BVPh

Abstract The BVPh (version 1.0) is a Mathematica package for highly nonlinear boundary-value/eigenvalue problems with singularity and/or multipoint boundary conditions. It is a combination of the homotopy analysis method (HAM) and the computer algebra system Mathematica, and provides us a convenient analytic tool to solve many nonlinear ordinary differential equations (ODEs) and even some nonlinear partial differential equations (PDEs). In this chapter, we briefly describe its scope, the basic mathematical formulas, and the choice of base functions, initial guess and the auxiliary linear operator, and so on, together with a simple users guide. As open resource, the BVPh 1.0 is given in the appendix of this chapter and free available (Accessed 25 Nov 2011, will be updated in the future) at

<http://numericaltank.sjtu.edu.cn/BVPh.htm>

7.1 Introduction

By means of high-performance computer and numerical techniques such as Runge-Kutta's method, it is convenient to gain accurate numerical approximations of most of nonlinear initial-value problems (IVPs) except those with chaos (Lorenz, 1963; Liao, 2009b). However, it is more difficult to solve boundary-value problems (BVPs), especially when there exist high nonlinearity, multiple solutions, singularity and an infinite interval.

The BVP4C is a famous software in MATLAB (Kierzenka and Shampine, 2001; Shampine et al., 2003, online) for multipoint boundary-value problems. For more details about the BVP4C, please refer to the website (Accessed 15 April 2011) at

<http://www.mathworks.com/help/techdoc/ref/bvp4c.html>

Many linear ordinary differential equations (ODEs) can be solved by means of the BVP4C. The shooting method (Shampine et al., 2003) is employed in BVP4C, which first transforms a boundary-value problem (BVP) into an initial value problem (IVP) by adding some guessed initial conditions and then correcting them step by step in such a way that all original boundary conditions are satisfied. So, BVP4C is a

numerical tool in essence. Using BVP4c, nonlinear problems are often solved by traditional iteration approaches such as Newton's iteration. Unfortunately, it is well-known that convergence of iteration can *not* be guaranteed by these traditional iteration approaches, especially when the nonlinearity is strong. Besides, by means of BVP4c, it is generally not easy to find out *all* multiple solutions of BVPs, especially when these solutions are very close, and/or rather sensitive to the guessed initial conditions of the shooting method (Shampine et al., 2003). Furthermore, based on numerical computations, it is difficult for BVP4c to resolve the singularity in governing equations and/or boundary conditions, such as $\sin(\pi z)/z$ at $z = 0$, even if $\sin(\pi z)/z$ has the limit π as $z \rightarrow 0$. In addition, BVP4c regards an infinite interval as a kind of singularity and replaces it by a finite one: this results in additional inaccuracy and uncertainty of solutions.

The Chebfun (Battles and Trefethen, 2004; Trefethen, 2007) is a collection of algorithms, and a software system in MATLAB (Kierzenka and Shampine, 2001; Shampine et al., 2003, online), developed by Nick Trefethen and Zachary Battles of Oxford University since 2002 and Toby Driscoll of the University of Delaware beginning in 2008. For more details about the Chebfun, please refer to Battles and Trefethen (2004), Trefethen (2007) and the website (Accessed 15 April 2011) at

<http://www2.maths.ox.ac.uk/~chebfun/>

As mentioned by the Chebfun team, the Chebfun is a powerful "numerical" tool based on Chebyshev expansions, fast Fourier transform, barycentric interpolation and so on. But, unlike BVP4c, solutions of differential equations given by Chebfun are expressed by a sum of some smooth base functions such as Chebyshev polynomials so that it can be *exactly* differentiated *any* times at *any* a given point! This is in essence different from BVP4c, although both of Chebfun and BVP4c are based on MATLAB. Especially, using such kind of base functions, it is natural and much easier to resolve singularities in equations and/or boundary conditions, and besides to solve differential equations in an infinite interval exactly. So, "computing numerically with functions instead of numbers" (Trefethen, 2007) is indeed a wonderful idea!

Although linear differential equations can be solved conveniently by Chebfun, only a few examples for nonlinear differential equations are given. This is mainly because, like BVP4c, the Chebfun uses Newton's iteration to solve non-linear problems, but it is well-known that convergence of Newton's iteration is strongly dependent upon initial guesses and thus is *not* guaranteed. Besides, Chebfun searches for multiple solutions of a nonlinear ODE by using different guess approximations. However, it is not very clear how to gain these different guess approximations. So, it seems difficult to solve highly nonlinear differential equations with multiple solutions by means of the Chebfun (version 4.0).

Inspired by the general validity of the homotopy analysis method (HAM) (Li et al., 2010; Liao, 1992, 1997, 1999a,b; Liao and Campo, 2002; Liao, 2003a,b, 2004, 2005, 2006, 2009a, 2010; Liao and Tan, 2007; Xu et al., 2010) for highly nonlinear problems in so many different fields (Abbas et al., 2008; Abbasbandy, 2006, 2007; Abbasbandy and Parkes, 2008; Abbasbandy, 2008; Abbasbandy et al., 2009;

Abbasbandy and Parkes, 2010; Abbasbandy and Shivanian, 2011; Akyildiz and Vajravelu, 2008; Akyildiz et al., 2009; Alizadeh-Pahlavan and Borjian-Boroujeni, 2008; Alizadeh-Pahlavan et al., 2009; Allan and Syam, 2005; Allan, 2007, 2009; Cai, 2006; Cheng, 2008; Gao, 2007; Hayat and Sajid, 2007; Jiao, 2009; Jiao et al., 2009; Kumari and Nath, 2010; Kumari et al., 2010; Liang, 2010; Liang and Jeffrey, 2009a,b, 2010; Liu, 2008; Liu and Li, 2009; Mahapatra et al., 2009; Marinca and Herişanu, 2008, 2009; Molabahrami and Khani, 2009; Motsa et al., 2010a,b; Niu and Wang, 2010; Pandey et al., 2011; Pirbodaghi et al., 2009; Sajid, 2006; Sajid and Hayat, 2008; Shidfar et al., 2010; Shidfar and Molabahrami, 2010; Siddheshwar, 2010; Singh et al., 2009; Song and Tao, 2007; Tao et al., 2007; Turkyilmazoglu, 2009, 2010a,b, 2011a,b,c,d; Van Gorder and Vajravelu, 2008, 2009; Van Gorder et al., 2010a,b; Van Gorder and Vajravelu, 2011; Wu, 2009; Wu and Cheung, 2008, 2009; Yabushita et al., 2007; Zand et al., 2009; Zand and Ahmadian, 2009; Zhao and Wong, 2008; Zhu, 2008, 2006a,b; Zou, 2008; Zou et al., 2007) and by the ability of “computing with functions instead of numbers” (Trefethen, 2007) provided by computer algebra system such as Mathematica (Abell and Braselton, 2004) and Maple, the author developed a Mathematica package BVPh (version 1.0) for highly nonlinear differential equations with multiple solutions and singularities. Our aim is to develop a kind of *analytic* tool for *as many* nonlinear BVPs *as possible* such that multiple solutions of highly nonlinear BVPs can be conveniently found out, and that the infinite interval and singularities of governing equations and/or boundary conditions can be easily resolved.

As shown in Part I of this book, the HAM has some obvious advantages over other traditional analytic approximation methods. First, based on the homotopy in topology, the HAM is independent of small/large physical parameters, and thus is valid even if a nonlinear problem does not contain any perturbed quantities at all. Besides, the so-called homotopy-control parameter c_0 introduced by Liao (1997) provides a convenient way to guarantee the convergence of series solution, so that, different from other analytic approximation methods, the HAM is valid for highly nonlinear problems. Furthermore, the HAM provides us extremely large freedom to choose initial guess and the auxiliary linear operator $\mathcal{L}$ so that multiple solutions of nonlinear problems can be easily found out. In addition, using the idea of *computing with functions instead of numbers* (Trefethen, 2007), the infinite interval and singularities of governing equation and/or boundary conditions can be resolved in a easy and natural way by means of computer algebra system such as Mathematica, Maple and so on. Therefore, in the frame of the HAM, it is possible to develop such a Mathematica package for nonlinear boundary value problems, which has the following characteristics:

- **Guarantee of convergence:** the convergence of series solution is guaranteed by choosing a proper convergence-control parameter c_0 in the frame of the HAM;
- **Multiple solutions:** multiple solutions of nonlinear problems are found out by means of the freedom of the HAM on the choice of different guess approximations and different auxiliary linear operators;

- **Singularity:** singularities of governing equations and/or boundary conditions are resolved analytically by computing with functions of the computer algebra system Mathematica, instead of numbers;
- **Infinite interval:** equations are solved in an infinite interval exactly by means of choosing proper base functions defined in the infinite interval.

The Mathematica package BVPh (version 1.0) is given in the appendix of this chapter and free available at

<http://numericaltank.sjtu.edu.cn/BVPh.htm>

Here, we briefly describe its scope, the basic mathematical formulas, and the choice of base functions, initial guess, the auxiliary linear operator $\mathcal{L}$, the auxiliary function and the convergence-control parameter c_0 , together with a simple users guide. Twelve examples are used to show its validity for some types of nonlinear ODEs and PDEs, with the corresponding files of input data for BVPh 1.0 given in the appendix of chapters in Part II, which are free available at the same website mentioned above.

7.1.1 Scope

The BVPh 1.0 provides us an analytic tool to solve some nonlinear ordinary differential equations (ODEs) and partial differential equations (PDEs), mainly related to the following problems:

1. a nonlinear boundary-value equation $\mathcal{F}[z, u] = 0$ in a finite interval $z \in [0, a]$,
2. a nonlinear boundary-value equation $\mathcal{F}[z, u] = 0$ in an infinite interval $z \in [b, \infty)$,
3. a nonlinear eigenvalue equation $\mathcal{F}[z, u, \lambda] = 0$ in a finite interval $z \in [0, a]$,
4. a nonlinear PDE related to non-similarity and/or unsteady boundary-layer flows,

where $\mathcal{F}$ denotes a nonlinear differential operator, $u(z)$ is an unknown function, λ is an unknown eigenvalue, $a > 0$ and $b \geq 0$ are known constants, respectively. The boundary conditions are linear, which may be defined at multipoints including the two endpoints.

Some examples are given in the following chapters of Part II to show the validity of the BVPh 1.0 for above mentioned problems. In Chapter 8, we illustrate how to find out multiple solutions of nonlinear ODEs by means of different initial guesses and base functions. Five examples are used in Chapter 9 to illustrate how to use BVPh 1.0 to solve highly nonlinear eigenvalue problems with singularity, and/or varying (real or complex) coefficients and/or multipoint boundary conditions. In Chapter 10, we illustrate how to solve nonlinear ODEs in an infinite interval by means of the BVPh 1.0, whose solutions decay either exponentially or algebraically at infinity. In Chapter 11 and Chapter 12, we illustrate that the BVPh 1.0 can be even applied to solve some nonlinear PDEs, such as non-similarity and/or unsteady boundary-layer flows. All of these examples show the validity of the BVPh 1.0 for

some types of highly nonlinear ODEs and PDEs with multiple solutions and singularity.

Note that the BVPh 1.0 is only the beginning of our attempt. Modified versions for more types of nonlinear ODEs and PDEs with more complicated and stronger singularities will be released in future. It is indeed very difficult to develop a general Mathematica package valid for as many nonlinear ODEs and PDEs as possible. For this reason, our codes are free available online as open resource so that researchers in different countries and different generations can do their contributions to finish this hard work together, since science belongs to the whole human-being.

7.1.2 Brief mathematical formulas

The BVPh 1.0 is based on the HAM. Here, the mathematical formulas are briefly described.

7.1.2.1 Boundary-value problems in a finite interval

Consider nonlinear boundary-value problems governed by a nonlinear n th-order ODE in a finite interval

$$\mathcal{F}[z, u] = 0, \quad z \in [0, a], \quad (7.1)$$

subject to the n linear boundary conditions

$$\mathcal{B}_k[z, u] = \gamma_k, \quad 1 \leq k \leq n, \quad (7.2)$$

where $\mathcal{F}$ is a n th-order nonlinear differential operator, $\mathcal{B}_k$ is a linear differential operator, $u(z)$ is a smooth function, $a > 0$ and γ_k are constants, respectively. The boundary conditions may be defined at multipoints including the two endpoints. Assume that at least one solution exists, and that all solutions are smooth.

Let $q \in [0, 1]$ denote the embedding parameter, $u_0(z)$ an initial guess of $u(z)$, respectively. In the frame of the HAM, we construct such a continuous variation (or deformation) $\phi(z; q)$ that, as q increases from 0 to 1, $\phi(z; q)$ varies continuously from the initial guess $u_0(z)$ to the solution $u(z)$ of (7.1) and (7.2). Such kind of continuous variation is governed by the so-called zeroth-order deformation equation

$$(1 - q)\mathcal{L}[\phi(z; q) - u_0(z)] = c_0 q H(z) \mathcal{F}[z, \phi(z; q)], \quad z \in [0, a], \quad q \in [0, 1], \quad (7.3)$$

subject to the boundary conditions

$$\mathcal{B}_k[z, \phi(z; q)] = \gamma_k, \quad 1 \leq k \leq n, \quad (7.4)$$

where $\mathcal{L}$ is an auxiliary linear operator, c_0 is the so-called convergence-control parameter, $H(z)$ is an auxiliary function, respectively.

As show in Part I, the HAM provides us extremely large freedom to choose the auxiliary linear operator $\mathcal{L}$, the convergence-control parameter c_0 and the auxiliary function $H(z)$. Assume that all of them are properly chosen so that the homotopy-Maclaurin series

$$\phi(z; q) = u_0(z) + \sum_{m=1}^{+\infty} u_m(z) q^m \quad (7.5)$$

absolutely converges at $q = 1$, where

$$u_m(z) = \mathcal{D}_m [\phi(z; q)] = \frac{1}{m!} \left. \frac{\partial^m \phi(z; q)}{\partial q^m} \right|_{q=0}.$$

Here, $\mathcal{D}_m$ is called the m th-order homotopy-derivative operator. Then, we have the so-called homotopy-series solution

$$u(z) = u_0(z) + \sum_{m=1}^{+\infty} u_m(z), \quad (7.6)$$

where $u_m(z)$ is governed by the so-called m th-order deformation equation

$$\mathcal{L} [u_m(z) - \chi_m u_{m-1}(z)] = c_0 H(z) \delta_{m-1}(z), \quad z \in [0, a], \quad (7.7)$$

subject to the n linear boundary conditions

$$\mathcal{B}_k [z, u_m(z)] = 0, \quad 1 \leq k \leq n, \quad (7.8)$$

where

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1, \end{cases} \quad (7.9)$$

and

$$\delta_k(z) = \mathcal{D}_k \{ \mathcal{F} [z, \phi(z; q)] \} = \left\{ \frac{1}{k!} \frac{\partial^k \mathcal{F} [z, \phi(z; q)]}{\partial q^k} \right\} \bigg|_{q=0} \quad (7.10)$$

can be easily obtained by means of the Theorems proved in Chapter 4.

Note that the m th-order deformation equation (7.7) is linear, subject to the n linear boundary conditions (7.8), and therefore is easy to solve in theory, especially by computer algebra system like Mathematica. Especially, the convergence-control parameter c_0 provides us a convenient way to guarantee the convergence of the homotopy-series (7.6), whose optimal value is determined by the minimum of the squared residual of the governing equation (7.1) at high enough order approximation of $u(z)$. For more details, please refer to Chapter 8.

Note that, for boundary-value problems in a finite interval $z \in [0, a]$, we should set the control parameter `TypeEQ = 1` for the BVPh 1.0.

7.1.2.2 Boundary-value problems in an infinite interval

Consider boundary-value problems governed by a nonlinear n th-order ODE in an infinite interval

$$\mathcal{F}[z, u] = 0, \quad z \in [b, +\infty), \quad (7.11)$$

subject to the n linear boundary conditions

$$\mathcal{B}_k[z, u] = \gamma_k, \quad 1 \leq k \leq n, \quad (7.12)$$

where $\mathcal{F}$ is a n th-order nonlinear differential operator, $\mathcal{B}_k$ is a linear differential operator, $u(z)$ is a smooth function, $b \geq 0$ and γ_k are constants, respectively. The boundary conditions may be defined at multipoints including the two endpoints. Assume that at least one solution exists, and that all solutions are smooth.

All related formulas are the same as those given above in Sect 7.1.2.1, except that the finite interval $z \in [0, a]$ is replaced by the infinite one $z \in [b, +\infty)$. However, completely different initial guess $u_0(z)$ and auxiliary linear operator $\mathcal{L}$ are used for these two types of boundary-value problems, because their solutions are expressed by completely different base functions. For more details, please refer to Chapter 10.

When the BVPh 1.0 is used to solve boundary-value problems in an infinite interval $z \in [b, +\infty)$, we should set `TypeEQ = 1` and `zR = infinity`.

7.1.2.3 Eigenvalue problems in a finite interval

Consider eigenvalue problems governed by a nonlinear n th-order ODE in a finite interval

$$\mathcal{F}[z, u, \lambda] = 0, \quad z \in [0, a], \quad (7.13)$$

subject to the n linear boundary conditions

$$\mathcal{B}_k[z, u] = \gamma_k, \quad 1 \leq k \leq n, \quad (7.14)$$

where $\mathcal{F}$ is a n th-order nonlinear differential operator, $\mathcal{B}_k$ is a linear differential operator, $u(z)$ is a smooth eigenfunction, λ is an unknown eigenvalue, $a > 0$ and γ_k are constants, respectively. The boundary conditions may be defined at multipoints including the two endpoints. Assume that at least one eigenfunction and one eigenvalue exist, and that all eigenfunctions are smooth.

Eigenvalue problems often have an infinite number of eigenfunctions and eigenvalues. To distinguish different eigenfunctions and eigenvalues, we should add one additional boundary condition

$$\mathcal{B}_0[z, u] = \gamma_0, \quad (7.15)$$

where $\mathcal{B}_0$ is a linear differential operator and γ_0 is a constant.

Let $q \in [0, 1]$ denote the embedding parameter, $u_0(z)$ an initial guess of the eigenfunction $u(z)$, λ_0 an initial guess of the eigenvalue λ , respectively. In the frame of the

HAM, we first construct such two continuous variations (or deformations) $\phi(z; q)$ and $\Lambda(q)$ that, as q increases from 0 to 1, $\phi(z; q)$ varies continuously from the initial guess $u_0(z)$ to the eigenfunction $u(z)$ of (7.13) and (7.14), so does $\Lambda(q)$ from the initial guess λ_0 to the eigenvalue λ , respectively. Such two continuous variations are governed by the so-called zeroth-order deformation equation

$$(1 - q)\mathcal{L}[\phi(z; q) - u_0(z)] = c_0 q H(z) \mathcal{F}[z, \phi(z; q), \Lambda(q)], \quad z \in [0, a], \quad (7.16)$$

subject to the n boundary conditions

$$\mathcal{B}_k[z, \phi(z; q)] = \gamma_k, \quad 1 \leq k \leq n, \quad (7.17)$$

and the additional boundary condition

$$\mathcal{B}_0[z, \phi(z; q)] = \gamma_0, \quad (7.18)$$

where $\mathcal{L}$ is the auxiliary linear operator, c_0 is the so-called convergence-control parameter, $H(z)$ is the auxiliary function, respectively.

As shown in Part II, the HAM provides us extremely large freedom to choose the auxiliary linear operator $\mathcal{L}$, the convergence-control parameter c_0 and the auxiliary function $H(z)$. Assume that all of them are properly chosen so that the homotopy-Maclaurin series

$$\phi(z; q) = u_0(z) + \sum_{m=1}^{+\infty} u_m(z) q^m, \quad \Lambda(q) = \lambda_0 + \sum_{m=1}^{+\infty} \lambda_m q^m, \quad (7.19)$$

absolutely converge at $q = 1$. Then, we have the so-called homotopy-series solution

$$u(z) = u_0(z) + \sum_{m=1}^{+\infty} u_m(z), \quad \lambda = \lambda_0 + \sum_{m=1}^{+\infty} \lambda_m, \quad (7.20)$$

where the unknown $u_m(z)$ is governed by the m th-order deformation equation

$$\mathcal{L}[u_m(z) - \chi_m u_{m-1}(z)] = c_0 H(z) \delta_{m-1}(z), \quad z \in [0, a], \quad (7.21)$$

subject to the n linear boundary conditions

$$\mathcal{B}_k[z, u_m(z)] = 0, \quad 1 \leq k \leq n, \quad (7.22)$$

where

$$\delta_k(z) = \mathcal{D}_k \{ \mathcal{F}[z, \phi(z; q), \Lambda(q)] \} = \left\{ \frac{1}{k!} \frac{\partial^k \mathcal{F}[z, \phi(z; q), \Lambda(q)]}{\partial q^k} \right\} \Big|_{q=0} \quad (7.23)$$

can be easily obtained by means of the Theorems proved in Chapter 4. Here, $\mathcal{D}_k$ is the so-called k th-order homotopy-derivative operator. Note that $\delta_{m-1}(z)$ in (7.21) contains

$$\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_{m-1}.$$

In addition, the unknown λ_{m-1} is determined by the additional boundary-condition

$$\mathcal{B}_0[z, u_m(z)] = 0. \quad (7.24)$$

For more details, please refer to Chapter 9.

In fact, the BVP 1.0 is valid for more generalized boundary conditions

$$\mathcal{B}_k[z, u, \lambda] = 0, \quad 1 \leq k \leq n, \quad (7.25)$$

which may contain the unknown eigenvalue λ .

Note that, when the BVP 1.0 is used to solve eigenvalue problems in a finite interval $z \in [0, a]$, we should set $\text{TypeEQ} = 2$.

7.1.3 Choice of base function and initial guess

In the frame of the HAM, we should first of all choose a set of base functions

$$\{e_0(z), e_1(z), e_2(z), \dots\},$$

which is good enough to approximate the unknown solutions $u(z)$ of a nonlinear boundary-value problem. In other words, we should choose such a kind of base functions that $u(z)$ can be expressed in the form

$$u(z) = \sum_{m=0}^{+\infty} a_m e_m(z), \quad (7.26)$$

where a_m is a coefficient. The above expression is called the *solution-expression* of $u(z)$, which plays an important role in the frame of the HAM for the choice of the auxiliary linear operator $\mathcal{L}$, the auxiliary function $H(z)$ and initial guess $u_0(z)$.

For boundary-value/eigenvalue problems in a finite interval $z \in [0, a]$, its solution can be expressed by different base functions, such as a power series

$$u(z) = \sum_{m=0}^{+\infty} A_m z^m, \quad (7.27)$$

or a Chebyshev series (Boyd, 2000; Mason and Handscomb, 2003)

$$u(z) = \sum_{m=0}^{+\infty} B_m T_m(z), \quad (7.28)$$

where $T_m(z)$ is the m th Chebyshev polynomial of the first kind, A_m and B_m are coefficients, respectively. Besides, it is well-known that a smooth function $u(z)$ in a finite interval $z \in [0, a]$ can be expressed by Fourier series

$$u(z) = \sum_{m=0}^{+\infty} \left(\bar{A}_m \cos \frac{m\pi z}{a} + \bar{B}_m \sin \frac{m\pi z}{a} \right), \quad (7.29)$$

where $\bar{A}_m, \bar{B}_m$ are coefficients. In addition, $u(z)$ can be expressed more efficiently by means of the so-called hybrid-base, i.e. a kind of combination of polynomials and trigonometric functions, as described in Sect. 7.2.3. Thus, the polynomials, trigonometric functions and their combination supply a complete set of base functions to approximate a smooth solution $u(z)$ in a finite interval $z \in [0, 1]$.

In addition, properties of a solution, which can be obtained frequently before solving a given BVP, are valuable to give a more accurate solution-expression. For example, if the solution of a BVP in a finite interval $z \in [0, a]$ is odd, we have the corresponding solution-expression

$$u(z) = \sum_{m=0}^{+\infty} A_{2m+1} z^{2m+1} \quad (7.30)$$

in power series, and

$$u(z) = \sum_{m=0}^{+\infty} \bar{B}_m \sin \frac{m\pi z}{a} \quad (7.31)$$

in Fourier series, respectively. Similarly, for an even solution of a BVP in $z \in [0, a]$, we have the solution-expression

$$u(z) = \sum_{m=0}^{+\infty} A_{2m} z^{2m} \quad (7.32)$$

in power series, and

$$u(z) = \sum_{m=0}^{+\infty} \bar{A}_m \cos \frac{m\pi z}{a} \quad (7.33)$$

in Fourier series, respectively.

For boundary-value/eigenvalue problems in an infinite interval $z \in [b, +\infty)$, where $b \geq 0$ is a bounded constant, $u(z)$ can be expressed by

$$u(z) = \sum_{k=0}^{+\infty} \sum_{m=0}^{+\infty} \alpha_{k,m} z^k \exp(-m \gamma z) \quad (7.34)$$

for exponentially decaying solutions (at infinity), or

$$u(z) = \sum_{m=0}^{+\infty} \frac{\beta_m}{(1 + \gamma z)^m} \quad (7.35)$$

for algebraically decaying solutions (at infinity), where $\gamma > 0$ is a parameter and $\alpha_{k,m}, \beta_m$ are coefficients.

The initial guess $u_0(z)$ should obey the solution-expression (7.26) and besides satisfy all boundary conditions, if possible. Thus, the initial guess $u_0(z)$ is often in

the form

$$u_0(z) = \sum_{m=1}^K b_m e_m(z), \quad K \geq n, \quad (7.36)$$

where n is the number of all boundary conditions, b_m is a coefficient, $e_m(z)$ denotes base functions, $K \geq n$ is a positive integer, respectively. In case of $K = n$, the initial guess is uniformly determined by the n boundary conditions. However, when $\mu = K - n > 0$, such kind of initial guess $u_0(z)$ provides us μ additional degree of freedom. This kind of unknown parameters are called *the multiple-solution-control parameters*, which provide us a convenient way to search for multiple solutions and/or to guarantee the convergence of homotopy-series, as illustrated in Chapter 8 and Chapter 9.

Note that, for the boundary-value/eigenvalue problems mentioned above, the choice of base functions is mainly determined by the interval and the properties of solution, not by governing equations in details. So, a smooth function in a *finite* interval $z \in [0, a]$ should be approximated by (7.27), (7.28) and (7.29), but a smooth function in an *infinite* interval $z \in [b, +\infty)$, where $b > 0$, should be approximated by either (7.34) for exponentially decaying solutions or (7.35) for algebraically decaying solutions, respectively. Given a nonlinear boundary-value problem, it is generally not difficult to derive some properties of solution by analyzing governing equations and/or boundary conditions before solving it, such as asymptotic properties of solution at infinity, its odd and even property, symmetry and so on. All of these properties of solution provide us valuable information to choose a good enough base functions.

It should be emphasized that, unlike perturbation techniques which regard small/large physical parameters as the starting point, we regard base functions of solution as the starting-point of the HAM. This is mainly because base function is a key for analytic approximation of solution: it is as important as *numbers* for numerical approximations. *Computing with functions instead of numbers* implies that the base functions are very important for BVPh 1.0.

Note that, using base functions instead of numbers and by means of a computer algebra system, many singular terms such as $\sin(\pi z)/z$ as $z \rightarrow 0$ can be easily resolved, because both of the numerator $\sin(\pi z)$ and the denominator z of $\sin(\pi z)/z$ as $z \rightarrow 0$ are now regarded as a *function* instead of the *number*¹ 0, and besides the limit

$$\lim_{z \rightarrow 0} \frac{\sin(\pi z)}{z} = \pi$$

is easily obtained by means of a computer algebra system like Mathematica. In addition, some types of base functions are well defined in an infinite interval and thus can be conveniently used to approximate solutions of a nonlinear boundary-value problem in the infinite interval exactly. In this way, many types of singularities can be easily resolved for BVPh 1.0 by means of the ability of *computing with functions instead of numbers* provided by the computer algebra system Mathematica.

¹ Note that $0/0$ has no meanings for numerical computations, and this results in singularities to many numerical tools such as BVPh4 c.

Since *all* smooth functions in a finite interval $z \in [0, a]$ can be expressed by Chebyshev series (7.28) and/or Fourier series (7.29), quite different boundary-value equations in a finite interval $z \in [0, a]$ may have the same solution-expression in the frame of the HAM. This suggests the possibility to develop a Mathematica package for nonlinear boundary-value problems in general.

7.1.4 Choice of the auxiliary linear operator

The choice of the auxiliary linear operator $\mathcal{L}$ is mainly determined by base-functions of solution $u(z)$. In principle, an auxiliary linear operator $\mathcal{L}$ should be chosen properly in such a way that

- all solution-expressions must be satisfied.
- all high-order deformation equations have unique solutions, and can be easily solved by computer algebra system like Mathematica.
- the convergence of all homotopy-series can be guaranteed by means of choosing proper convergence-control parameters.

For n th-order boundary-value/eigenvalue equations in a *finite* interval $z \in [0, a]$, we choose the auxiliary linear operator

$$\mathcal{L}(u) = \frac{d^n u(z)}{dz^n} \quad (7.37)$$

when $u(z)$ is expressed by power series (7.27) or Chebyshev series (7.28). In this case, we should choose `TypeL = 1` for BVPh 1.0.

However, when $u(z)$ is expressed by Fourier series (7.29) or by the so-called hybrid-base functions described in Sect. 7.2.3, we often choose the following auxiliary linear operator

$$\begin{aligned} \mathcal{L}(u) &= u'' + \omega_1^2 u, & \text{when } n = 2, \\ \mathcal{L}(u) &= u''' + \omega_1^2 u', & \text{when } n = 3, \\ \mathcal{L}(u) &= u'''' + (\omega_1^2 + \omega_2^2) u'' + \omega_1^2 \omega_2^2 u, & \text{when } n = 4, \\ \mathcal{L}(u) &= u''''' + (\omega_1^2 + \omega_2^2) u''' + \omega_1^2 \omega_2^2 u', & \text{when } n = 5, \\ &\vdots \end{aligned}$$

for a n th-order boundary-value equation in a finite interval $z \in [0, a]$, where the prime denotes the differentiation with respect to z , and $\omega_i > 0$ is a frequency. Depending on the related boundary conditions, $\omega_i > 0$ may be different each other, such as

$$\omega_i = i \left(\frac{\kappa \pi}{a} \right), \quad (7.38)$$

or the same, such as

$$\omega_i = \frac{\kappa\pi}{a}, \quad (7.39)$$

where $\kappa \geq 1$ and $i \geq 1$ are positive integers. The above auxiliary linear operators can be expressed in a general form

$$\mathcal{L}(u) = \left[\prod_{i=1}^m \left(\frac{d^2}{dx^2} + \omega_i^2 \right) \right] u, \quad \text{when } n = 2m, \quad (7.40)$$

or

$$\mathcal{L}(u) = \left[\prod_{i=1}^m \left(\frac{d^2}{dx^2} + \omega_i^2 \right) \right] u', \quad \text{when } n = 2m + 1. \quad (7.41)$$

Note that all of them have the property

$$\mathcal{L}[\cos(\omega_i z)] = \mathcal{L}[\sin(\omega_i z)] = 0,$$

where $\omega_i > 0$ is the frequency mentioned above. The auxiliary linear operator (7.40) or (7.41) with different values of κ in (7.38) or (7.39) provides us a convenient way to search for multiple solutions of many nonlinear boundary-value/eigenvalue equations, as illustrated in Sect. 9.3.1, Sect. 9.3.2 and Sect. 9.3.3. Note that such kind of auxiliary linear operator corresponds to `TypeL` = 2 for `BVPh` 1.0.

Note that all examples in Chapter 8 and Chapter 9 for the boundary-value or eigenvalue problems in a finite interval $z \in [0, a]$ are solved by means of either (7.37) or (7.40), (7.41). So, for a nonlinear boundary-value or eigenvalue problem in a finite interval $z \in [0, a]$, it is strongly suggested to attempt these two kinds of auxiliary linear operators first.

For n th-order boundary-value problems in an *infinite* interval $z \in [b, +\infty)$ with an exponentially decaying solution, where $b \geq 0$ is a bounded constant, we often (but *not* always) choose such an auxiliary linear operator $\mathcal{L}$ in the form

$$\mathcal{L}(u) = \frac{d^n u}{dz^n} + \sum_{m=0}^{n-1} \check{a}_m \frac{d^m u}{dz^m}, \quad (7.42)$$

that

$$\mathcal{L}[\exp(m \gamma z)] = 0, \quad 1 + n' - n \leq m \leq n' \quad (7.43)$$

holds, where n' is the number of the boundary conditions at infinity, $\gamma > 0$ is a parameter to be chosen later, $\check{a}_m$ is a coefficient, respectively. The n unknown coefficients $\check{a}_m$ in (7.42) are uniquely determined by the n linear algebraic equations given by (7.43).

For n th-order boundary-value problems in an infinite interval $z \in [b, +\infty)$ with an algebraically decaying solution $u \sim z^\beta$ as $z \rightarrow +\infty$, where $b > 0$ and β are bounded constants, we often (but *not* always) choose such an auxiliary linear operator $\mathcal{L}$ in the form

$$\mathcal{L}(u) = z^n \frac{d^n u}{dz^n} + \sum_{m=0}^{n-1} \check{b}_m z^m \frac{d^m u}{dz^m} \quad (7.44)$$

that

$$\mathcal{L}\left(z^{\beta+m}\right)=0, \quad 1-n \leq m \leq 0 \quad (7.45)$$

holds, where $\check{b}_m$ is an unknown constant. The n unknown coefficients in (7.44) are uniquely determined by the n linear algebraic equations given by (7.45). For boundary-value problems in the infinite interval $z \in [0, +\infty)$, we should first use such a transform

$$\xi = 1 + \gamma z$$

so that $\xi \in [1, +\infty)$, where $\gamma > 0$ is a parameter, and then use the auxiliary linear operator suggested above. In this way, we can also regard γ as a kind of convergence-control parameter, as shown in Sect. 10.3. Note that, for boundary-value problems in an infinite interval $z \in [b, +\infty)$, we must set `zR=infinity` for BVPh 1.0.

In the frame of the HAM and by means of the ability of “computing with functions instead of numbers” provided by computer algebra system like Mathematica, an infinite interval $[b, +\infty)$ has essentially no difference from a finite interval $z \in [0, a]$: only different base functions, different auxiliary linear operators and different initial guess are used. By means of BVPh 1.0, the two endpoints of an infinite interval $z \in [b, +\infty)$ are regarded as the same without fundamental difference: all boundary conditions at the two endpoints are regarded as a kind of the limit as either $z \rightarrow b$ or $z \rightarrow +\infty$, respectively. In this way, the infinite interval and many types of singularities in governing equations and/or boundary conditions of BVPs can be resolved easily.

Since the HAM provides us extremely large freedom to choose the auxiliary linear operator $\mathcal{L}$, we can choose an auxiliary operator $\mathcal{L}$ in other forms, when necessary. In this case, the auxiliary linear operator must be explicitly defined in the input data file for the BVPh 1.0.

Note that, for the boundary-value problems, the choice of auxiliary linear operators is mainly determined by base functions. As mentioned above, the choice of the base functions is mainly determined by the interval and asymptotic properties of solution. So, the choice of the auxiliary linear operator is mainly determined by the interval and asymptotic properties of solution.

7.1.5 Choice of the auxiliary function

In essence, the choice of the auxiliary function $H(z)$ may be regarded as a part of choosing the auxiliary linear operator $\mathcal{L}$, since the auxiliary linear operator $\mathcal{L}$ and the auxiliary function $H(z)$ in a zeroth-order deformation equation such as (7.3) can be combined as one, i.e.

$$\check{\mathcal{L}}(u) = \frac{\mathcal{L}(u)}{H(z)}.$$

So, like the choice of the auxiliary linear operator $\mathcal{L}$, the choice of an auxiliary function $H(z)$ is also mainly determined by the interval and the properties of solution. In principle, the auxiliary function $H(z)$ should be chosen in such a way that

- the solution-expression must be obeyed;
- solutions of high-order deformation equations exist and are unique;
- the convergence of homotopy-series solutions is guaranteed by means of proper convergence-control parameters.

Mostly, we can simply choose the auxiliary function $H(z) = 1$, especially for BVPs in a finite interval $z \in [0, a]$. For BVPs in an infinite interval $[b, +\infty)$, we may sometimes gain exponentially decaying solutions in the form

$$u(z) = \sum_{m=0}^{+\infty} a_m \exp(-m\gamma z)$$

by means of the auxiliary linear operator $H(z) = \exp(\kappa\gamma z)$, or algebraically decaying solution in the form

$$u(z) = \sum_{m=\mu}^{+\infty} \frac{b_m}{z^m}$$

by means of $H(z) = z^\kappa$, where $\gamma > 0$ and $\mu \geq 0$ are parameters, and κ is a parameter determined by the asymptotic properties of $u(z)$ at infinity and the so-called “rule of coefficient ergodicity” suggested by the author in his first book (see Sect. 2.3.4 of (Liao, 2003b)), i.e. each coefficient a_m and b_m of above solution-expressions may be modified as $m \rightarrow +\infty$.

7.1.6 Choice of the convergence-control parameter c_0

Different from all other analytic approximation techniques, the HAM provides us a convenient way to guarantee the convergence of series solution by means of introducing the so-called convergence-control parameter c_0 in zeroth-order deformation equation such as (7.3) and (7.16). In fact, it is the convergence-control parameter c_0 that essentially differs the HAM from all other analytic techniques, as shown in Part I.

As shown in Chapter 3, at the m th-order homotopy-approximation, the optimal convergence control parameter is determined by the minimum of the squared residual E_m of governing equation, corresponding to

$$\frac{dE_m}{dc_0} = 0, \quad (7.46)$$

where

$$E_m = \int_{\Omega} \mathcal{F} \left[z, \sum_{k=0}^m u_k(z) \right]^2 dz \quad (7.47)$$

is a squared residual for BVPs in either a finite interval $\Omega : z \in [0, a]$ or an infinite interval $\Omega : [b, +\infty)$, and

$$E_m = \int_{\Omega} \mathcal{F} \left[z, \sum_{k=0}^m u_k(z), \sum_{k=0}^{m-1} \lambda_k \right]^2 dz \quad (7.48)$$

is a squared residual for eigenvalue problems in a finite interval $z \in [0, a]$, respectively. Here, $\mathcal{F}[z, u] = 0$ and $\mathcal{F}[z, u, \lambda] = 0$ denote governing equations of BVPs and eigenvalue problems, respectively. For the sake of computation efficiency, these integrals are numerically calculated by means of some discrete points (we set the default `Nintegral = 50`).

In most cases, the squared residual E_m is dependent upon the convergence-control parameter c_0 only. However, to search for multiple solutions in the frame of the HAM, we often use the so-called multiple-solution-control parameter σ in an initial guess $u_0(z)$, which provides us one more degree of freedom to gain multiple solutions. In this case, the optimal convergence-control parameter c_0 and the optimal multiple-solution-control parameter σ are determined by

$$\frac{\partial E_m}{\partial c_0} = 0, \quad \frac{\partial E_m}{\partial \sigma} = 0. \quad (7.49)$$

For examples, please refer to Chapter 8 and Chapter 9.

7.2 Approximation and iteration of solutions

Although the high-order deformation equations (7.7) and (7.21) are always linear, they are still not easy to solve in general, because the right-hand side term $\delta_{m-1}(z)$ may be rather complicated. For example, when the auxiliary linear operator (7.40) or (7.41) is used, the linear differential equation

$$\mathcal{L}(u) = z^i \left(\cos \frac{j\kappa\pi z}{a} + \sin \frac{j\kappa\pi z}{a} \right) \quad (7.50)$$

has a short closed-form solution for arbitrary integers $i \geq 0$ and $j \geq 1$. However, the term $\delta_0(z) = \mathcal{F}[z, u_0(z)]$ or $\delta_0(z) = \mathcal{F}[z, u_0(z), \lambda_0]$ maybe be very complicated, since the nonlinear differential operator $\mathcal{F}$ is rather general.

For instance, in case of $\mathcal{F}[z, u, \lambda] = u'' + \lambda \sin u$ and using the auxiliary linear operator (7.40) and the initial guess $u_0 = \cos(\kappa\pi z/a)$, we had to solve a linear differential equation in the form

$$u'' + \left(\frac{\kappa\pi z}{a} \right)^2 u = \sin \left(\cos \frac{\kappa\pi z}{a} \right),$$

which has no closed-form solution, since the right-hand side term must be expanded into an infinite series

$$\sin\left(\cos\frac{\kappa\pi z}{a}\right) = \cos\frac{\kappa\pi z}{a} - \frac{1}{3!}\cos^3\frac{\kappa\pi z}{a} + \cdots.$$

In this case, the length of $u_m(z)$ is infinite so that it is hard to gain high-order approximations. To avoid this, we must approximate the term $\delta_{m-1}(z)$ by means of a reasonable number of properly chosen base-functions so that the m th-order deformation equation (7.7) or (7.21) can be solved efficiently with high-enough accuracy.

More importantly, as illustrated in Chapter 2, iteration can greatly accelerate the convergence of homotopy-series. However, the necessary condition of using iteration approach in the frame of the HAM is to approximate solutions of high-order deformation equations by a reasonable number of base-functions in a high-enough accuracy, i.e.

$$u_m(z) \approx \sum_{m=0}^{N_t} a_m e_m(z),$$

where $e_m(z)$ denotes the corresponding base-function, N_t denotes the number of truncated terms, a_m is a coefficient, respectively. As mentioned above, the term $\delta_{m-1}(z)$ may be rather complicated. So, to gain $u_m(z)$ in the above form, we must approximate $\delta_{m-1}(z)$ in the form

$$\delta_{m-1}(z) \approx \sum_{m=0}^{N_t} b_m e_m(z),$$

where the coefficient b_m is uniquely determined by $\delta_{m-1}(z)$ and the base-function $e_m(z)$.

As illustrated in Chapter 2, the HAM provides us great freedom to choose the initial guess $u_0(z)$. Since $u_m(z)$ has a fixed length, it is rather convenient to employ the iteration approach by using the M th-order approximation as a new initial guess $u_0^*(z)$, i.e.

$$u_0(z) + \sum_{m=1}^M u_m(z) \rightarrow u_0^*(z). \quad (7.51)$$

The above expression provides us the so-called M th-order iteration formula. In this way, the convergence of the homotopy-series can be greatly accelerated, as illustrated in Chapter 8 and Chapter 9.

The iteration approach of the BVPh 1.0 is currently possible only for nonlinear boundary-value/eigenvalue problems in a finite interval $z \in [0, a]$. So, we consider here the approximation of a smooth function $f(z)$ in a finite interval $\in [0, a]$ only.

7.2.1 Polynomials

It is well-known that a smooth solution $u(z)$ in a finite interval $z \in [0, a]$ can be well approximated by a polynomial

$$u(z) \approx \sum_{k=0}^{N_t} a_k z^k, \quad (7.52)$$

where a_k is a coefficient, N_t is the number of truncated terms. Besides, we have the so-called best approximation by

$$u(z) \approx \frac{b_0}{2} + \sum_{k=1}^{N_t} b_k T_k(z) \quad (7.53)$$

where $T_k(z)$ is the k th Chebyshev polynomial of the first kind, N_t denotes the number of Chebyshev polynomials, and

$$b_k = \frac{2}{\pi} \int_0^\pi u \left[\frac{a}{2} (1 + \cos \theta) \right] \cos(k\theta) d\theta.$$

When polynomial is used to solve the boundary-value/eigenvalue problems in a finite interval $z \in [0, a]$ by means of the BVPh 1.0, we should set

$$\text{TypeBase} = 0, \text{TypeL} = 1$$

with $\text{ApproxQ} = 0$ for the polynomial (7.52), and $\text{ApproxQ} = 1$ for the Chebyshev polynomial (7.53), respectively.

7.2.2 Trigonometric functions

It is well-known that the Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{+\infty} \left(b_n \sin \frac{n\pi z}{a} + a_n \cos \frac{n\pi z}{a} \right)$$

of a continuous function $f(z)$ in $z \in (-a, a)$ converges to $f(z)$ in the interval $z \in (-a, a)$, where

$$a_n = \frac{1}{a} \int_a^a f(t) \cos \frac{n\pi t}{a} dt, \quad b_n = \frac{1}{a} \int_a^a f(t) \sin \frac{n\pi t}{a} dt.$$

For a continuous function $f(z)$ in $[0, a]$, we can define $f(-z) = f(z)$ in $z \in (0, a)$ and the corresponding even Fourier series reads

$$f(z) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos \frac{n\pi z}{a}. \quad (7.54)$$

Alternatively, defining $f(z) = -f(-z)$ in $z \in (0, a)$, we have the odd Fourier series

$$f(z) = \sum_{n=1}^{+\infty} b_n \sin \frac{n\pi z}{a}. \quad (7.55)$$

In practice, we have the corresponding approximations

$$f(z) = \frac{a_0}{2} + \sum_{n=1}^{N_l} a_n \cos \frac{n\pi z}{a} \quad (7.56)$$

or

$$f(z) = \sum_{n=1}^{N_l} b_n \sin \frac{n\pi z}{a}, \quad (7.57)$$

where N_l denotes the number of truncated terms.

When the above-mentioned trigonometric functions are used to solve boundary-value/eigenvalue problems in a finite interval $z \in [0, a]$ by means of the BVPh 1.0, we should set

$$\text{ApproxQ} = 1, \text{TypeL} = 2$$

with $\text{TypeBase} = 1$ for the odd expression (7.57) and $\text{TypeBase} = 2$ for the even expression (7.56), respectively.

7.2.3 Hybrid-base functions

Note that the first-order derivative of the even Fourier series (7.54) equals to zero at the two endpoints $x = 0$ and $x = a$, but the original function $f(z)$ may have arbitrary values of $f'(0)$ and $f'(a)$. So, in case of $f'(0) \neq 0$ and/or $f'(a) \neq 0$, one had to use many terms of the even Fourier series (7.54) so as to obtain an accurate approximation near the two endpoints. To overcome this disadvantage, we first express $f(z)$ by such a combination

$$f(z) \approx Y(z) + w(z), \quad (7.58)$$

where

$$Y(z) = \left\{ f'(0)z - \frac{[f'(0) + f'(a)]}{2a} z^2 \right\} \cos \frac{\pi z}{a} \quad (7.59)$$

and then approximate $w(z) = f(z) - Y(z)$ by the even Fourier series

$$w(z) \sim \frac{\bar{a}_0}{2} + \sum_{n=1}^{N_T} \bar{a}_n \cos \frac{n\pi z}{a}, \quad (7.60)$$

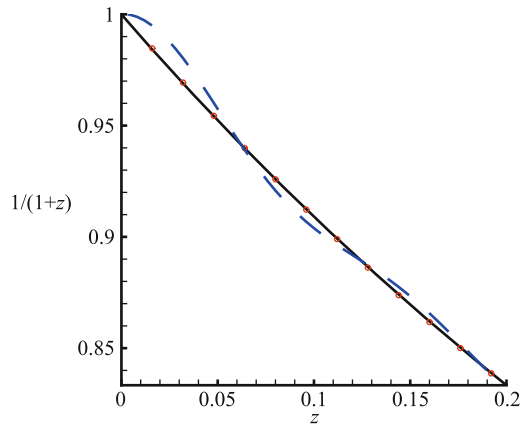
with the Fourier coefficient

$$\bar{a}_n = \frac{2}{a} \int_0^a [f(t) - Y(t)] \cos \frac{n\pi t}{a} dt.$$

Here, N_T is the number of truncated terms. Note that $Y(z)$ satisfies

$$Y'(0) = f'(0), \quad Y'(a) = f'(a)$$

Fig. 7.1 Comparison of different approximations of $1/(1+z)$ in $z \in [0, \pi]$. Solid line: $1/(1+z)$; Symbols: hybrid-base approximation with 15 terms; Dashed line: traditional Fourier approximation with 50 terms.



so that $w'(0) = w'(a) = 0$. Therefore, we often need a few terms of the even Fourier series to accurately approximate $w(z)$. Note also that both of the trigonometric and polynomial functions are used in (7.58) to approximate $f(z)$. It is found that, by means of such kind of approximations based on hybrid-base functions, one often needs much less terms to approximate a given smooth function $f(z)$ in $[0, a]$ than the traditional Fourier series. For example, the 15-term hybrid-base approximation of $f(z) = 1/(1+z)$ in $z \in [0, \pi]$ is much better than its 50-term traditional Fourier approximation, especially near $z = 0$, as shown in Fig. 7.1.

Alternatively, for a continuous function $f(z)$ in $[0, a]$, we can use

$$Y(z) = f(0) + \frac{[f(a) - f(0)]}{a} z \quad (7.61)$$

or

$$Y(z) = \frac{[f(0) + f(a)]}{2} + \frac{[f(0) - f(a)]}{2} \cos \frac{\pi z}{a}, \quad (7.62)$$

and approximate $w(z)$ by the odd Fourier series

$$w(z) \sim \sum_{n=1}^{N_t} \bar{b}_n \sin \frac{n\pi z}{a}, \quad (7.63)$$

where

$$\bar{b}_n = \frac{2}{a} \int_0^a [f(t) - Y(t)] \sin \frac{n\pi t}{a} dt.$$

It is suggested to use the hybrid-base approximation (7.58) with (7.59) for an even function $f(z)$, and (7.58) with (7.61) or (7.62) for an odd function $f(z)$, respectively. If $f(z)$ is neither an odd nor even function, both of them work.

By means of the BVPh 1.0, the above-mentioned hybrid-base approximation method is possible to solve the m th-order deformation equation (7.7) and (7.21). We first approximate $\delta_i(z)$ by means of the above-mentioned hybrid-base approximation method with a reasonable number of truncated terms, for instance,

$$\delta_i(z) \approx Y_i(z) + w_i(z), \quad (7.64)$$

where

$$Y_i(z) = \left\{ \delta'_i(0) z - \frac{[\delta'_i(0) + \delta'_i(a)]}{2a} z^2 \right\} \cos \frac{\pi z}{a} \quad (7.65)$$

and

$$w_i(z) \sim \frac{\check{a}_0}{2} + \sum_{n=1}^{N_i} \check{a}_n \cos \frac{n\pi z}{a}, \quad (7.66)$$

with the Fourier coefficient given by

$$\check{a}_n = \frac{2}{a} \int_0^a [\delta_i(t) - Y_i(t)] \cos \frac{n\pi t}{a} dt.$$

Alternatively (especially when the solution is odd), we can also use

$$Y_i(z) \approx \delta_i(0) + \frac{[\delta_i(a) - \delta_i(0)]}{a} z, \quad (7.67)$$

or

$$Y_i(z) \approx \frac{[\delta_i(0) + \delta_i(a)]}{2} + \frac{[\delta_i(0) - \delta_i(a)]}{2} \cos \frac{\pi z}{a}, \quad (7.68)$$

where

$$w_i(z) \sim \sum_{n=1}^{N_i} \check{b}_n \sin \frac{n\pi z}{a}, \quad (7.69)$$

and

$$\check{b}_n = \frac{2}{a} \int_0^a [\delta_i(t) - Y_i(t)] \sin \frac{n\pi t}{a} dt.$$

In this way, as long as N_i is large enough, the original high-order deformation equation (7.7) and (7.21) can be replaced by

$$\mathcal{L}[u_m(z) - \chi_m u_{m-1}(z)] \approx c_0 H(z) [Y_{m-1}(z) + w_{m-1}(z)] \quad (7.70)$$

with high accuracy, where we choose the auxiliary linear operator (7.40) or (7.41). Then, the above linear equation can be divided into a finite number of linear differential equations in the form of (7.50), which are easy to solve by means of the computer algebra system Mathematica (Abell and Braselton, 2004). More importantly, given N_T , i.e. the number of truncated terms, $u_m(z)$ has always a fixed length, which does not increase even when the approximation order is very high. Therefore, by means of this hybrid-base approximation method, the high-order deformation equation (7.7) and (7.21) can be solved rather efficiently, no matter how complicated the nonlinear operator $\mathcal{F}$ is.

When the above-mentioned hybrid-base approximation is used, we have even larger freedom to choose the initial guess $u_0(z)$. For example, for a 2nd-order boundary-value/eigenvalue problem in a finite interval $z \in [0, a]$, we may choose such an initial guess in the form

$$u_0(z) = B_0 + B_1 \cos \frac{\kappa \pi z}{a} + B_2 \sin \frac{\kappa \pi z}{a}, \quad (7.71)$$

or

$$u_0(z) = (B_0 + B_1 z + B_2 z^2) \cos \frac{\kappa \pi z}{a}, \quad (7.72)$$

where B_0, B_1 and B_2 are determined by linear boundary conditions.

When the above-mentioned hybrid-base approximation is used to solve boundary-value/eigenvalue problems in a finite interval $z \in [0, a]$ by means of the BVPh 1.0, we should set

$$\text{ApproxQ} = 1, \text{TypeL} = 2$$

with $\text{TypeBase} = 1$ for the odd expression (7.63) and $\text{TypeBase} = 2$ for the even expression (7.60), respectively.

7.3 A simple users guide of the BVPh 1.0

7.3.1 Key modules

BVPh The module `BVPh[k_, m_]` gives the k th to m th-order homotopy approximations of a nonlinear boundary-value problem (when $\text{TypeEQ} = 1$) or a nonlinear eigenvalue problem (when $\text{TypeEQ} = 2$), as defined above. It is the basic module. For example, `BVPh[1, 10]` gives the 1st to 10th-order homotopy-approximations. Thereafter, `BVPh[11, 15]` further gives the 11th to 15th-order homotopy-approximations.

iter The module `iter[k_, m_, r_]` provides us homotopy-approximations of the k th to m th iteration by means of the r th-order iteration formula (7.51). It calls the basic module `BVPh`. For example, `iter[1, 10, 3]` gives homotopy-approximations of the 1st to 10th iteration by the 3rd-order iteration formula. Furthermore, `iter[11, 20, 3]` gives the homotopy-approximations of the 11th to 20th iterations. For eigenvalue problems, the initial guess of eigenvalue is determined by an algebraic equation. Thus, if there are more than one initial guesses of eigenvalue, it is required to choose one among them by inputting an integer, such as 1 or 2, corresponding to the 1st or the 2nd initial guess of the eigenvalue, respectively. If the convergence-control parameter c_0 is unknown at the beginning of iteration, curves of squared residual of governing equation at the up-to 3rd-order approximations versus c_0 are given at the 1st iteration, in order to choose a proper value of c_0 . This value of c_0 will be renewed after `Nupdate` times iterations. The iteration stops when the squared residual of governing equation is less than a critical value `ErrReq`, whose default is 10^{-20} .

GetErr The module `GetErr[k_]` gives the squared residual of the governing equation at the k th-order homotopy-approximation gained by the module `BVPh`, or at the k th-iteration homotopy-approximation obtained by the mod-

ule `iter`. Note that, `error[k]` provides the residual of governing equation at the k th-order homotopy-approximation gained by BVPh, or at the k th-iteration homotopy-approximation obtained by `iter`, and `Err[k]` gives the averaged squared residual of the governing equation at the k th-order homotopy-approximation gained by BVPh, and `ERR[k]` gives the averaged squared residual of the governing equation at the k th-iteration homotopy-approximation obtained by `iter`, respectively.

GetMin1D The module `GetMin1D[f_, x_, a_, b_, Npoint_]` searches for the local minimums of a real function $f(x)$ in the interval $x \in [a, b]$ by means of dividing the interval $[a, b]$ into `Npoint` equal parts. If `Npoint` is large enough, it gives all local minimums with the corresponding position x . In general, `Npoint = 20` is suggested. This module is often used to search for the optimal convergence-control parameter c_0 , or multiple solutions of a nonlinear boundary-value problem. It calls the module `GetMin1D0[f_, x_, a_, b_, Npoint_]`.

GetMin2D Using `GetMin2D[f_, x_, a_, b_, y_, c_, d_, Npoint_]`, we can search for the local minimums of a real function $f(x, y)$ in the interval $a \leq x \leq b$, $c \leq y \leq d$ by dividing it into `Npoint` $\times$ `Npoint` equal parts. If `Npoint` is large enough, it gives all local minimums with the corresponding position (x, y) . In general, `Npoint = 20` is suggested. This module is often used to search for multiple solutions of a nonlinear boundary-value problem. It calls the module `GetMin2D0[f_, x_, a_, b_, c_, d_, Npoint_]`.

hp The module `hp[f_, m_, n_]` gives the $[m, n]$ homotopy-Padé approximation of the homotopy-approximation `f`, where `f[0]`, `f[1]`, `f[2]` denotes the zeroth, first and 2nd-order homotopy-approximation of `f`. For details about the homotopy-Padé approximation, please refer to Chapter 2.

7.3.2 Control parameters

TypeEQ A control parameter for the type of governing equation: `TypeEQ = 1` corresponds to a nonlinear boundary-value equation, `TypeEQ = 2` corresponds to a nonlinear eigenvalue problem, respectively.

TypeL A control parameter for the type of base functions: `TypeL = 1` corresponds to polynomial (7.52) or Chebyshev polynomial (7.53), and `TypeL = 2` corresponds to a trigonometric approximation or a hybrid-base approximation described in Sect.7.2.3, respectively. It is valid only for boundary-value/eigenvalue problems in a finite interval $z \in [0, a]$, where $a > 0$ is a constant.

ApproxQ A control parameter for approximation of solutions. When `ApproxQ = 1`, the right-hand side term of all high-order deformation equations is approx-

imated by Chebyshev polynomial (7.53), or by the hybrid-base approximation described in Sect. 7.2.3. When `ApproxQ` = 0, there is no such kind of approximation. When the module `iter` is employed, `ApproxQ` = 1 is automatically assigned. For the BVPh (version 1.0), this parameter is valid only for boundary-value/eigenvalue problems in a finite interval $z \in [0, a]$, where $a > 0$ is a constant.

TypeBase A control parameter for the type of base functions to approximate solutions: `TypeBase` = 0 corresponds to the Chebyshev polynomial (7.53), `TypeBase` = 1 corresponds to the hybrid-base (7.58) with the odd Fourier approximation (7.63), `TypeBase` = 2 corresponds to the hybrid-base (7.58) with the even Fourier approximation (7.60), respectively. This parameter is valid only when `ApproxQ` = 1 for boundary-value/eigenvalue problems in a finite interval $z \in [0, a]$.

Ntruncated A control parameter to determine the positive integer $N_t > 0$, corresponding to the number of truncated terms in (7.53), (7.60) and (7.63). The larger the number N_t , the better the approximations, but the more CPU time. It is valid only when `ApproxQ` = 1 for boundary-value/eigenvalue problems in a finite interval $z \in [0, a]$. The default is 10.

NtermMax A positive integer used in the module `truncated`, which ignores all polynomial terms whose order is higher than `NtermMax`. The default is 90.

ErrReq A critical value of squared residual of governing equation to stop the iteration of the module `iter`. The default is 10^{-10} .

NgetErr A positive integer used in the module BVPh. The squared residual of governing equation is calculated when the order of approximation divided by `NgetErr` is an integer. The default is 2.

Nupdate A critical value of the times of iteration to update the convergence-control parameter c_0 . The default is 10.

Nintegral Number of discrete points with equal space, which are used to numerically calculate the integral of a function. It is used in the module `int`. The default is 50.

ComplexQ A control parameter for complex variables. `ComplexQ` = 1 corresponds to the existence of complex variables in governing equations and/or boundary conditions. `ComplexQ` = 0 corresponds to the nonexistence of such kind of complex variables. The default is 0.

c0L A real number to determine the interval of the convergence-control parameter c_0 for plotting curves of the squared residual of the governing equation versus c_0 in the module `iter`. The default is -2, corresponding to the interval

$-2 \leq c_0 \leq 0$. The value of $c_0 L$ can be positive, such as $c_0 L = 1$, corresponding to the interval $0 \leq c_0 \leq 1$.

7.3.3 Input

f [z_, u_, lambda_] The governing equation, corresponding to $\mathcal{F}[z, u] = 0$ for nonlinear boundary-value problems in either a finite interval $z \in [0, a]$ or an infinite interval $z \in [b, +\infty)$, or corresponding to $\mathcal{F}[z, u, \lambda] = 0$ for nonlinear eigenvalue problems in a finite interval $z \in [0, a]$, where $a > 0, b \geq 0$ are bounded constants. Note that, **lambda** denotes the eigenvalue for nonlinear eigenvalue problems, but has no meanings at all for nonlinear boundary-value problems.

BC [k_, z_, u_, lambda_] The k th boundary condition, corresponding to either $\mathcal{B}_k[z, u] = 0$ for nonlinear boundary-value problems, or $\mathcal{B}_k[z, u, \lambda] = 0$ for nonlinear eigenvalue problems, respectively, where $0 \leq k \leq n$. Note that, **lambda** denotes the eigenvalue for nonlinear eigenvalue problems, but has no meanings at all for nonlinear boundary-value problems.

u [0] The initial guess $u_0(z)$.

L [f_] The auxiliary linear operator. For boundary-value/eigenvalue problems in a finite interval $z \in [0, a]$, the auxiliary linear operator (7.37) is automatically chosen when **TypeL**=1, and the auxiliary linear operator (7.40) or (7.41) is used otherwise. For boundary-value problems in an infinite interval $z \in [b, +\infty)$, where $b \geq 0$ is a bounded constant, one may choose either (7.42) for exponentially decaying solutions or (7.44) for algebraically decaying solutions, respectively. In any cases, the auxiliary linear operator must be clearly defined and properly chosen.

H [z_] The auxiliary function. The default is $H[z_] := 1$.

OrderEQ The order of the boundary-value equation $\mathcal{F}[z, u] = 0$ or the eigenvalue equation $\mathcal{F}[z, u, \lambda] = 0$.

zL The left end-point of the interval $z \in [a, b]$, corresponding to $z = a$. For example, **zL**=1 corresponds to $a = 1$. For boundary-value/eigenvalue problems in a finite interval $z \in [0, a]$, **zL** = 0 is automatically assigned. The default is 0.

zR The right end-point of the interval $z \in [a, b]$, corresponding to $z = b$. For example, **zR**=1 corresponds to $b = 1$. For boundary-value problems in an infinite interval, **zR** = infinity must be used for BVPh (version 1.0).

7.3.4 Output

`U[k]` The k th-order homotopy-approximation of $u(z)$ given by the basic module BVPh.

`V[k]` The k th-iteration homotopy-approximation of $u(z)$ given by the iteration module `iter`.

`Uz[k]` The k th-order homotopy-approximation of $u'(z)$ given by the basic module BVPh.

`Vz[k]` The k th-iteration homotopy-approximation of $u'(z)$ given by the iteration module `iter`.

`Lambda[k]` The k th-order homotopy-approximation of the eigenvalue λ given by the basic module BVPh.

`LAMBDA[k]` The k th-iteration homotopy-approximation of the eigenvalue λ given by the iteration module `iter`.

`error[k]` The residual of governing equation given by either the k th-order homotopy-approximation (obtained by the basic module BVPh) or the k th-iteration homotopy-approximation (obtained by the iteration module `iter`).

`Err[k]` The averaged squared residual of governing equation given by the k th-order homotopy-approximation (obtained by the basic module BVPh).

`ERR[k]` The averaged squared residual of governing equation given by the k th-iteration homotopy-approximation (obtained by the iteration module `iter`).

7.3.5 Global variables

All control parameters and output variables mentioned above are global. Except theses, the following variables and parameters are also global.

`z` The independent variable z .

`u[k]` The solution $u_k(z)$ of the k th-order deformation equation.

`lambda[k]` A constant variable, corresponding to λ_k .

`delta[k]` A function dependent upon z , corresponding to the right-hand side term $\delta_k(z)$ in the high-order deformation equation.

`L` The auxiliary linear operator $\mathcal{L}$.

`Linv` The inverse operator of $\mathcal{L}$, corresponding to $\mathcal{L}^{-1}$.

`nIter` The number of iteration, used in the module `iter`.

`sNum` A positive integer, which determines which initial guess λ_0 of eigenvalue is chosen when there exist multiple solutions of λ_0 .

Appendix 7.1 Mathematica package BVPh (version 1.0)

The BVPh (version 1.0) is a HAM-based Mathematica package for some types of n th-order nonlinear boundary-value/eigenvalue problems, developed by Shijun Liao of the Shanghai Jiao Tong University beginning in 2010. It is an open resource and free available at

<http://numericaltank.sjtu.edu.cn/BVPh.htm>

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 Mathematica package BVPh (version 1.0)

by Shijun LIAO
Shanghai Jiao Tong University
August 2010

```
<<Calculus`Pade`;
<<Graphics`Graphics`;

(*****)
(* Default of control parameters *)
(*****)
Ntruncated = 10;
Nupdate    = 10;
NtermMax   = 90;
Nintegral  = 50;
ErrReq     = 10^(-20);
NgetErr    = 2;
ComplexQ   = 0;
kappa      = 1;
PRN        = 1;
c0L        = -2;
zL         = 0;
c0         = .;

(*****)
(* Default auxiliary function *)
(*****)
H[z_] := 1;

U[0] = u[0];
(*****)
(* Define Linv *)
(* Find solution of linear equation: L[u] = f *)
(*****)
Linv[f_] := Module[{temp,w},
temp = DSolve[ L[w[z]] == f, w[z], z];
temp[[1,1,2]] /. C[_]->0 // TrigReduce
];

(*****)
(* Property of the inverse operator of L *)
(* Linv[f_+g_] := Linv[f] + Linv[g] *)
(*****)
Linv[p_Plus] := Map[Linv,p];
Linv[c_f_] := cLinv[f] /. FreeQ[c,z];

(*****)
(* Define Getdelta[k] *)
(* Get delta[k] automatically based on *)
(* the definition of f[z,u,lambda] *)
(*****)
```

```

Getdelta[k_] := Module[{temp, phi, LAMBDA, w,
                        lamb, q, eq, m, n, coeff, Coeff},
eq = f[z, phi[z, q], LAMBDA[q]];
temp[0] = D[eq, {q, k}]/k! // Expand;
temp[1] = temp[0]/.D[phi[z, q], {z, m_}, {q, n_}]
        ->n!*diff[w[n], {z, m}];
temp[2] = temp[1]/.D[LAMBDA[q], {q, n_}]
        ->n!*lamb[n];
temp[3] = temp[2]/.{phi[z, q]->w[0],
                    LAMBDA[q]->lamb[0]};
temp[4] = temp[3]/.diff[w[m_], {z, 0}]->w[m];
temp[5] = temp[4]/.{w->u, diff->D};
delta0[k] = temp[5]/. lamb->lambda;
If[ApproxQ == 0,
    temp[-1] = delta0[k]//Expand//TrigReduce
];
If[ApproxQ == 1 && TypeL == 1,
If[TypeEQ == 1,
temp[-1]=ChebyApproxA[delta0[k], z, 0, zR, Ntruncated],
For[m=0, m<=k, m++,
    coeff[m]=Coefficient[temp[5], lamb[m]]
];
coeff[-1] = temp[5]/.lamb[_]->0;
For[n=-1, n<=k, n++,
Coeff[n]=ChebyApproxA[coeff[n], z, 0, zR, Ntruncated];
];
lamb[-1] = 1;
temp[-1] = Sum[Coeff[n]*lamb[n], {n, -1, k}]
        /. lamb->lambda;
];
];
If[ApproxQ == 1 && TypeL == 2,
If[TypeEQ == 1,
temp[-1] = TrigApprox[delta0[k], z, zR,
                      Ntruncated, TypeBase],
For[m=0, m<=k, m++,
    coeff[m]=Coefficient[temp[5], lamb[m]]
];
coeff[-1] = temp[5]/.lamb[_]->0;
For[n=-1, n<=k, n++,
    Coeff[n] = TrigApprox[coeff[n], z, zR,
                          Ntruncated, TypeBase];
];
lamb[-1] = 1;
temp[-1] = Sum[Coeff[n]*lamb[n], {n, -1, k}]
        /. lamb->lambda;
];
];
delta[k] = temp[-1]//Expand//GetDigit;
];

```

```

(*****)
(* Define GetBC[k] *)
(* Get boundary conditions automatically *)
(* based on the definition BC[n,z,u,lambd] *)
(*****)
GetBC[i_,k_] := Module[{temp,j,phi,q,LAMB},
phi = Sum[u[j]*q^j,{j,0,k}];
LAMB = Sum[lambd[j]*q^j,{j,0,k}];
temp[0] = BC[i,z,phi,LAMB];
temp[1] = Series[temp[0],{q,0,k}]/Normal;
temp[2] = Coefficient[temp[1],q^k];
temp[2]//Expand//GetDigit
];

(*****)
(* Define functions chi[m] *)
(*****)
chi[m_] := If[m <= 1, 0, 1];

(*****)
(* Define truncated[z^n_] *)
(*****)
truncated[a_] := a /; FreeQ[a,z];
truncated[a_*f_] := a*truncated[f] /; FreeQ[a,z];
truncated[p_Plus] := Map[truncated,p];
truncated[z^n_] := If[n > NtermMax, 0, z^n];

(*****)
(* Define GetReal[] and GetDigit[] *)
(*****)
Naccu = 100;
GetReal[c_] := N[IntegerPart[c*10^Naccu]
/10^Naccu, Naccu] /; NumberQ[c];
GetDigit[c_] := N[IntegerPart[c*10^Naccu]
/10^Naccu, Naccu] /; NumberQ[c];
Default[GetDigit, 1] = 0;
GetDigit[p_Plus] := Map[GetDigit, p];
GetDigit[c_.*f_] := GetReal[c]*f /; NumberQ[c];
GetDigit[c_] := 0 /; NumberQ[c]
&& Abs[c] < 10^(-Naccu+1);

(*****)
(* Define int[f,x,x0,x1,Nintegral] *)
(* Integration of f in the interval [x0,x1] *)
(* by Nintegral points *)
(*****)
int[f_, x_, x0_, x1_, Nintegral_]
:= Module[{temp,dx,s,t,i,M},
M = Nintegral;
dx = N[x1-x0,100]/M;
temp[0] = Series[f,{x,0,2}]/Normal;
temp[0] = temp[0] /. x^_ -> 0 ;
s = temp[0];
For[i=1,i<=M,i=i+1,

```



```

t = x0 + i*dx;
temp[i] = f /.x->t;
s = s + temp[i]/Expand;
];
temp[M+1] = (temp[0]+temp[M])/2;
(s - temp[M+1])*dx//Expand
];

(*****)
(* Define ChebyApproxA[f,x,a,b,M] *)
(* Approximate a function by Chebyshev polynomial *)
(*****)
ChebyApproxA[f_,x_,a_,b_,Ntruncated_]
:= Module[{temp,n,z,t,F,A},
temp[0] = f/. x-> a + (b-a)*(z+1)/2;
F = temp[0] /. z -> Cos[t];
For[n=0, n<=Ntruncated, n++,
A[n] = 2*int[F*Cos[n*t],t,0,Pi,100]/Pi
];
For[n = 0, n<=Ntruncated, n++,
temp[0] = A[n]/Expand;
A[n] = GetDigit[temp[0]];
];
temp[1] = A[0]/2 + Sum[A[n]*ChebyshevT[n,z],
{n,1,Ntruncated}];
temp[1] /. z-> -1 + 2*(x-a)/(b-a)//Expand
];

(*****)
(* Define TrigApprox[f,x,Ntruncated,TypeBase] *)
(* Approximate f[x] in [0,xR] : *)
(* TypeBase = 0 : Fourier series *)
(* TypeBase = 1 : hybrid-base *)
(* Ntruncated : number of truncated terms *)
(*****)
TrigApprox[f_,x_,xR_,Ntruncated_,TypeBase_]
:= Module[{temp,a,b,c,y,i,j,t},
If[TypeBase == 0, temp[0] = f/.x->t];
If[TypeBase == 1,
temp[0] = Series[f,{x,0,2}]/Normal;
temp[1] = temp[0] /. x^_. -> 0;
temp[2] = f /. x->xR;
temp[3] = temp[1]+(temp[2]-temp[1])/xR*x//Expand;
y = GetDigit[temp[3]];
temp[0] = f - y /. x->t;
];
If[TypeBase == 2,
temp[0] = D[f,x]//Expand;
temp[1] = Series[temp[0],{x,0,2}]/Normal;
temp[1] = temp[1] /. x^_. -> 0;
temp[2] = temp[0] /. x->xR;
a = temp[1];
b = -(temp[1] + temp[2])/2/xR;
temp[3] = (a*x+b*x^2)*Cos[Pi*x/xR]//Expand;

```

```

y = GetDigit[temp[3]];
temp[0] = f - y /. x -> t ;
];
If[TypeBase == 0 || TypeBase == 2,
  For[i = 0, i <= Ntruncated, i++,
    c[i] = 2*int[temp[0]*Cos[i*t*Pi/xR],
      t, 0, xR, Nintegral]/xR;
  ];
temp[4] = c[0]/2
  + Sum[c[j]*Cos[j*x*Pi/xR], {j, 1, Ntruncated}];
];
If[TypeBase == 1,
  For[i = 1, i <= Ntruncated, i++,
    c[i] = 2*int[temp[0]*Sin[i*t*Pi/xR],
      t, 0, xR, Nintegral]/xR;
  ];
temp[4] = Sum[c[j]*Sin[j*x*Pi/xR],
  {j, 1, Ntruncated}];
];
If[TypeBase == 0,
  temp[5] = GetDigit[temp[4]],
  temp[5] = GetDigit[temp[4] + y]
];
temp[5]
];

(*****
(* Define GetMin1D[f,x,a,b,Npoint] *)
(*****
GetMin1D[f_,x_,x0_,x1_,Npoint_] := Module[
{temp,fmin,Xmin,xmin0,xmin1,dx,Num,i,j,X,s},
Num = 0;
dx = (x1-x0)/Npoint //GetDigit;
For[i = 0, i <= Npoint, i++,
  X[i] = x0 + i*dx //GetDigit;
  temp[i] = f /. x -> X[i];
];
For[i = 1, i <= Npoint-1, i++,
  If[temp[i] < temp[i-1] && temp[i] < temp[i+1],
    Num = Num + 1;
    fmin[Num] = temp[i];
    Xmin[Num] = X[i];
  ];
];
For[i = 1, i <= Num, i++,
  j = 0;
  xmin0 = Xmin[i];
  Label[100];
  j = j + 1;
  temp[0] = xmin0 - dx/5^(j-1) //GetDigit;
  temp[1] = xmin0 + dx/5^(j-1) //GetDigit;
  s = GetMin1D0[f,x,temp[0],temp[1],10];
  xmin1 = s[[2]];
  If[Abs[xmin1 - xmin0] < 10^(-20),

```

```

        Xmin[i] = xmin1;
        fmin[i] = f /. x-> xmin1;
        Goto[200];
    ];
    xmin0 = xmin1;
    Goto[100];
    Label[200];
    Print[" Minimum = ",fmin[i]/N, " at ",
          x, " = ",N[Xmin[i],20]];
];

];

GetMin1D0[f_,x_,x0_,x1_,Npoint_]
:= Module[{temp, fmin, Xmin, dx, i,z,F},
dx = (x1-x0)/Npoint // GetDigit;
fmin = 10^100;
Xmin = 0;
For[i = 1, i <= Npoint, i++,
    z = x0 + i*dx // GetDigit;
    F = f /. x->z // GetDigit;
    If[F < fmin,
        fmin = F;
        Xmin = z;
    ];
];
{fmin,Xmin}
];

(*****
(* Define GetMin2D[f,x,x0,x1,y,y0,y1,Npoint] *)
(*****
GetMin2D[f_,x_,x0_,x1_,y_,y0_,y1_,Npoint_]
:=Module[{temp,fmin,Xmin,Ymin,xmin0,xmin1,
        ymin0,ymin1,dx,dy,Num,i,j,X,Y,s},
Num = 0;
dx = (x1-x0)/Npoint //GetDigit;
dy = (y1-y0)/Npoint //GetDigit;
For[i = 0, i <= Npoint, i++,
    X[i] = x0 + i*dx //GetDigit;
    For[j = 0, j <= Npoint, j++,
        Y[j] = y0 + j*dy //GetDigit;
        temp[i,j] = f /. {x -> X[i], y->Y[j]};
    ];
];
For[i = 1, i <= Npoint-1, i++,
    For[j = 1, j <= Npoint-1, j++,
        If[temp[i,j] < temp[i-1,j]
            && temp[i,j] < temp[i+1,j]
            && temp[i,j] < temp[i,j-1]
            && temp[i,j] < temp[i,j+1],
            Num = Num + 1;
            fmin[Num] = temp[i,j];
            Xmin[Num] = X[i];
            Ymin[Num] = Y[j];
        ];
    ];
];

```

```

];
];
];
For[i = 1, i <= Num, i++,
  j = 0;
  xmin0 = Xmin[i];
  ymin0 = Ymin[i];
  Label[100];
  j = j + 1;
  X[0] = xmin0 - dx/5^(j-1) //GetDigit;
  X[1] = xmin0 + dx/5^(j-1) //GetDigit;
  Y[0] = ymin0 - dy/5^(j-1) //GetDigit;
  Y[1] = ymin0 + dy/5^(j-1) //GetDigit;
  s = GetMin2D0[f,x,X[0],X[1],y,Y[0],Y[1],10];
  xmin1 = s[[2]];
  ymin1 = s[[3]];
  If[Abs[xmin1 - xmin0] < 10^(-20)
    && Abs[ymin1 - ymin0] < 10^(-20),
    Xmin[i] = xmin1;
    Ymin[i] = ymin1;
    fmin[i] = f /. {x->xmin1, y->ymin1};
    Goto[200];
  ];
  xmin0 = xmin1;
  ymin0 = ymin1;
  Goto[100];
  Label[200];
  Print[" Minimum = ",fmin[i]//N];
  Print[" at ", x, " = ",N[Xmin[i],20]];
  Print[" ", y, " = ",N[Ymin[i],20]];
  ];
];

GetMin2D0[f_,x_,x0_,x1_,y_,y0_,y1_,Npoint_]
:= Module[{temp,fmin,Xmin,Ymin,dx,dy,i,X,Y,F},
dx = (x1-x0)/Npoint // GetDigit;
dy = (y1-y0)/Npoint // GetDigit;
fmin = 10^10;
Xmin = 10^10;
Ymin = 10^10;
For[i = 1, i <= Npoint, i++,
  X = x0 + i*dx // GetDigit;
  For[j = 1, j <= Npoint, j++,
    Y = y0 + j*dy // GetDigit;
    F = f /. {x->X, y->Y} // GetDigit;
    If[F < fmin,
      fmin = F;
      Xmin = X;
      Ymin = Y
    ];
  ];
];
{fmin,Xmin,Ymin}
];

```

```

(*****)
(* Define GetErr[m] *)
(* This module gives averaged squared residual *)
(*****)
GetErr[k_] := Module[{temp},
error[k] = f[z,U[k],Lambda[k-1]];
If[ComplexQ == 0, temp[0] = error[k]^2 ];
If[ComplexQ == 1,
temp[1] = Re[error[k]];
temp[2] = Im[error[k]];
temp[0] = temp[1]^2 + temp[2]^2;
];
If[!NumberQ[zRintegral],
If[FreeQ[zR,infinity] && zR < 100,
zRintegral = zR,
Print["Squared residual is integrated
in the interval [",zL,"b"];
Print[" The value of b = ? "];
zRintegral = Input[];
];
];
temp[1] = Abs[zRintegral-zL];
Err[k] = int[temp[0],z,zL,zRintegral,Nintegral]
/temp[1];
];

(*****)
(* Define hp[f_,m_,n_] *)
(*****)
hp[f_,m_,n_] := Block[{k,i,df,res,q},
df[0] = f[0];
For[k=1,k<=m+n,k++,df[k]=f[k]-f[k-1]//Expand];
res = df[0] + Sum[df[i]*q^i,{i,1,m+n}];
Pade[res,{q,0,m,n}]/.q->1
];

(*****)
(* Main Code *)
(* HAM approach without iteration *)
(*****)
BVPh[begin_,end_] := Block[
{uSpecial,CC,temp,ss,w,i,j,EQ,Unknown,phi,LAM,
sLeng,sMin,sMinI,RHS,base,a},
time[0] = SessionTime[];
For[k=begin,k<=end,k=k+1,
If[PRN == 1, Print["k = ",k]];
If[k == 1 && NumberQ[c0], c0 = GetDigit[c0]];
If[k == 1, Clear[lambda]];

(* Get special solution *)
Getdelta[k-1];
RHS = H[z]*delta[k-1]//Expand;
temp[1] = Linv[RHS];

```

```

uSpecial = c0*temp[1] + chi[k]*u[k-1]//Expand;

(* Get lambda[k-1] and u[k] *)
temp[1] = DSolve[L[w[z]]==0,w[z],z];
temp[2] = temp[1][[1,1,2]];
Unknown = {};
For[i = 1, i <= OrderEQ, i++,
  base[i] = Coefficient[temp[2],C[i]];
  If[FreeQ[base[i],Exp[n_.*z]],
    Unknown = Union[Unknown,{CC[i,k]}],
    If[(Abs[base[i]] /. z -> 1.) < 1,
      Unknown = Union[Unknown,{CC[i,k]}],
      CC[i,k] = 0;
      base[i] = 0;
    ]
  ];
];

ss = Sum[base[i]*CC[i,k],{i,1,OrderEQ}];
u[k] = uSpecial + ss // Expand;
For[i = 1, i <= OrderEQ, i++,
  eq[i] = GetBC[i,k]//Expand//GetDigit
];
EQ = {};
For[i=1, i <= OrderEQ, i++,
  If[FreeQ[BC[i,z,w[z],a],infinity],
    EQ = Union[EQ,{eq[i] == 0}];
  ];
];

If[TypeEQ != 1,
  eq[0] = GetBC[0,k]//Expand//GetDigit;
  EQ = Union[EQ,{eq[0] == 0}];
  Unknown = Union[Unknown,{lambda[k-1]}];
];

ss = Solve[EQ,Unknown];
sLeng = Length[ss];
If[sLeng == 1, sNum = 1];
If[k == 1 && !NumberQ[sNum],
  For[j = 1, j <= sLeng, j++,
    temp[1] = lambda[0] /. ss[[j]];
    Print[j," th solution of lambda[0] = ",
      temp[1]//N];
  ];
  Print["which solution is chosen ? "];
  sNum = Max[1,Input[]];
];

If[k == 1 && TypeEQ != 1 && NumberQ[sNum]
  && NumberQ[LAMBDA[nIter-1]],
  sMin = 10^10;
  For[j = 1, j <= sLeng, j++,
    temp[1] = lambda[0] /. ss[[j]];

```

```

        temp[2] = Abs[temp[1]-LAMBDA[nIter-1]];
        If[ temp[2] < sMin,
            sNum = j;
            sMin = temp[2];
        ];
    ];
];

For[i = 1, i <= OrderEQ, i++,
    temp[1] = CC[i,k] /. ss[[sNum]]//Expand;
    CC[i,k] = temp[1]//GetDigit;
];
u[k]      = u[k]//Expand//GetDigit;
U[k]      = Expand[U[k-1] + u[k]];
Uz[k]     = D[U[k],z];

If[TypeEQ != 1,
    temp[1] = lambda[k-1] /. ss[[sNum]]//Expand;
    lambda[k-1] = temp[1]//GetDigit;
    Lambda[k-1] = Sum[lambda[i],{i,0,k-1}]
                  //Expand//GetDigit;
    If[PRN == 1,
        Print[k-1,"th-order approx. of eigenvalue
                = ",Lambda[k-1]/N ];
    ];
];

(* Print results *)
If[PRN == 1 && NumberQ[c0] && TypeEQ == 1,
    output[z,U,k]
];
If[IntegerQ[k/NgetErr] && PRN == 1,
    GetErr[k];
    If[NumberQ[Err[k]],
        Print["Squared residual = ",Err[k]/N];
    ];
    If[Err[k] < ErrReq,
        Print["Congratulation: the required
                squared residual is satisfied !"];
        Goto[end]
    ];
];

If[PRN == 1,
    time[k] = SessionTime[];
    temp[0] = time[k]-time[0];
    Print["Used CPU time   = ",temp[0],
          "      (seconds) "];
];

];
If[PRN == 1, Print[" Successful !"]];
Label[end];
];

```

```

(*****)
(*           Main Code           *)
(*  HAM approach with iteration  *)
(*  begin :      number of starting iteration  *)
(*  end :        number of stopping iteration  *)
(*  OrderIter :  order of iteration formula    *)
(*****)
iter[begin_, end_, OrderIter_] := Block[{temp, time},
PRN = 0;
If[TypeL != 1, ApproxQ = 1];
For[nIter = Max[1, begin], nIter <= end, nIter = nIter + 1,
  If[nIter == 1,
    sNum = .;
    If[NumberQ[c0], c0 = GetDigit[c0]];
    Time[0] = SessionTime[];
    V[0] = u[0] // Expand // GetDigit;
    Vz[0] = D[V[0], z];
  ];
  If[nIter > 1,
    u[0] = V[nIter - 1];
    U[0] = u[0];
  ];
  Print[ nIter, " th iteration:"];

  If[NumberQ[c0], BVPh[1, OrderIter]];
  If[!NumberQ[c0],
    BVPh[1, Max[3, OrderIter]];
    GetErr[1];
    GetErr[2];
    GetErr[3];
    Print[" Squared Residual of Governing EQ:"];
    Print["   Red line   : 1st-order approx. "];
    Print["   Green line  : 2nd-order approx. "];
    Print["   Blue line   : 3rd-order approx. "];
    LogPlot[{Abs[Err[1]], Abs[Err[2]], Abs[Err[3]]},
      {c0, c0L, 0}, PlotStyle -> {RGBColor[1, 0, 0],
        RGBColor[0, 1, 0], RGBColor[0, 0, 1]}];
    Print[" Adjust the interval of c0 by choosing
      a better end-point "];
    Print[" New end-point on left (c0 < 0)
      or right (c0 > 0) = ?
      (Input 0 to skip it) "];
    c0L = Input[];
    If[c0L != 0,
      LogPlot[{Abs[Err[1]], Abs[Err[2]], Abs[Err[3]]},
        {c0, c0L, 0}, PlotStyle -> {RGBColor[1, 0, 0],
          RGBColor[0, 1, 0], RGBColor[0, 0, 1]}];
    ];
    Print[" Choose the value of
      convergence-control parameter c0"];
    Print[" c0 = ? (Input 0 to stop the code)"];
    temp[0] = Input[];
    c0 = GetDigit[temp[0]];
    Print[" c0 = ", c0/N];
  ];
];

```



```

If[c0==0,c0 =. ;Print["Stop!"];Goto[stop]];
];

temp[0] = U[OrderIter]//Expand;
If[PolynomialQ[temp[0],z],
  temp[0] = truncated[temp[0]]
];
V[nIter] = GetDigit[temp[0]];
Vz[nIter] = D[V[nIter],z];
If[TypeEQ == 1,
  output[z,V,nIter],
  LAMBDA[nIter]=Lambda[OrderIter-1]//Expand;
  Print[" Eigenvalue = ",LAMBDA[nIter]//N ];
];

GetErr[OrderIter];
ERR[nIter] = Err[OrderIter];
Print[" Squared Residual = ",ERR[nIter]//N];
If[ERR[nIter] < ErrReq,
  Print["STOP:Required accuracy satisfied!"];
  Goto[stop];
];
If[nIter > 1 && ERR[nIter] > ERR[nIter-1]
  && Err[nIter] < 10^(-7),
  Print["Squared residual does NOT decrease!"];
  Goto[stop];
];
Time[nIter] = SessionTime[];
Temp[-1] = Time[nIter]-Time[0];
Print[" Used CPU time = ", Temp[-1],
      " (seconds) "];
If[IntegerQ[nIter/Nupdate]&&Abs[c0]< 1/2,c0 =.];
];
Print[" End of iteration ! "];
Label[stop];
PRN = 1;
];

(*****)
(* Print equation, boundary conditions *)
(* and control parameters *)
(*****)
PrintInput[s_] := Module[{} ,
Print["-----"];
Print["The values of control parameters: "];
Print[" Nupdate = ", Nupdate];
Print[" Ntruncated = ", Ntruncated];
Print[" NtermMax = ", NtermMax];
Print[" Nintegral = ", Nintegral];
Print[" ErrReq = ", ErrReq//N];
Print[" NgetErr = ", NgetErr];
Print[" ComplexQ = ", ComplexQ];
Print[" PRN = ", PRN];
Print[" c0L = ", c0L];

```

```

Print[" zL          = ", zL];
Print["-----"];
Print[" Governing Equation      : ",
      f[z,s,lambda], " = 0 "];
Print[" Interval of z          : ",
      zL, " < z < ", zR];
If[TypeEQ != 1,
  Print[" ",0,"th Boundary Condition : ",
        BC[0, z, s, lambda], " = 0"];
];
For[i=1, i<= OrderEQ, i++,
  Print[" ",i,"th Boundary Condition : ",
        BC[i, z, s, lambda], " = 0"];
];
Print["-----"];
If[ApproxQ == 0,
  Print[" Delta[k] is NOT approximated
        by other base functions ! "],
  If[TypeL == 1,
    Print[" Base function      : Chebyshev "],
    If[TypeBase == 1,
      Print[" Base function      : hybrid-base
            with sine and polynomial"],
      Print[" Base function      : hybrid-base
            with cosine and polynomial "];
    ];
  ];
];
Print[" Auxiliary linear operator L[u] = ",L[s]];
If[TypeL != 1,
  Print["          kappa          = ",kappa]
];
Print[" Initial guess          u[0] = ",u[0]];
Print[" Auxiliary function      H[z] = ",H[z]];
If[NumberQ[c0],
  Print[" Convergence-control parameter c0 = ",c0]
];
Print["-----"];
];

```

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Chapter 8

Nonlinear Boundary-value Problems with Multiple Solutions

Abstract In this chapter, using three different types of nonlinear boundary-value equations with multiple solutions, we verify the validity of the HAM-based Mathematica package BVPh (version 1.0) for n th-order nonlinear boundary-value equations $\mathcal{F}[z, u] = 0$ in a finite interval $0 \leq z \leq a$, subject to the n linear boundary conditions $\mathcal{B}_k[z, u] = \gamma_k$ ($1 \leq k \leq n$), where $\mathcal{F}$ is a n th-order nonlinear differential operator, $\mathcal{B}_k$ is a linear operator, γ_k is a constant, respectively. Especially, the so-called multiple-solution-control parameter is introduced into initial guess in order to search for multiple solutions. We illustrate that, using the BVPh 1.0 as a tool, multiple solutions of some nonlinear boundary-value equations can be found out by means of such kind of multiple-solution-control parameter.

8.1 Introduction

In this chapter, we illustrate the general validity of the Mathematica package BVPh (version 1.0) for nonlinear boundary-value problems with multiple solutions, governed by a n th-order nonlinear differential equation in a finite interval

$$\mathcal{F}[z, u] = 0, \quad z \in [0, a], \quad (8.1)$$

subject to the n linear multipoint boundary conditions

$$\mathcal{B}_k[z, u] = \gamma_k \quad (8.2)$$

at the two endpoints and some separated points in the interval $z \in (0, a)$, where $\mathcal{F}$ is a nonlinear n th-order differential operator, $\mathcal{B}_k$ is a linear differential operator, z is an independent variable, $u(z)$ is the unknown function, γ_k is a constant, respectively. Assume that this kind of nonlinear boundary-value problems has at least one smooth solution.

A nonlinear boundary-value equation has often multiple solutions, which are difficult to find out by numerical techniques in general, especially when they are close

each other and/or numerically unstable. Based on the homotopy analysis method (HAM) (Liao, 1992, 1997, 1999a,b; Liao and Campo, 2002; Liao, 2003a,b, 2004, 2005, 2006, 2009, 2010a,b; Liao and Tan, 2007; Li et al., 2010; Xu et al., 2010), some analytic approaches were suggested to gain multiple solutions of nonlinear boundary-value problems. In 2005, Liao (2005) applied the HAM to successfully obtain two branches of solutions of the boundary-layer equation

$$F''' + \frac{1}{2}FF'' - \beta F'^2 = 0, \quad F(0) = 0, \quad F'(0) = 1, \quad F'(+\infty) = 0,$$

where $-1 < \beta < +\infty$ is a constant, by means of introducing an additional unknown quantity $\delta = F(+\infty)$. Solving a nonlinear algebraic equation related to this unknown quantity δ , Liao (2005) found one new branch of solutions when $\beta > 1$, which had been *never* reported by other analytic methods and even neglected by numerical methods, mainly because the difference between the values of $F''(0)$ of the two branches of solutions is so small that it is hard to distinguish them. This illustrates the great potential of the HAM for nonlinear problems with multiple solutions. In addition, it verifies that multiple solutions of some nonlinear problems can be found out by introducing an unknown parameter properly.

In 2009, Abbasbandy et al. (2009) applied the HAM to gain the multiple solutions of a 2nd-order reaction-diffusion equation by solving a nonlinear algebraic equation about the convergence-control parameter. In 2010, Xu et al. (2010) used the HAM to obtain multiple solutions of viscous boundary-layer flows between two moving parallel walls by means of a shooting-like technique.

In this chapter, we illustrate that, using the BVPh 1.0, multiple solutions of nonlinear boundary-value equations can be found out by means of introducing a so-called *multiple-solution-control parameter* in initial guess. Three examples are used to illustrate the validity of this strategy, and their input data files for BVPh 1.0 are given in the appendix of this chapter and free available (Accessed 25 Nov 2011, will be updated in the future) at

<http://numericaltank.sjtu.edu.cn/BVPh.htm>

8.2 Brief mathematical formulas

Let $u_0(z)$ denote an initial guess of the solution $u(z)$ of (8.1) and (8.2), $q \in [0, 1]$ an embedding parameter, respectively. In the frame of the HAM, we should first construct such a continuous deformation (homotopy) $\phi(z; q)$ that, as q increases from 0 to 1, $\phi(z; q)$ varies from the initial guess $u_0(z)$ to the solution $u(z)$ of (8.1) and (8.2). Such a kind of continuous deformation is constructed by the so-called zeroth-order deformation equation

$$(1 - q) \mathcal{L}[\phi(z; q) - u_0(z)] = c_0 q \mathcal{F}[z, \phi(z; q)], \quad (8.3)$$

subject to the n linear boundary conditions

$$\mathcal{B}_k[z, \phi(z; q)] = \gamma_k, \quad (8.4)$$

where $\mathcal{L}$ is an auxiliary linear operator with the property $\mathcal{L}(0) = 0$, whose highest order of derivative is exactly n , and $c_0 \neq 0$ is a convergence-control parameter, respectively. Obviously, it holds

$$\phi(z; 0) = u_0(z), \quad \phi(z; 1) = u(z), \quad (8.5)$$

at $q = 0$ and $q = 1$, respectively. So, if the auxiliary linear operator $\mathcal{L}$ and the convergence-control parameter c_0 are so properly chosen that the Maclaurin–homotopy series

$$\phi(z; q) = u_0(z) + \sum_{m=1}^{+\infty} u_m(z) q^m \quad (8.6)$$

converges to $\phi(z; 1)$, we have the homotopy-series solution

$$u(z) = u_0(z) + \sum_{m=1}^{+\infty} u_m(z), \quad (8.7)$$

where $u_m(z)$ is governed by the m th-order deformation equation

$$\mathcal{L}[u_m(z) - \chi_m u_{m-1}(z)] = c_0 \delta_{m-1}(z), \quad (8.8)$$

subject to the n linear boundary conditions

$$\mathcal{B}_k[z, u_m] = 0, \quad (8.9)$$

where

$$\delta_k(z) = \mathcal{D}_k \{ \mathcal{F}[z, \phi(z; q)] \}$$

and

$$\mathcal{D}_k[\phi] = \frac{1}{k!} \left. \frac{\partial^k \phi}{\partial q^k} \right|_{q=0}, \quad \chi_k = \begin{cases} 0, & k \leq 1, \\ 1, & k > 1. \end{cases}$$

Here, $\mathcal{D}_k$ is the k th-order homotopy-derivative operator. Equation (8.8) is given by means of Theorem 4.15. For details, please refer to Chapter 4. Note that $\delta_k(z)$ only depends upon the nonlinear differential operator $\mathcal{F}[z, u]$, and can be easily deduced by the theorems proved in Chapter 4. In fact, for nonlinear ordinary differential equations (ODEs), it is unnecessary for users to deduce the term $\delta_k(z)$ by means of the BVPh 1.0.

At the m th-order homotopy-approximation

$$\tilde{u}(z) \approx u_0(z) + \sum_{k=1}^m u_k(z), \quad (8.10)$$

the discrete squared residual E_m of the governing equation (8.1) is defined by

$$E_m = \frac{1}{N_p + 1} \sum_{k=0}^{N_p} \{ \mathcal{F} [z_k, \tilde{u}(z_k)] \}^2, \quad (8.11)$$

where

$$z_k = k \left(\frac{a}{N_p} \right)$$

denotes the coordinate of the $(N_p + 1)$ separate points. The optimal convergence-control parameter c_0 is determined by the minimum of the squared residual E_m .

In the frame of the HAM, we have extremely large freedom to choose the auxiliary linear operator $\mathcal{L}$ and the initial guess $u_0(z)$. It is well-known that the continuous function $f(z)$ in a finite interval $z \in [0, a]$ can be well approximated by either Chebyshev series or Fourier series. If the power or Chebyshev polynomial is used to approximate $u(z)$, we often choose the auxiliary linear operator

$$\mathcal{L}(u) = \frac{d^n u}{dz^n}, \quad (8.12)$$

where n is the order of the highest-derivative of (8.1). If the hybrid-base approximation method described in Sect. 7.2.3 is used to approximate the solution $u(z)$, we choose the following auxiliary operator

$$\begin{aligned} \mathcal{L}(u) &= u'' + \omega_1^2 u, & \text{when } n = 2, \\ \mathcal{L}(u) &= u''' + \omega_1^2 u', & \text{when } n = 3, \\ \mathcal{L}(u) &= u'''' + (\omega_1^2 + \omega_2^2) u'' + \omega_1^2 \omega_2^2 u, & \text{when } n = 4, \\ \mathcal{L}(u) &= u''''' + (\omega_1^2 + \omega_2^2) u''' + \omega_1^2 \omega_2^2 u', & \text{when } n = 5, \\ &\vdots \end{aligned}$$

where $\omega_k > 0$ is a frequency of the corresponding base function $\cos(\omega_k z)$ and $\sin(\omega_k z)$, i.e.

$$\mathcal{L} [\cos(\omega_k z)] = \mathcal{L} [\sin(\omega_k z)] = 0.$$

The above auxiliary linear operators can be expressed in a general form

$$\mathcal{L}(u) = \left[\prod_{i=1}^m \left(\frac{d^2}{dx^2} + \omega_i^2 \right) \right] u, \quad \text{when } n = 2m, \quad (8.13)$$

or

$$\mathcal{L}(u) = \left[\prod_{i=1}^m \left(\frac{d^2}{dx^2} + \omega_i^2 \right) \right] u', \quad \text{when } n = 2m + 1, \quad (8.14)$$

where the frequency ω_i may be different each other, such as

$$\omega_i = i \left(\frac{\kappa\pi}{a} \right), \quad (8.15)$$

or the same, like

$$\omega_i = \frac{\kappa\pi}{a}, \quad (8.16)$$

depending on the boundary conditions (8.2). Here, $\kappa \geq 1$ is a positive integer.

Note that the initial guess $u_0(z)$ should satisfy the n linear boundary-conditions (8.2) at least, and should be expressed by either polynomial or trigonometric functions, or the hybrid-base functions.

For the choice of the auxiliary linear operator $\mathcal{L}$ and the initial guess $u_0(z)$, please refer to Chapter 7.

8.3 Examples

Many nonlinear boundary-value equations have multiple solutions, which are difficult to obtain by numerical techniques in general. In this section, three nonlinear boundary-value problems with multiple solutions are used to illustrate the validity of the BVPh 1.0, which is given in Appendix 7.1 and free available at

<http://numericaltank.sjtu.edu.cn/BVPh.htm>

The input data files of all these examples for BVPh 1.0 are given in the appendix of this chapter and free available at the same website.

8.3.1 Nonlinear diffusion-reaction model

First of all, let us consider a nonlinear diffusion-reaction model in porous catalysts, governed by

$$u u'' - \gamma^2 = 0, \quad u'(0) = 0, \quad u(1) = 1, \quad (8.17)$$

where γ is a given parameter. The above equation has the implicit closed-form solution

$$z = \frac{u(0)}{i\gamma} \sqrt{\frac{\pi}{2}} \operatorname{Erf} \left[i \sqrt{\ln \frac{u(z)}{u(0)}} \right], \quad (8.18)$$

where $i = \sqrt{-1}$, $\operatorname{Erf}(z)$ denotes the error function, and $u(0)$ is determined by

$$1 = \frac{u(0)}{i\gamma} \sqrt{\frac{\pi}{2}} \operatorname{Erf} \left[i \sqrt{\ln \frac{1}{u(0)}} \right], \quad (8.19)$$

respectively.

This example is a special case of (8.1) when

$$\mathcal{F}[z, u] = u u'' - \gamma^2.$$

In this case, the m th-order homotopy-approximation is given by (8.10), where $u_k(z)$ is governed by the k th-order deformation equation

$$\mathcal{L}[u_k(z) - \chi_k u_{k-1}(z)] = c_0 \delta_{k-1}(z), \quad (8.20)$$

subject to the boundary conditions

$$u'_k(0) = 0, \quad u_k(1) = 0, \quad (8.21)$$

where

$$\delta_n(z) = \sum_{j=0}^n u_j u''_{n-j} - (1 - \chi_{n+1}) \gamma^2 \quad (8.22)$$

is given by Theorem 4.1.

Equation (8.17) has two solutions when $0 \leq \gamma \leq 0.775152$, as pointed out by Magyari (2008). Using the HAM with polynomial as base function, Abbasbandy et al. (2009) gained the two solutions by solving a nonlinear algebraic equation about the convergence-control parameter c_0 . In this chapter, we suggest a new optimal HAM approach to find out multiple solutions by introducing an unknown parameter in initial guess, called *the multiple-solution-control parameter*.

Without loss of generality, let us first consider the case of $\gamma = 3/5$. The nonlinear algebraic equation (8.19) has two solutions $u(0) = 0.101046$ and $u(0) = 0.779034$ when $\gamma = 3/5$, corresponding to the two exact solutions in the implicit form (8.18), respectively.

First of all, let us apply the hybrid-base approximation method mentioned in Sect. 7.2.3 to solve this nonlinear boundary value problem with multiple solutions. According to (8.13), we choose the auxiliary linear operator

$$\mathcal{L}(u) = u'' + \pi^2 u. \quad (8.23)$$

Note that, if $u(z)$ satisfies (8.17), then $u(-z) = u(z)$ is also its solution. So, $u(z)$ is an even function. This information is useful for us to choose the initial guess $u_0(z)$. So, we choose the initial guess

$$u_0(z) = \frac{1}{2}(\sigma + 1) + \frac{1}{2}(\sigma - 1) \cos(\pi z), \quad (8.24)$$

where $\sigma = u_0(0)$ is an unknown parameter. Note that $u_0(z)$ is dependent upon the unknown parameter σ and thus different values of σ give different initial guesses. The unknown parameter σ is introduced here to search for the multiple solutions, as shown later.

Using the auxiliary linear operator (8.23) and the initial guess (8.24), we gain the 10th-order homotopy-approximation by means of the BVP4 1.0. Note that, the approximation contains two unknown parameters: the convergence-control parameter c_0 and the unknown parameter σ in (8.24). However, given c_0 , the squared residual E_m is only dependent upon σ . For example, in case of $c_0 = -1$, the curves of E_m versus σ are as shown in Fig. 8.1, which clearly indicates that, in a large range $-0.8 < \sigma < 0.5$, the squared residual E_m decreases as the order of approximation increases, and the optimal value of σ (for $c_0 = -1$) is about -0.2 , corresponding to

Fig. 8.1 Squared residual E_m of (8.17) versus $\sigma = u_0(0)$ by means of the auxiliary linear operator (8.23) and the initial guess (8.24) when $\gamma = 3/5$ and $c_0 = -1$. Dashed line: 5th-order approximation; Dash-dotted line: 6th-order approximation; Solid line: 8th-order approximation.

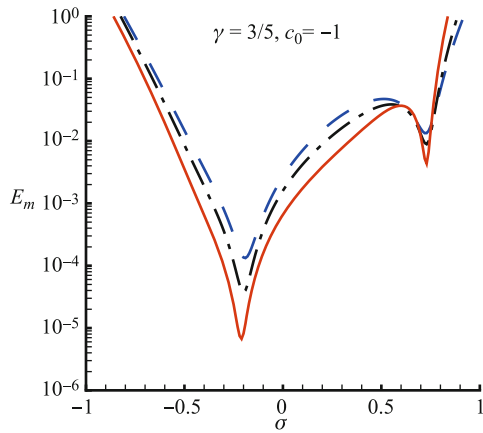


Fig. 8.2 Squared residual E_m of (8.17) versus $\sigma = u_0(0)$ by means of the auxiliary linear operator (8.23) and the initial guess (8.24) when $\gamma = 3/5$ and $c_0 = -1$. Dashed line: 10th-order approximation; Dash-dotted line: 15th-order approximation; Solid line: 20th-order approximation.

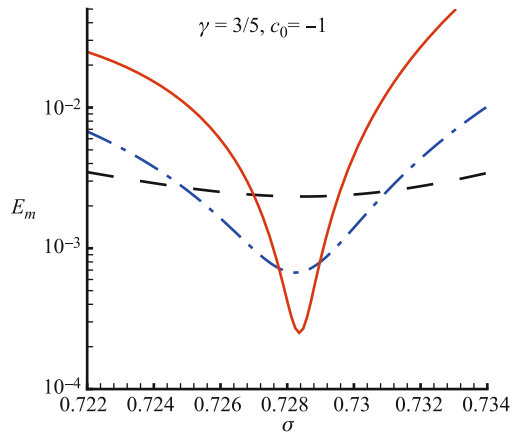


Fig. 8.3 Approximations of two branches of solutions $u(z)$ of (8.17) when $\gamma = 3/5$ by means of the auxiliary linear operator (8.23) and the initial guess (8.24). Solid line: 5th-order homotopy-approximation of the low-branch solution given by $c_0 = -1$ and $\sigma = -1/5$; Dashed line: 10th-order homotopy-approximation of the up-branch solution given by $c_0 = -0.71848810$ and $\sigma = 0.72834766$; Symbols: exact solution.

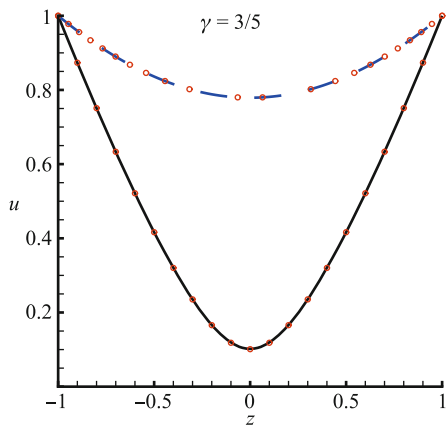


Table 8.1 The approximation of $u(0)$ at the m th iteration by means of the auxiliary linear operator (8.23) and the initial guess (8.24) when $N_t = 20$, $\gamma = 3/5$, $c_0 = -1$ with different values of σ .

m , number of iteration	$\sigma = -1/2$	$\sigma = -1/5$	$\sigma = 1/2$
5	0.113536	0.101924	0.138348
10	0.100775	0.101008	0.099292
15	0.101057	0.101046	0.101032
20	0.101046	0.101045	0.101041
25	0.101046	0.101046	0.101045
30	0.101046	0.101046	0.101046

Table 8.2 Squared residual E_m of (8.17) at the m th iteration by means of the auxiliary linear operator (8.23) and the initial guess (8.24) when $N_t = 20$, $\gamma = 3/5$, $c_0 = -1$ with different values of σ .

m , number of iteration	$\sigma = -1/2$	$\sigma = -1/5$	$\sigma = 1/5$
5	4.6×10^{-4}	3.5×10^{-6}	3.2×10^{-3}
10	4.6×10^{-7}	1.5×10^{-8}	1.8×10^{-5}
15	1.5×10^{-8}	2.1×10^{-9}	1.5×10^{-7}
20	2.3×10^{-9}	1.9×10^{-9}	6.1×10^{-9}
25	1.9×10^{-9}	1.8×10^{-9}	2.0×10^{-9}
30	1.9×10^{-9}	1.8×10^{-9}	1.9×10^{-9}

the fastest convergent series. This is indeed true. It is found that, when $c_0 = -1$, the 3rd-order iteration approach gives convergent result $u(0) = 0.101046$ by means of $\sigma = -1/2, -1/5, +1/2$, and the iteration approach by means of $\sigma = -1/5$ corresponds to the fastest convergence among them, as shown in Table 8.1 and Table 8.2, respectively. Besides, the corresponding 5th-order approximation agrees well with the low-branch exact solution, as shown in Fig. 8.3. Thus, when $c_0 = -1$, there exists such a interval $\Omega_1 \supset [-1/2, +1/2]$ that for any $\sigma \in \Omega_1$, the corresponding homotopy-series converges to the low-branch solution of (8.17). Note that one can not gain accurate approximation of the low-branch solution if $\sigma \notin \Omega_1$. In this meaning, like c_0 , the unknown parameter σ in the initial guess (8.24) can be regarded as a kind of convergence-control parameter, too.

According to Fig. 8.1, there exists another region $\Omega_2 \subset (0.5, 1)$ such that the squared residual E_m decreases as the order of approximation increases. However, when $c_0 = -1$, this interval is rather small, i.e. $\Omega_2 \subset (0.7276, 0.7292)$, as shown in Fig. 8.2. It is found that such a kind of interval becomes smaller and smaller as the order of approximation increases, so that it is time-consuming to determine its exact boundary. Besides, even using the optimal value $\sigma^* = 0.7284$ (when $c_0 = -1$) at the 20th-order of approximation, we still can not gain convergent series solution. Using other values of c_0 , such as $c_0 = -1/4$, such kind of interval Ω_2 is still very small. So, when $c_0 = -1$, the corresponding interval Ω_2 is so small that it is very difficult to gain the accurate enough approximation of the up-branch solution of (8.17) by means of iteration approach.

To gain an accurate enough approximation of the up-branch solution of (8.17), we regard both of c_0 and σ as the unknown parameters and search for the minimum of the squared residual of the governing equation at a given order of approximation. By means of the Mathematica command `Minimize` under a restriction $\sigma > 1/2$ (otherwise, we always find the low-branch solution), the minimum of the squared residual and the corresponding optimal parameters c_0^* and σ^* are easily obtained, as shown in Table 8.3. Alternatively, we can use the module `GetMin2D` of the package `BVPh 1.0` to search for the local minimum of the squared residual. At the 10th-order approximation, we have the optimal convergence-control parameters $c_0^* = -0.71848810$ and $\sigma^* = 0.72834766$, corresponding to the minimum of the squared residual $E_{10} = 2.68 \times 10^{-8}$. Note that the corresponding 10th-order homotopy-approximation of $u(z)$ agrees well with the up-branch exact solution, as shown in Fig. 8.3, and besides it gives the accurate approximation $u(0) = 0.779034$ that is equal to the exact value, as show in Table 8.3.

Table 8.3 Optimal parameters c_0^* and σ^* of the m th-order approximation of the up-branch solution of (8.17) by means of the auxiliary linear operator (8.23) and the initial guess (8.24) when $\gamma = 3/5$.

m , order of approx.	c_0^*	σ_0^*	E_m	$u(0)$
4	-0.71489261	0.72844802	2.59×10^{-5}	0.778979
5	-0.71891704	0.72832456	1.51×10^{-6}	0.779088
6	-0.71353321	0.72835306	3.32×10^{-7}	0.779022
8	-0.71350498	0.72834803	3.20×10^{-8}	0.779032
10	-0.71848810	0.72834766	2.68×10^{-8}	0.779034

In this way, we successfully gain the two solutions of (8.17) by means of introducing the unknown parameter σ in the initial guess (8.24). We call such kind of unknown parameter *the multiple-solution-control parameter*. Like the convergence-control parameter c_0 which provides us a convenient way to guarantee the convergence of solution series, the so-called multiple-solution-control parameter provides us a convent way to search for multiple solutions of nonlinear differential equations, as shown later.

Secondly, since the HAM provides us extremely large freedom to choose the auxiliary linear operator and the initial guess, we can also choose the auxiliary linear operator

$$\mathcal{L}(u) = u'', \quad (8.25)$$

and the initial guess

$$u_0(z) = \sigma + (1 - \sigma) z^2 \quad (8.26)$$

so as to approximate the solutions of (8.17) by means of polynomial, where $\sigma = u_0(0)$ is an unknown parameter, which is used here as a multiple-solution-control parameter.

Using the `BVPh 1.0`, it takes a few seconds CPU time to gain the corresponding 5th-order homotopy-approximation. By means of the Mathematica command

Minimize, the optimal convergence-control parameter $c_0^* = -1.16236$ and the optimal multiple-solution-control parameter $\sigma^* = 0.76500$ are found, corresponding to the minimum of the squared residual of the governing equation (at the 5th-order homotopy-approximation) 6.3×10^{-12} . By means of $c_0 = -1.16236$ and $\sigma = 0.76500$, the corresponding homotopy-series converges rather quickly to the up-branch solution: even the 8th-order homotopy-approximation gives the exact value $u(0) = 0.779034$, as shown in Table 8.4. Setting $c_0 = -1$, the curves of the squared residual E_m at up-to 5th-order of approximation versus σ are as shown in Fig 8.4. It indicates that, when $c_0 = -1$, there exists an interval Ω_2 such that, for any $\sigma \in \Omega_2$, E_m decreases as the order of approximation increases. It is found that the homotopy-series converges by means of $c_0 = -1$ and $0.6 \leq \sigma \leq 1$, therefore it holds $\Omega_2 \supset [0.6, 1]$ when $c_0 = -1$. It is interesting that, by means of 3rd-order iteration approach with *any* a multiple-solution-control parameter $\sigma \in [0.6, 1]$, we also gain accurate approximation of the up-branch solution of (8.17).

Table 8.4 Squared residual E_m of (8.17) and $u(0)$ at the m th-order approximation when $\gamma = 3/5$ by means of the auxiliary linear operator (8.25) and the initial guess (8.26) with $c_0 = -1.16236$ and $\sigma = 0.765$.

m , order of approximation	Squared residual E_m	$u(0)$
4	1.5×10^{-9}	0.779002
8	2.4×10^{-13}	0.779034
12	1.1×10^{-17}	0.779034
16	8.2×10^{-23}	0.779034
20	3.4×10^{-25}	0.779034

According to Fig. 8.4, there exists an interval Ω_1 near $\sigma = 0$ such that the squared residual E_m of the governing equation decreases as the order of approximation increases. Unfortunately, this interval is rather small, as shown more clearly in Fig. 8.5, so that it is time-consuming to determine the boundary of the interval Ω_1 exactly. To avoid it, we use the Mathematica command Minimize with the restriction $\sigma < 1/10$ (otherwise, we always find the up-branch solution) to gain the optimal homotopy-approximation. It is found that, the minimum of the squared residual at the 25th-order homotopy-approximation arrives 2.7×10^{-9} by means of the optimal convergence-control parameter $c_0^* = -1.6951993251$ and the optimal multiple-solution-control parameter $\sigma^* = 0.05122982326872$, which give an accurate approximation $u(0) = 0.101046$ of the low-branch solution, as shown in Table 8.5. So, using the auxiliary linear operator (8.25) and the initial guess (8.26), we also gain the two solutions of (8.17) by means of the so-called multiple-solution-control parameter σ .

All of these illustrate that the so-called *multiple-solution-control parameter* σ introduced in the initial guess (8.24) and (8.26) indeed provides us a convenient way to search for multiple solutions of nonlinear boundary value problems. For (8.17), given a reasonable convergence-control parameter c_0 , there exist two intervals Ω_1 and Ω_2 such that, as long as $\sigma \in \Omega_1$ or $\sigma \in \Omega_2$, the corresponding homotopy-

Fig. 8.4 Squared residual E_m of (8.17) versus $\sigma = u_0(0)$ in the interval $\sigma \in [0, 1]$ by means of the auxiliary linear operator (8.25) and the initial guess (8.26) when $\gamma = 3/5$ and $c_0 = -1$. Dashed line: 3rd-order approximation; Dash-dotted line: 4th-order approximation; Solid line: 5th-order approximation.

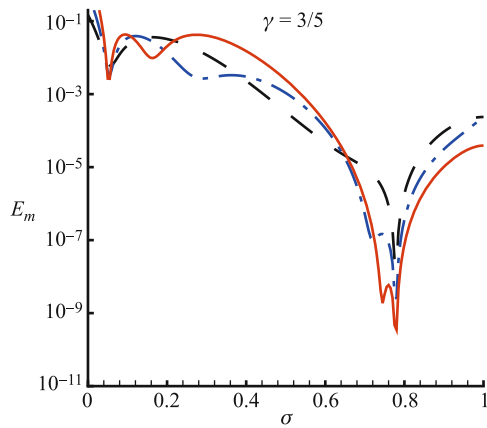


Fig. 8.5 Squared residual E_m of (8.17) versus $\sigma = u_0(0)$ in the local interval $\sigma \in [0, 0.15]$ by means of the auxiliary linear operator (8.25) and the initial guess (8.26) when $\gamma = 3/5$ and $c_0 = -1$. Dashed line: 3rd-order approximation; Dash-dotted line: 4th-order approximation; Solid line: 5th-order approximation.

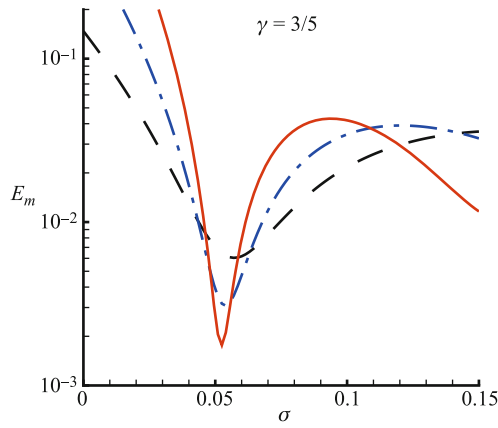


Table 8.5 Optimal parameters c_0^* and σ^* for the m th-order approximation of the low-branch solution of (8.17) by means of the auxiliary linear operator (8.25) and the initial guess (8.26) when $\gamma = 3/5$.

m , order of approx.	c_0^*	σ_0^*	E_m	$u(0)$
5	-1.52954	0.05137	6.3×10^{-4}	0.102379
10	-1.6299959583	0.05123009640906	2.3×10^{-5}	0.101165
15	-1.6648354484	0.05122982392509	1.1×10^{-6}	0.101060
20	-1.6833563121	0.05122982327004	5.2×10^{-8}	0.101047
25	-1.6951993251	0.05122982326872	2.7×10^{-9}	0.101046

series converges to either the low-branch or the up-branch solution, respectively. It is interesting that the intervals Ω_1 and Ω_2 depend upon not only the convergence-control parameter c_0 but also the auxiliary linear operator and the initial guess. By means of the auxiliary linear operator (8.23) and the initial guess (8.24), the interval Ω_1 is much larger than Ω_2 so that it is easier to gain the low-branch solution than the

up-branch ones. However, using the auxiliary linear operator (8.25) and the initial guess (8.26), the interval Ω_2 is much larger than Ω_1 so that it is easier to gain the up-branch solution than the low-branch ones. Since the HAM provides us extremely large freedom to choose the auxiliary linear operator and the initial guess, we can use the auxiliary linear operator (8.23) and the initial guess (8.24) to gain the low-branch of solution, and the auxiliary linear operator (8.25) and the initial guess (8.26) to obtain the up-branch solution! Using this strategy, we gain the two branches¹ of solutions for the possible values of γ . This also illustrates that, using the BVPh 1.0, we can search for multiple solutions of nonlinear boundary-value equations (8.1) by means of different types of auxiliary linear operators and different types of initial guesses that contains an unknown parameter, called the multiple-solution-control parameter.

8.3.2 A three-point nonlinear boundary-value problem

Let us further consider a three-point nonlinear boundary-value problem

$$u'''' = \beta z(1 + u^2), \quad u(0) = u'(1) = u''(1) = 0, \quad u''(0) - u''(\alpha) = 0, \quad (8.27)$$

where $\alpha \in (0, 1)$ and β are given constants. Graef et al. (2003, 2004) proved that the above equation has at least two positive solutions when $\alpha = 1/5$ and $\beta = 10$.

This problem is a special case of (8.1) when

$$\mathcal{F}[z, u] = u'''' - \beta z(1 + u^2).$$

Similarly, the m th-order homotopy-approximation is given by (8.10), where $u_k(z)$ is governed by the k th-order deformation equation

$$\mathcal{L}[u_k(z) - \chi_k u_{k-1}(z)] = c_0 \delta_{k-1}(z) \quad (8.28)$$

subject to the boundary conditions

$$u_k(0) = 0, \quad u'_k(1) = 0, \quad u''_k(1) = 0, \quad u''_k(0) - u''_k(\alpha) = 0, \quad (8.29)$$

where

$$\delta_i(z) = u_i'''' - (1 - \chi_{i+1}) \beta z - \beta z \sum_{j=0}^i u_j u_{i-j} \quad (8.30)$$

is gained by Theorem 4.1.

Note that the governing equation (8.27) contains the term z . So, the solution $u(z)$ in the finite interval $z \in [0, 1]$ can be expressed by polynomial. Thus, we choose the

¹ More truncated terms are necessary in order to gain accurate low-branch solution for small γ , because there exists the singularity as $\gamma \rightarrow 0$.

auxiliary linear operator

$$\mathcal{L}(u) = u'''' \quad (8.31)$$

and the initial guess in the form

$$u_0(z) = a_0 + \sum_{i=1}^4 a_i z^i,$$

where a_k is an unknown constant. Let $\sigma = u(1)$ denote the unknown multiple-solution-control parameter, which provides us the additional condition

$$u(1) = \sigma. \quad (8.32)$$

Then, the five unknown constants a_k ($0 \leq k \leq 4$) of the initial guess $u_0(z)$ are determined by the four boundary conditions (8.27) and the additional boundary condition (8.32). Thus, we have the initial guess

$$u_0(z) = \frac{\sigma}{(2\alpha - 3)} [2(3\alpha - 4)z + 6(1 - \alpha)z^2 + 2\alpha z^3 - z^4]. \quad (8.33)$$

Note that the above initial guess is dependent upon the position $z = \alpha$, where there exists the multipoint boundary-condition $u''(0) = u''(\alpha)$.

The multipoint boundary conditions can be easily resolved by means of computer algebra system like Mathematica: all boundary conditions are satisfied in a similar way. Let $u_k^*(z)$ denote a special solution of (8.28). According to (8.31), its general solution reads

$$u_k(z) = u_k^*(z) + C_0 + C_1 z + C_2 z^2 + C_3 z^3,$$

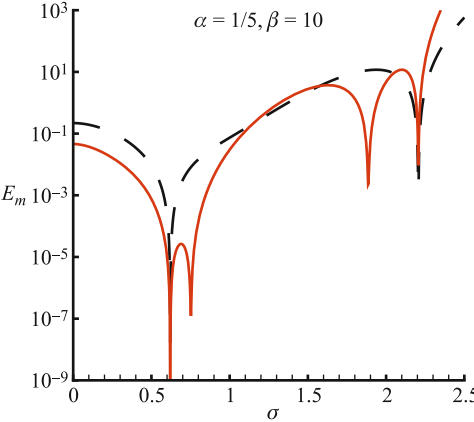
where the four coefficients C_0, C_1, C_2 and C_3 are determined by the four linear boundary conditions (8.29). This is mainly because computer algebra system like Mathematica provide us the ability to “compute with functions instead of numbers” (Trefethen, 2007).

Let us first consider the case of $\alpha = 1/5$ and $\beta = 10$. Regarding both of the convergence-control parameter c_0 and the multiple-solution-control parameter σ as unknowns, the squared residual E_m of the governing equation at the m th-order homotopy-approximation is dependent upon c_0 and σ . It is found that, at the 5th-order homotopy-approximation, the minimum of the squared residual reads 1.6×10^{-20} , corresponding to the optimal convergence-control parameter $c_0^* = -1.00661$ and the optimal multiple-solution-control parameter $\sigma^* = 0.62018$. The curves of E_m versus σ when $c_0 = -1$ are as shown in Fig. 8.6, which indicates that there exists an interval Ω_1 such that, for any $\sigma \in \Omega_1$, the squared residual E_m decreases as m increases. It is found that the *same* convergent series solution can be obtained by means of $c_0 = -1$ and $\sigma = 0, 0.5$ and 1.1 , so that $\Omega_1 \supset [0, 1.1]$. Especially, using $\sigma = 0.62$ and $c_0 = -1$, we gain the accurate result with $u(1) = 0.627315$ in a few seconds of CPU time by means of the 3rd-order iteration approach, as shown in Table 8.6.

Table 8.6 Squared residual E_m of (8.27) and $u(1)$ at the m th iteration approximation when $\alpha = 1/5$ and $\beta = 10$ by means of the auxiliary linear operator (8.31) and the initial guess (8.33) with $c_0 = -1$ and $\sigma = 0.62$.

m , times of iteration	E_m	$u(1)$
1	1.7×10^{-8}	0.627299
2	1.2×10^{-10}	0.627313
3	8.7×10^{-13}	0.627315
5	4.4×10^{-17}	0.627315
10	8.2×10^{-28}	0.627315

Fig. 8.6 Squared residual E_m of (8.27) versus σ in the interval $\sigma \in [0, 2.5]$ by means of the auxiliary linear operator (8.31) and the initial guess (8.33) when $\alpha = 1/5$, $\beta = 10$ and $c_0 = -1$. Dashed line: 3rd-order approximation; Solid line: 5th-order approximation.



According to Fig. 8.6, there should exist a solution near $\sigma = 2$. Let Ω_2 denote such an interval that, for any $\sigma \in \Omega_2$ and $c_0 = -1$, we gain a convergent homotopy-series of the 2nd solution. However, it is found that the interval Ω_2 is so small that it is time-consuming to determine it exactly. To avoid it, we search for the 2nd solution by means of the minimum of the squared residual at the high enough order of approximation. Using the Mathematica command `Minimize` with the restriction $\sigma > 2$ (otherwise we always gain the 1st solution), it is found that, at the 5th-order homotopy-approximation, the minimum of the squared residual reads 1.7×10^{-15} , corresponding to the optimal convergence-control parameter $c_0^* = -1.0202828199$ and the optimal multiple-solution-control parameter $\sigma^* = 2.2062238124$. Furthermore, at the 10th-order of approximation, the minimum of the squared residual reads 7.4×10^{-31} , corresponding to the optimal convergence-control parameter $c_0^* = -1.0166130822$ and the optimal multiple-solution-control parameter $\sigma^* = 2.206223812318637$. It is found that, at $z = 1$, the 5th to 10th homotopy-approximations tend to the same value $u(1) = 2.2411770066$, as shown in Table 8.7. Therefore, we indeed obtain two different positive solutions of the three-point nonlinear boundary-value equation (8.27) when $\alpha = 1/5$ and $\beta = 10$, as shown in Fig. 8.7 and listed in Table 8.8.

Fig. 8.7 Approximations of two solutions $u(z)$ of (8.27) when $\alpha = 1/5$ and $\beta = 10$ by means of the auxiliary linear operator (8.31) and the initial guess (8.33). Solid line: homotopy-approximation of the 1st solution given by 10th iteration with $c_0 = -1$ and $\sigma = 0.62$; Dashed line: 10th-order homotopy-approximation of the 2nd solution given by the optimal parameters $c_0 = -1.0166130822$ and $\sigma = 2.206223812318637$.

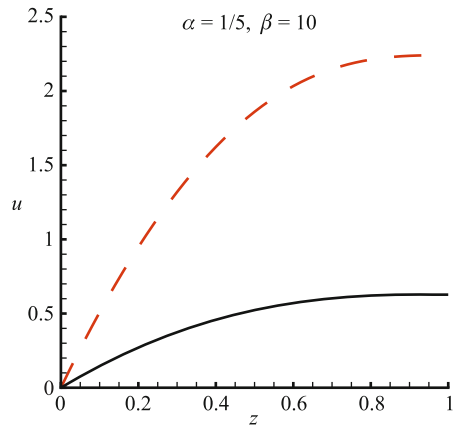


Table 8.7 Optimal convergence-control parameters c_0^* and the optimal multiple-solution-control parameter σ^* at the m th-order approximation of the 2nd solution of (8.27) when $\alpha = 1/5$ and $\beta = 10$ by means of the auxiliary linear operator (8.31) and the initial guess (8.33).

m	c_0^*	σ_0^*	E_m	$u(1)$
3	-1.0156751650	2.206225834958724	3.2×10^{-9}	2.2411852031
5	-1.0202828199	2.206223812447131	1.7×10^{-15}	2.2411770076
6	-1.0186293138	2.206223812311513	6.1×10^{-19}	2.2411770076
7	-1.0147412057	2.206223812318337	2.2×10^{-21}	2.2411770076
8	-1.0216642167	2.206223812318337	3.8×10^{-25}	2.2411770076
9	-1.0198947262	2.206223812318337	2.1×10^{-28}	2.2411770076
10	-1.0166130822	2.206223812318337	7.4×10^{-31}	2.2411770076

Table 8.8 Convergent homotopy-approximation of the two positive solutions of (8.27) when $\alpha = 1/5$ and $\beta = 10$ by means of the auxiliary linear operator (8.31) and the initial guess (8.33).

z	1st solution	2nd solution
0	0	0
0.1	0.144853	0.508335
0.2	0.269514	0.948159
0.3	0.374042	1.319559
0.4	0.458715	1.623039
0.5	0.524156	1.859952
0.6	0.571473	2.033131
0.7	0.602414	2.147651
0.8	0.619519	2.211668
0.9	0.626286	2.237253
1.0	0.627315	2.241177

Fig. 8.8 $u(1)$ of two positive solutions $u(z)$ of (8.27) when $\alpha = 1/5$ with different values of β by means of the auxiliary linear operator (8.31) and the initial guess (8.33). Solid line: low-branch solution; Dashed line: up-branch solution.

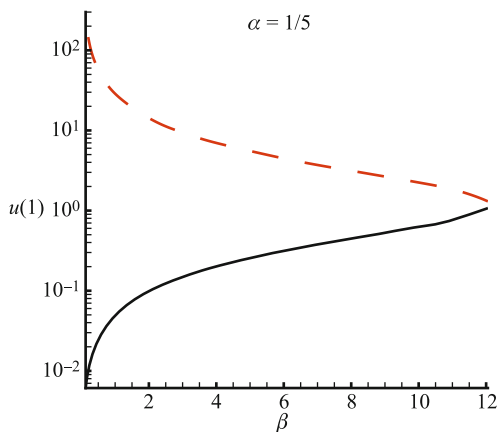
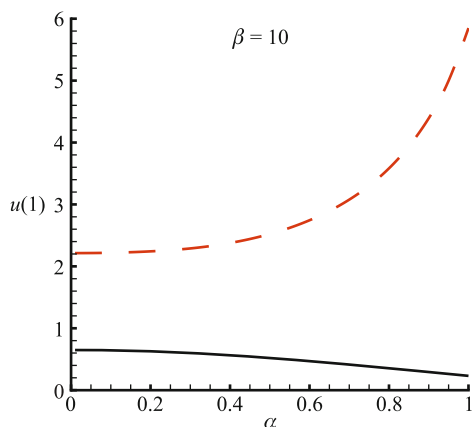


Fig. 8.9 $u(1)$ of two positive solutions $u(z)$ of (8.27) when $\beta = 10$ with different values of α by means of the auxiliary linear operator (8.31) and the initial guess (8.33). Solid line: low-branch solution; Dashed line: up-branch solution.



Similarly, we can find out the multiple solutions of (8.27) for different values of α and β by means of the BVPh 1.0. For example, when $\alpha = 1/5$, it is found that (8.27) has two positive solutions in the interval $\beta \in (0, 12.05]$, as listed in Table 8.9. Note that, as β tends to zero, $u(1)$ of the up-branch solution increases quickly, but $u(1)$ of the low-branch solution tends to zero, as shown in Fig. 8.8.

Similarly, when $\beta = 10$, we find out the multiple solutions of (8.27) for different values of α in the boundary condition

$$u''(0) - u''(\alpha) = 0.$$

It is found that there exist at least two positive solutions in the interval $\alpha \in (0, 1]$, as shown in Fig. 8.9. The corresponding values of $u(1)$ versus α are listed in Table 8.10. They show the influence of the above boundary condition at a separate point $z = \alpha$ to the solutions of nonlinear boundary-value equation (8.27).

Note that (8.27) can be first transferred into a nonlinear eigenvalue equation and then be solved by the BVPh 1.0, as shown in Example 9.3.4.

Table 8.9 The value of $u(1)$ of the two positive solutions of (8.27) when $\alpha = 1/5$ with different values of β by means of the auxiliary linear operator (8.31) and the initial guess (8.33).

β	1st solution	2nd solution
0.125	0.0061113	230.072656
0.25	0.012224	115.027019
0.5	0.024455	57.494885
1	0.048974	28.710135
2	0.098462	14.279968
3	0.149017	9.435354
4	0.201271	6.985695
6	0.314218	4.474595
8	0.447555	3.141431
9	0.528490	2.660305
10	0.627315	2.241177
11	0.762509	1.843780
12	1.054690	1.332973
12.05	1.10240	1.275268

Table 8.10 The value of $u(1)$ of the two positive solutions of (8.27) when $\beta = 10$ with different values of α by means of the auxiliary linear operator (8.31) and the initial guess (8.33).

α	1st solution	2nd solution
0.01	0.648176	2.213774
0.05	0.646932	2.215093
0.1	0.643033	2.219512
0.2	0.627315	2.241177
0.3	0.601033	2.289274
0.4	0.564806	2.377541
0.5	0.520086	2.522182
0.6	0.468852	2.744448
0.7	0.413081	3.077674
0.8	0.354329	3.583159
0.9	0.293562	4.391860
0.95	0.262562	4.999196
1	0.231157	5.848855

This example illustrates that, using the BVPh 1.0, we can gain multiple solutions of nonlinear multipoint boundary-value problems by introducing the so-called multiple-solution-control parameter in initial guess.

8.3.3 Channel flows with multiple solutions

The laminar flow of a viscous incompressible fluid in a two-dimensional porous channel with orthogonally moving walls (Xu et al., 2010) is described by a nonlinear boundary-value differential equation

$$F'''' + \alpha (z F''' + 3F'') + (F F''' - F' F'') = 0, \quad 0 \leq z \leq 1, \quad (8.34)$$

subject to the boundary conditions

$$F(0) = 0, \quad F''(0) = 0, \quad F(1) = R, \quad F'(1) = 0, \quad (8.35)$$

where the prime denotes the differentiation with respect to z , R is the cross-flow Reynolds number, α is a physical constant related to the wall expansion ratio, respectively. Here, $F(z)$ is related to the stream function

$$\psi(x, y) = \frac{Vx}{d} F(z), \quad z = y/d,$$

where d is the half of the distance between the two parallel walls. For details, please refer to Berman (1953) and Xu et al. (2010). Writing $u(z) = F(z)/R$, the above equation becomes

$$u'''' + \alpha (z u''' + 3u'') + R (u u''' - u' u'') = 0, \quad 0 \leq z \leq 1, \quad (8.36)$$

subject to the boundary conditions

$$u(0) = 0, \quad u''(0) = 0, \quad u(1) = 1, \quad u'(1) = 0. \quad (8.37)$$

In the frame of the HAM, Xu et al. (2010) developed an analytic approach to find out the multiple solutions of this problem, which is in principle based on the shooting method. Here, we illustrate that, using the BVPh 1.0, we can find out its multiple solutions by introducing the so-called multiple-solution-control parameter in the initial guess.

This problem is a special case of (8.1) when

$$\mathcal{F}[z, u] = F'''' + \alpha (z F''' + 3F'') + (F F''' - F' F'').$$

The m th-order homotopy-approximation is given by (8.10), where $u_k(z)$ is governed by the k th-order deformation equation

$$\mathcal{L}[u_k(z) - \chi_k u_{k-1}(z)] = c_0 \delta_{k-1}(z) \quad (8.38)$$

subject to the boundary conditions

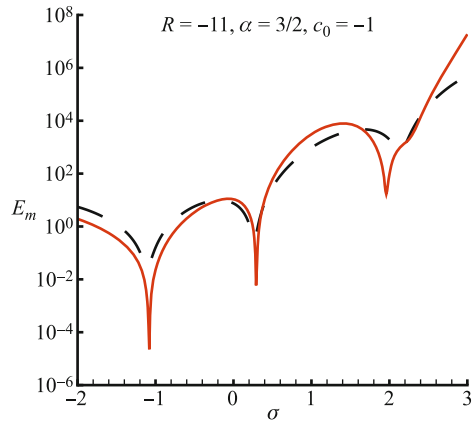
$$u_k(0) = 0, \quad u_k''(0) = 0, \quad u_k(1) = 0, \quad u_k'(1) = 0, \quad (8.39)$$

where

$$\delta_i(z) = u_i'''' + \alpha (z u_i''' + 3u_i'') + R \sum_{j=0}^i (u_j u_{i-j}''' - u_j' u_{i-j}'') \quad (8.40)$$

is gained by Theorem 4.1.

Fig. 8.10 Squared residual E_m of (8.36) versus σ when $R = -11$ and $\alpha = 3/2$ by means of the auxiliary linear operator (8.41) and the initial guess (8.42) with $c_0 = -1$. Solid line: E_{10} ; Dashed line: E_6 .



Analyzing the governing equation (8.36) and the boundary conditions (8.37), it is easy to find that the solution is an odd function. Besides, noticing that (8.36) contains the term z , it is convenient to express $u(z)$ by polynomials. So, we choose the auxiliary linear operator

$$\mathcal{L}(u) = u'''' \quad (8.41)$$

and the initial guess in the form

$$u_0(z) = a_1 z + a_2 z^3 + a_3 z^5,$$

where a_1, a_2, a_3 are constants. Note that this initial guess automatically satisfies the boundary conditions at $z = 0$. Thus, there are only *two* boundary conditions left at $z = 1$ for the *three* unknown constants a_1, a_2 and a_3 so that one of them is undetermined. This provides us one additional degree of freedom in the initial guess, which is useful for us to find out the multiple solutions, as shown later. To search for the multiple solutions, we introduce the so-called multiple-solution-control parameter $\sigma = u'_0(0)$. Then, the three unknown constants are uniquely determined by $\sigma = u'_0(0)$ and the two boundary conditions (8.37) at $z = 1$. Thus, we have the initial guess

$$u_0(z) = \sigma z + \frac{1}{2}(5 - 4\sigma)z^3 - \frac{1}{2}(3 - 2\sigma)z^5. \quad (8.42)$$

Without loss of generality, let us consider the case $R = -11$ and $\alpha = 3/2$. Regarding both of the convergence-control parameter c_0 and the multiple-solution-control parameter σ as unknowns, the squared residual E_m of the governing equation at the m th-order approximation is dependent upon both of c_0 and σ . Obviously, the smaller the value of E_m , the better the corresponding m th-order homotopy approximation. So, the optimal approximation is given by the minimum of E_m . Besides, any a very small value of E_m corresponds to an accurate approximation of $u(z)$. To gain some information about the multiple solutions of (8.36) in case of $R = -11$ and $\alpha = 3/2$, we first plot the curves of E_6 and E_{10} versus σ when $c_0 = -1$, as shown in Fig. 8.10.

Note that there are three local minimums, which suggest that there might exist three solutions in the interval $\sigma < 0$, $0 < \sigma < 1$ and $\sigma > 1$, respectively. To confirm this guess, we use the Mathematica command `Minimize` first with the restriction $\sigma < 0$ to search for the solution of (8.36). The minimum of the squared residual decreases to 1.8×10^{-7} and 6.9×10^{-17} at the 10th and 20th-order approximation, respectively, and the corresponding value of $u'(0)$ converges to -1.02377 , as shown in Table 8.11. So, there indeed exists a solution in the interval $\sigma < 0$. It is found that, when $c_0 = -1$, there exists such an interval $\Omega_1 \supset [-2, -0.7]$ that, for any $\sigma \in \Omega_1$, we gain the same homotopy-approximation with the same value $u'(0) = -1.02377$ by means of the BVPh 1.0 without iteration. Especially, using $\sigma = -1$ and $c_0 = -1$, which are close to their optimal values, we gain the convergent solution in a few seconds of CPU time by a laptop computer.

Similarly, using the Mathematica command `Minimize` with the restriction $0 < \sigma < 1$, we find out the 2nd solution of (8.36) when $R = -11$ and $\alpha = 3/2$ by means of the BVPh 1.0. The minimum of the squared residual decreases to 1.8×10^{-10} at the 20th-order approximation, and the optimal homotopy-approximation given by the corresponding optimal convergence-control parameter c_0^* and the optimal multiple-solution-control parameter σ^* gives a convergent value $u'(0) = 0.16935$, as shown in Table 8.12. Let Ω_2 denote such an interval that, for any $\sigma \in \Omega_2$, the corresponding homotopy-series converges to the 2nd solution. However, it is found

Table 8.11 $u'(0)$ and the optimal parameters c_0^* and σ^* at the m th-order approximation of the 1st solution of (8.36) when $R = -11$ and $\alpha = 3/2$ by means of the auxiliary linear operator (8.41) and the initial guess (8.42).

m	c_0^*	σ_0^*	E_m	$u'(0)$
10	-1.15185	-1.081208	1.8×10^{-7}	-1.02375
12	-1.16501	-1.081396	2.3×10^{-9}	-1.02377
14	-1.25436	-1.081389	2.5×10^{-10}	-1.02377
16	-1.17126	-1.081372	3.6×10^{-13}	-1.02377
18	-1.24314	-1.08137198	4.6×10^{-14}	-1.02377
20	-1.17586	-1.08137151	6.9×10^{-17}	-1.02377

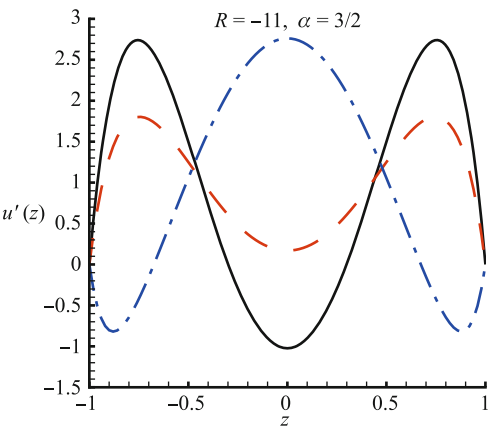
Table 8.12 $u'(0)$ and the optimal parameters c_0^* and σ^* at the m th-order approximation of the 2nd solution of (8.36) when $R = -11$ and $\alpha = 3/2$ by means of the auxiliary linear operator (8.41) and the initial guess (8.42).

m	c_0^*	σ_0^*	E_m	$u'(0)$
10	-0.87023	0.293224	4.9×10^{-4}	0.17103
12	-0.96316	0.292173	2.2×10^{-5}	0.16900
14	-0.90210	0.292288	1.1×10^{-6}	0.16926
16	-0.91428	0.292329	5.6×10^{-8}	0.16937
18	-0.94502	0.292321	3.0×10^{-9}	0.16935
20	-0.89768	0.292322	1.8×10^{-10}	0.16935

Table 8.13 Squared residual E_m of (8.36) and $u'(0)$ of the m th-iteration approximation when $R = 11$ and $\alpha = 3/2$ by means of the auxiliary linear operator (8.41) and the initial guess (8.42) with $c_0 = -1/2$ and $\sigma = 2$, corresponding to the 3rd solution.

m , time of iteration	E_m	$u'(0)$
1	6.9×10^3	2.31018
5	1.8×10^2	2.67366
10	4.5	2.75021
15	9.8×10^{-2}	2.75977
20	1.9×10^{-3}	2.76094
25	3.2×10^{-5}	2.76108
30	5.1×10^{-7}	2.76110
35	7.7×10^{-9}	2.76111
40	1.0×10^{-10}	2.76111

Fig. 8.11 Multiple solutions of (8.36) when $R = -11$ and $\alpha = 3/2$ gained by means of the auxiliary linear operator (8.41) and the initial guess (8.42). Solid line: the 1st solution; Dashed line: the 2nd solution; Dash-dotted line: the 3rd solution.



that the interval Ω_2 is so small that it is time-consuming to find its boundary exactly. So, it is more convenient to gain the 2nd solution by means of the BVPh 1.0 *without* iteration.

To gain the 3rd solution, we use the Mathematica command `Minimize` with the restriction $\sigma > 1$ to search for the minimum of the squared residual. It is found that the minimum of the squared residual E_m indeed decreases as the order of approximation increases, but rather slowly. Even at the 20th-order of approximation, the minimum of the squared residual reads 1.6 by means of the optimal convergence-control parameter $c_0^* = -0.75716$ and the optimal multiple-solution-control parameter $\sigma^* = 1.960920$. To accelerate the convergence, the 3rd-order iteration approach is used. It is found that, when $c_0 = -1/2$, there exists such an interval $\Omega_3 \supset [0.5, 3]$ that, for any $\sigma \in \Omega_3$, we obtain the same convergent homotopy-approximation with $u'(0) = 2.76111$. For example, when $c_0 = -1/2$ and $\sigma = 2$, the squared residual of the governing equation decreases quickly, as shown in Table 8.13.

In this way, we successfully obtain the three solutions of (8.36), corresponding to $u'(0) = -1.02377$, $u'(0) = 0.16935$ and $u'(0) = 2.76111$, respectively, as shown in

Fig. 8.11. This confirms once again that, using the BVPh 1.0, multiple solutions of nonlinear boundary-value problems can be found out by introducing the so-called multiple-solution-control parameter in initial guess.

8.4 Concluding remarks

In this chapter, using three nonlinear boundary-value equations as examples, we illustrate the validity of the BVPh 1.0 for nonlinear boundary-value equations with multiple solutions, governed by n th-order nonlinear boundary-value equation $\mathcal{F}[z, u] = 0$ in a finite interval $0 \leq z \leq a$, subject to the n linear boundary conditions $\mathcal{B}_k[z, u] = \gamma_k$ ($1 \leq k \leq n$).

An unknown parameter, namely *multiple-solution-control parameter*, is introduced into initial guess, for the first time, so as to search for multiple solutions. We illustrate that, using the BVPh 1.0 as a tool, multiple solutions of some n th-order nonlinear boundary-value problems can be found out by means of the so-called multiple-solution-control parameter, whose optimal value is determined by the minimum of the squared residual of governing equation at a high-enough order homotopy-approximation. In essence, the unknown multiple-solution-control parameter provides us one additional degree of freedom in an initial guess. Note that, it is the HAM that provides us great freedom to introduce such a kind of unknown parameter in the initial guess $u_0(z)$, whose different values correspond to different initial guesses. So, in addition to *the convergence-control parameter* c_0 that provides us a convenient way to guarantee the convergence of homotopy-series, we introduce in this chapter a new concept, *the multiple-solution-control parameter*, that provides us a convenient way to search for multiple solutions, although, as mentioned in Chapter 10, it also has influence on the convergence of homotopy-series. Our examples illustrate that, using an optimal multiple-solution-control parameter and an optimal convergence-control parameter c_0 determined by the minimum of squared residual of governing equation, we can find out multiple solutions of some n th-order nonlinear multipoint boundary-value problems in a finite interval $[0, a]$ by means of the BVPh 1.0.

Note that, different from numerical software BVP4, our approach is in principle an analytic ones, so that these two control-parameters are used as unknowns to gain homotopy-approximations until optimal values are found for them. This is an advantage of analytic approaches over numerical ones. From this point of view, the BVPh 1.0 provides us more freedom and flexibility to guarantee the convergence of series solution and to search for multiple solutions of nonlinear problems than the numerical package BVP4.

Based on the HAM, the BVPh 1.0 provides us great freedom to use different base functions to approximate solutions of a n th-order nonlinear boundary-value equation $\mathcal{F}[z, u] = 0$ in a finite interval $0 \leq z \leq a$. This is mainly because the HAM provides us extremely large freedom to choose the auxiliary linear operator $\mathcal{L}$ and

the initial guess $u_0(z)$. In fact, it is due to such kind of freedom of the HAM that we can introduce the so-called multiple-solution-control parameter in the initial guess.

Finally, it must be emphasized that the n th-order nonlinear boundary-value equation (8.1) and the n linear boundary conditions (8.2) are so general that it is quite difficult to develop a package valid for all of them. Note that, as mentioned in Chapter 7, our aim is to develop a package valid for as many (but *not* all) n th-order nonlinear multipoint boundary-value problems as possible. The Mathematica package BVPh (version 1.0) provides us a convenient analytic tool to search for multiple solutions of nonlinear boundary-value equations, although further modifications (see the Problems in this chapter) and more applications are needed in future.

Note that the Chebfun 4.0 also provides us the ability to “compute with functions instead of numbers” (Trefethen, 2007). So, it should be valuable to establish a similar HAM-based package for highly nonlinear multipoint BVPs with singularities by means of Chebfun, an open resource available (Accessed 15 April 2011) at

<http://www2.maths.ox.ac.uk/chebfun/>

Appendix 8.1 Input data of BVPh for Example 8.3.1

```
(* Input Mathematica package BVPh version 1.0 *)
<<BVPh.txt;

(* Define the physical and control parameters *)
TypeEQ      = 1;
TypeL       = 1;
TypeBase    = 2;
ApproxQ     = 1;
Ntruncated  = 20;

(* Define the governing equation *)
f[z_,u_,lambda_] := u*D[u,{z,2}] - gamma^2;
gamma = 3/5 ;

(* Define Boundary conditions *)
zR      = 1;
OrderEQ = 2;
BC[1,z_,u_,lambda_] := Limit[D[u,z], z->0 ];
BC[2,z_,u_,lambda_] := u - 1 /. z->zR;

(* Define initial guess *)
U[0] = u[0];
If[TypeL == 1,
  u[0] = sigma + (1-sigma)*z^2,
  u[0] = (sigma+1)/2 + (sigma-1)/2*Cos[kappa*Pi*z];
];
sigma = .;
```

```

(* Define output term *)
output[z_,u_,k_] := Print["output          = ",
                          u[k] /. z->0//N];

(* Define the auxiliary linear operator *)
omega[1] = Pi/zR;
L[f_] := Module[{temp,numA,numB,i},
If[TypeL == 1,
  temp[1] = D[f,{z,OrderEQ}],
  numA = IntegerPart[OrderEQ/2];
  numB = OrderEQ - 2* numA//Expand;
  temp[0] = D[f,{z,numB}];
  For[i=1, i<=numA, i++,
    temp[1] = D[temp[0],{z,2}]
      + (kappa*omega[i])^2*temp[0];
    temp[0] = temp[1];
  ];
];
temp[1]//Expand
];

(* Print input and control parameters *)
PrintInput[u[z]];

(* Gain 3rd-order approximation *)
BVPh[1,3];

(* Gain squared residual at 3rd-order approx. *)
GetErr[3];

(* Gain optimal values of c0 and sigma *)
res=Minimize[Err[3],{c0,sigma}];
Print["Minimum square residual = ", res];

(* Set optimal c0 and sigma *)
c0=-1;
sigma = 3/4;
Print["c0 = ",c0,"      sigma  = ",sigma];

(* Gain 10th-order HAM approximation *)
BVPh[1,10];

```

The Mathematica package BVPh (version 1.0) and the above input data are free available at

<http://numericaltank.sjtu.edu.cn/BVPh.htm>

Note that, for nonlinear boundary-value/eigenvalue problems defined in a finite interval, BVPh (version 1.0) has the default $zL = 0$.

Appendix 8.2 Input data of BVPh for Example 8.3.2

```

(* Input Mathematica package BVPh version 1.0 *)
<<BVPh.txt;

(* Define the physical and control parameters *)
TypeEQ      = 1;
TypeL       = 1;
TypeBase    = 2;
ApproxQ     = 0;

(* Define the governing equation *)
f[z_,u_,lambda_] := D[u,{z,4}]-beta*z*(1+u^2);
beta = 10 ;

(* Define Boundary conditions*)
zR      = 1;
OrderEQ = 4;
alpha   = 1/5 ;
BC[1,z_,u_,lambda_] := Limit[u, z->0 ];
BC[2,z_,u_,lambda_] := D[u,z] /. z->zR;
BC[3,z_,u_,lambda_] := D[u,{z,2}] /. z->zR;
BC[4,z_,u_,lambda_] := Module[{temp},
    temp[1]=D[u,{z,2}]/.z->0;
    temp[2]=D[u,{z,2}]/.z->alpha;
    temp[1]-temp[2] //Expand
];

(* Define initial guess *)
U[0] = u[0];
u[0] = sigma/(2*alpha-3)*((6*alpha-8)*z
    +6*(1-alpha)*z^2+2*alpha*z^3-z^4);
sigma = .;

(* Define output term *)
output[z_,u_,k_] := Print["output" = ",
    N[ u[k] /. z->1, 24] ];

(* Define the auxiliary linear operator *)
L[f_] := Module[{temp,numA,numB,i},
If[TypeL == 1,
    temp[1] = D[f,{z,OrderEQ}],
    numA = IntegerPart[OrderEQ/2];
    numB = OrderEQ - 2* numA//Expand;
    temp[0] = D[f,{z,numB}];
    For[i=1, i<=numA, i++,
        temp[1] = D[temp[0],{z,2}]
            + (kappa*omega[i])^2*temp[0];
        temp[0] = temp[1];
    ];
];
temp[1]//Expand
];

```

```

(* Print input and control parameters *)
PrintInput[u[z]];

(* Gain 3rd-order HAM approximation *)
BVPh[1,3];

(* Gain squared residual at 3rd-order approx. *)
GetErr[3];

(* Gain optimal values of c0 and sigma *)
res=Minimize[Err[3],{c0,sigma}];
Print["Minimum square residual = ", res];

(* Set optimal c0 and sigma *)
c0=-1;
sigma = 62/100;
Print["c0 = ",c0,"      sigma = ",sigma];

(*Gain 10th-order HAM approximation *)
BVPh[1,10];

```

The Mathematica package BVPh (version 1.0) and the above input data are free available at

<http://numericaltank.sjtu.edu.cn/BVPh.htm>

Note that, for nonlinear boundary-value/eigenvalue problems defined in a finite interval, BVPh (version 1.0) has the default $zL = 0$.

Appendix 8.3 Input data of BVPh for Example 8.3.3

```

(* Input Mathematica package BVPh version 1.0 *)
<<BVPh.txt;

(* Define the physical and control parameters *)
TypeEQ      = 1;
TypeL       = 1;
TypeBase    = 2;
ApproxQ     = 0;
NgetErr     = 1;

(* Define the governing equation *)
f[z_,u_,lambda_] := D[u,{z,4}]
+ alpha*( z*D[u,{z,3}]+3*D[u,{z,2}] )
+ R( u*D[u,{z,3}] - D[u,z]*D[u,{z,2}] );
alpha = 3/2;

```

```

R      =  -11;

(* Define Boundary conditions *)
zR  = 1;
OrderEQ = 4;
BC[1,z_,u_,lambda_] := Limit[u, z->0] ;
BC[2,z_,u_,lambda_] := Limit[D[u,{z,2}], z->0 ];
BC[3,z_,u_,lambda_] := u - 1 /. z->zR;
BC[4,z_,u_,lambda_] := D[u,z] /. z->zR;

(* Define initial guess *)
u[0]=sigma*z+(5-4*sigma)/2*z^3-(3-2*sigma)/2*z^5;
sigma = .;

(* Define output term *)
output[z_,u_,k_] := Print["output          = ",
                          D[u[k],z]/.z->0/N];

(* Define the auxiliary linear operator *)
omega[1] = Pi/zR;
omega[2] = Pi/zR;
L[f_] := Module[{temp,numA,numB,i},
If[TypeL == 1,
  temp[1] = D[f,{z,OrderEQ}],
  numA = IntegerPart[OrderEQ/2];
  numB = OrderEQ - 2* numA//Expand;
  temp[0] = D[f,{z,numB}];
  For[i=1, i<=numA, i++,
    temp[1] = D[temp[0],{z,2}]
              + (kappa*omega[i])^2*temp[0];
    temp[0] = temp[1];
  ];
];
temp[1]//Expand
];

(* Print input and control parameters *)
PrintInput[u[z]];

(* Set c0 and sigma *)
c0    =-1/2;
sigma = 2;
Print[" c0  = ",c0, "      sigma  = ",sigma];

(* Gain approximation by 3rd-order iteration *)
iter[1,40,3];

```

The Mathematica package BVPh (version 1.0) and the above input data are free available at

<http://numericaltank.sjtu.edu.cn/BVPh.htm>

Note that, for nonlinear boundary-value/eigenvalue problems defined in a finite interval, BVPh (version 1.0) has the default $zL = 0$.

Problems

8.1. Boundary-value problems with nonlinear boundary conditions

Develop a HAM-based analytic approach for the n th-order nonlinear boundary-value equation in a finite interval:

$$\mathcal{F}[z, u] = 0, \quad 0 \leq z \leq a,$$

subject to the n nonlinear multipoint boundary conditions

$$\mathcal{B}_k[z, u] = \gamma_k,$$

where $\mathcal{B}_k$ is a nonlinear differential operator and γ_k is a constant. Assume that the above equation has at least one smooth solution. Modify the Mathematica package BVPh (version 1.0) for this problem.

8.2. Coupled nonlinear boundary-value problems

Develop a HAM-based analytic approach for n coupled nonlinear boundary-value equations in a finite interval $z \in [0, a]$:

$$\mathcal{F}_k[z, u] = 0, \quad 1 \leq k \leq n,$$

subject to some linear/nonlinear multipoint boundary conditions, where $n \geq 2$. Assume that there exists at least one smooth solution. Modify the Mathematica package BVPh (version 1.0) for this kind of problems in general.

8.3. Boundary-value problems in an infinite interval

Develop a HAM-based analytic approach for the n th-order nonlinear boundary-value equation in an infinite interval

$$\mathcal{F}[z, u] = 0, \quad 0 \leq z < +\infty,$$

subject to the n multipoint nonlinear boundary conditions

$$\mathcal{B}_k[z, u] = \gamma_k,$$

where $\mathcal{B}_k$ is a nonlinear differential operator and γ_k is a constant. Assume that there exists at least one smooth solution. Modify the Mathematica package BVPh (version 1.0) for this kind of problems in general.

8.4. Coupled nonlinear boundary-value problems in an infinite interval

Develop a HAM-based analytic approach for n coupled nonlinear boundary-value equations in an infinite interval $z \in [0, +\infty)$:

$$\mathcal{F}_k[z, u] = 0, \quad 1 \leq k \leq n,$$

subject to some linear/nonlinear boundary conditions, where $n \geq 2$. Assume that there exists at least one smooth solution. Modify the Mathematica package BVPh (version 1.0) for this kind of problems in general.

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Chapter 9

Nonlinear Eigenvalue Equations with Varying Coefficients

Abstract Five different types of examples are used to illustrate the validity of the HAM-based Mathematica package BVPh (version 1.0) for nonlinear eigenvalue equations $\mathcal{F}[z, u, \lambda] = 0$ in a finite interval $0 \leq z \leq a$, subject to the n linear boundary conditions $\mathcal{B}_k[z, u] = \gamma_k$ ($1 \leq k \leq n$), where $\mathcal{F}$ denotes a n th-order nonlinear ordinary differential operator, $\mathcal{B}_k$ is a linear differential operator, γ_k is a constant, $u(z)$ and λ denote eigenfunction and eigenvalue, respectively. These examples verify that, using the BVPh 1.0, multiple solutions of some highly nonlinear eigenvalue equations with singularity and/or multipoint boundary conditions can be found by means of different initial guesses and different types of base functions.

9.1 Introduction

In science and engineering, one often needs to solve nonlinear differential equations about eigenfunction and eigenvalue. Many nonlinear eigenvalue equations have multiple solutions. However, it is well known that multiple solutions of nonlinear boundary-value problems are not easy to gain by means of numerical techniques such as the shooting method (Shampine et al., 2003).

In 2005, Liao (2005) applied the homotopy analysis method (HAM) (Liao, 1992, 1997, 1999a,b; Liao and Campo, 2002; Liao, 2003a,b, 2004, 2005, 2006, 2009a,b, 2010a,b; Liao and Tan, 2007; Xu et al., 2010) to successfully obtain two branches of solutions of the boundary-layer equation

$$F''' + \frac{1}{2}FF'' - \beta F'^2 = 0, \quad F(0) = 0, \quad F'(0) = 1, \quad F'(+\infty) = 0,$$

where $-1 < \beta < +\infty$ is a constant, by means of introducing an additional unknown quantity $\delta = F(+\infty)$. Solving a nonlinear algebraic equation related to this unknown quantity δ , Liao (2005) found one new branch of solutions when $\beta > 1$, which had been *never* reported by other analytic methods and even neglected by numerical methods, mainly because the difference between the values of $F''(0)$ of the two

branches of solutions is so small that it is hard to distinguish them. It verifies that multiple solutions of some nonlinear problems can be found out by introducing an unknown parameter properly.

In Chapter 8, we illustrate that, using the HAM-based Mathematica package BVPh (version 1.0) as a analytic tool, multiple solutions of some highly nonlinear boundary-value problems in a finite interval $z \in [0, a]$ can be found by introducing an unknown parameter into initial guess, called the *multiple-solution-control parameter*. In this chapter, we further illustrate the validity of the BVPh 1.0 for nonlinear eigenvalue problems governed by a n th-order nonlinear ordinary differential equation in a finite interval $z \in [0, a]$:

$$\mathcal{F}[z, u, \lambda] = 0, \quad z \in [0, a], \quad (9.1)$$

subject to the n linear boundary conditions

$$\mathcal{B}_k[z, u] = \gamma_k, \quad 1 \leq k \leq n, \quad (9.2)$$

where $\mathcal{F}$ is a n th-order nonlinear ordinary differential operator, $\mathcal{B}_k$ is a linear differential operator, z is an independent variable, $u(z)$ is an eigenfunction in the finite interval $z \in [0, a]$, λ is an unknown eigenvalue, γ_k and $a > 0$ are bounded constants, respectively. Note that, when $n \geq 3$, the n linear boundary conditions (9.2) can be defined at separated points in the interval $[0, a]$ (including the two endpoints). The linear operator $\mathcal{B}_k$ is in the general form

$$\mathcal{B}_k[z, u] = \sum_{i=0}^n a_{k,n-i}(z) \frac{d^i u(z)}{dz^i}, \quad (9.3)$$

where $a_{k,i}(z)$ is a smooth function of z . Note that at least one of $a_{k,n-i}(z)$ is non-zero. Assume that the above eigenvalue problem has at least one smooth eigenfunction and the corresponding eigenvalue.

In general, there exist many different non-zero eigenfunctions and eigenvalues, depending on a term such as $u(a)$, $u'(0)$, and so on. Therefore, we add such an additional linear boundary condition

$$\mathcal{B}_0[z, u] = \gamma_0, \quad \text{at } z = z^*, \quad (9.4)$$

where $z^* \in [0, a]$ and γ_0 is a constant, to distinguish different eigenfunctions and eigenvalues. Note that the above boundary condition must be linearly independent of the n original boundary conditions (9.2).

Some numerical techniques are developed to solve nonlinear eigenvalue problems. One of them is the famous BVP4C in MATLAB (Shampine et al., 2003), which is based on the shooting method (Shampine et al., 2003). In 2009, by means of trigonometric functions as base functions, Liao (2009b) employed the HAM to analytically solve a nonlinear eigenvalue problem about a non-uniform beam acted by an axial force. In 2011, using polynomial as base functions, Abbasbandy and Shirzadi (2011) proposed a HAM-based analytic approach for linear eigenvalue

problems, and suggested a way to gain multiple eigenfunctions by using different values of the convergence-control parameter c_0 governed by a nonlinear algebraic equation.

In this chapter, we propose a HAM-based analytic approach for the nonlinear eigenvalue equation (9.1) with the multipoint boundary conditions (9.2) in general. Then, we illustrate that such kind of nonlinear eigenvalue problems can be solved by means of the BVPh 1.0. The validity and generality of the BVPh 1.0 are shown by five different types of eigenvalue equations.

9.2 Brief mathematical formulas

Note that both of the eigenfunction $u(z)$ and eigenvalue λ are unknown. Let $q \in [0, 1]$ denote the homotopy-parameter, $u_0(z)$ and λ_0 denote the initial guess of the eigenfunction $u(z)$ and eigenvalue λ , respectively, where $u_0(z)$ satisfies the n original boundary conditions (9.2) and the additional boundary condition (9.4). In the frame of the HAM, we should first construct two continuous deformations (homotopies) $\phi(z; q)$ and $\Lambda(q)$ such that, as $q \in [0, 1]$ increases from 0 to 1, $\phi(z; q)$ varies from the initial guess $u_0(z)$ to the eigenfunction $u(z)$, and $\Lambda(q)$ from the initial guess λ_0 to the eigenvalue λ , respectively. Such kind of continuous deformations (homotopies) are constructed by the so-called zeroth-order deformation equation

$$(1 - q)\mathcal{L}[\phi(z; q) - u_0(z)] = c_0 q \mathcal{F}[z, \phi(z; q), \Lambda(q)], \quad q \in [0, 1], \quad (9.5)$$

subject to the n linear boundary conditions

$$\mathcal{B}_k[z, \phi(z; q)] = \gamma_k, \quad 1 \leq k \leq n, \quad (9.6)$$

and the additional linear boundary condition

$$\mathcal{B}_0[z, \phi(z; q)] = \gamma_0, \quad (9.7)$$

where $c_0 \neq 0$ is the so-called convergence-control parameter, and $\mathcal{L}$ is an auxiliary linear operator with the property $\mathcal{L}(0) = 0$, whose highest order of derivative is n . Since the initial guess $u_0(z)$ satisfies all boundary conditions, obviously,

$$\phi(z; 0) = u_0(z), \quad \Lambda(0) = \lambda_0$$

and

$$\phi(z; 1) = u(z), \quad \Lambda(1) = \lambda$$

are solutions of (9.5) to (9.7) when $q = 0$ and $q = 1$, respectively. If the initial guess $u_0(z)$, the auxiliary linear operator $\mathcal{L}$ and the convergence-control parameter c_0 are properly chosen so that the homotopy-Maclaurin series

$$\phi(z; q) = u_0(z) + \sum_{m=1}^{+\infty} u_m(z) q^m, \quad (9.8)$$

$$\Lambda(q) = \lambda_0 + \sum_{m=1}^{+\infty} \lambda_m q^m, \quad (9.9)$$

are absolutely convergent at $q = 1$, we have the homotopy-series

$$u(z) = u_0(z) + \sum_{m=1}^{+\infty} u_m(z), \quad (9.10)$$

$$\lambda = \lambda_0 + \sum_{m=1}^{+\infty} \lambda_m, \quad (9.11)$$

respectively. At the M th-order of approximation, we have the homotopy approximations

$$u(z) \approx \sum_{m=0}^M u_m(z), \quad \lambda \approx \sum_{m=0}^M \lambda_m. \quad (9.12)$$

According to Theorem 4.15, $u_m(z)$ and λ_{m-1} are governed by the m th-order deformation equation

$$\mathcal{L}[u_m(z) - \chi_m u_{m-1}(z)] = c_0 \delta_{m-1}(z), \quad (9.13)$$

where

$$\delta_k(z) = \mathcal{D}_k \{ \mathcal{F}[z, \phi(z; q), \Lambda(q)] \}$$

and

$$\chi_k = \begin{cases} 0, & k \leq 1, \\ 1, & k > 1. \end{cases}$$

Here, $\mathcal{D}_k$ is the k th-order homotopy-derivative operator, defined by

$$\mathcal{D}_k = \frac{1}{k!} \frac{\partial^k}{\partial q^k} \bigg|_{q=0}.$$

Substituting the homotopy-Maclaurin series (9.8) into the n boundary conditions (9.6) and (9.7), then equating the like-power of q , we have the $(n+1)$ linear boundary conditions

$$\mathcal{B}_i[z, u_m(z)] = 0, \quad 0 \leq i \leq n. \quad (9.14)$$

For details, please refer to Chapter 2 and Chapter 4.

Note that $\delta_k(z)$ in (9.13) only depends upon the nonlinear operator $\mathcal{F}$. By means of the basic properties of the homotopy-derivative operator $\mathcal{D}_k$ proved in Chapter 4, it is easy to deduce an explicit expression of $\delta_k(z)$ in most cases. For example, in case of

$$\mathcal{F}[z, u, \lambda] = \mathcal{L}_0(u) + \lambda f(z, u),$$

where $\mathcal{L}_0$ is a n th-order linear differential operator and $f(z, u)$ is a smooth function of z and u , using the basic property of the homotopy-derivative operator $\mathcal{D}_n$ described by Theorem 4.1, the linearity property described by Theorem 4.2 and the commutativity property described by Theorem 4.3, we have the explicit expression

$$\delta_k(z) = \mathcal{L}_0[u_k(z)] + \sum_{n=0}^k \lambda_{k-n} \mathcal{D}_n \{f[z, \phi(z; q)]\} \quad (9.15)$$

with the recurrence formulas

$$\mathcal{D}_0 \{f[z, \phi(z; q)]\} = f(z, u_0), \quad (9.16)$$

$$\mathcal{D}_n \{f[z, \phi(z; q)]\} = \sum_{j=0}^{n-1} \left(1 - \frac{j}{n}\right) u_{n-j}(z) \frac{\partial \mathcal{D}_j \{f[z, \phi(z; q)]\}}{\partial u_0}, \quad (9.17)$$

given by Theorem 4.10. In this case, we can always gain the explicit expression of $\delta_k(z)$ for arbitrary linear n th-order differential operator $\mathcal{L}_0$ and arbitrary nonlinear function $f(z, u)$. For a general nonlinear operator $\mathcal{F}[z, u, \lambda]$, it is easy to gain $\delta_k(z)$ efficiently by means of computer algebraic system like Mathematica (Abell and Braselton, 2004). In fact, to solve nonlinear ODEs in a finite interval $z \in [0, a]$ by means of the BVPh 1.0, it is unnecessary for us to deduce the expression of $\delta_k(z)$, since it is given automatically by the package. For details, please refer to the BVPh 1.0 given in the appendix of Chapter 7, which is free available (Accessed 25 Nov 2011, will be updated in the future) at

<http://numericaltank.sjtu.edu.cn/BVPh.htm>

Note that both of the m th-order deformation equation (9.13) and the $(n+1)$ boundary conditions (9.14) are linear. Let $u_m^*(z)$ denote a special solution of (9.13). One has

$$u_m^*(z) = \chi_m u_{m-1}(z) + c_0 \mathcal{L}^{-1} [\delta_{m-1}(z)],$$

where $\mathcal{L}^{-1}$ is the inverse operator of the auxiliary linear operator $\mathcal{L}$. Note that the special solution $u_m^*(z)$ contains the unknown λ_{m-1} . The general solution $u_m(z)$ of (9.13) reads

$$u_m(z) = \chi_m u_{m-1}(z) + c_0 \mathcal{L}^{-1} [\delta_{m-1}(z)] + \sum_{i=1}^n A_i \varphi_i(z),$$

where A_i is an unknown coefficient and $\varphi_i(z)$ is a known function, which satisfies the equation

$$\mathcal{L}[\varphi_i(z)] = 0, \quad 1 \leq i \leq n.$$

The unknown λ_{m-1} and the n unknown integral coefficients A_i ($1 \leq i \leq n$) are exactly determined by the $(n+1)$ linear boundary conditions (9.14). Therefore, given the initial guess $u_0(z)$, we can gain $\lambda_0, u_1(z)$, then $\lambda_1, u_2(z)$, and so on. It is extremely efficient to do such a kind of work by means of computer algebra systems like Mathematica.

In the frame of the HAM, we have extremely large freedom to choose the auxiliary linear operator $\mathcal{L}$ and the base function for the general eigenvalue equation (9.1). Therefore, the BVPb 1.0 provides us great freedom to choose the auxiliary linear operator $\mathcal{L}$ and the initial guess $u_0(z)$, so that different types of nonlinear eigenvalue problems can be solved.

It is well-known that a continuous function $u(z)$ in $z \in [0, a]$ can be approximated by Fourier series (Boyd, 2000). Leonhard Euler (15 April 1707 — 18 September 1783) solved a simple but famous eigenvalue problem, governed by the 2nd-order linear differential equation

$$u'' + \lambda u = 0, \quad u(0) = 0, \quad u(a) = 0,$$

whose eigenfunction and eigenvalue read

$$u = A \sin \frac{\kappa \pi z}{a}, \quad \lambda = \left(\frac{\kappa \pi}{a} \right)^2$$

with a positive integer $\kappa \geq 1$ and an arbitrary constant A . The above linear eigenvalue problem has an infinite number of eigenfunctions and eigenvalues. Note that it is a special case of (9.1) and (9.2). In order to include such kind of eigenfunctions, we often choose the auxiliary linear operator

$$\mathcal{L}(u) = u'' + \left(\frac{\kappa \pi}{a} \right)^2 u \quad (9.18)$$

for the 2nd-order eigenvalue differential equation (9.1), which has the property

$$\mathcal{L} \left(C_1 \cos \frac{\kappa \pi z}{a} + C_2 \sin \frac{\kappa \pi z}{a} \right) = 0$$

for arbitrary constants C_1, C_2 and positive integer $\kappa \geq 1$.

Similarly, for a 3rd-order eigenvalue differential equation (9.1), we can choose the auxiliary linear operator

$$\mathcal{L}(u) = u''' + \left(\frac{\kappa \pi}{a} \right)^2 u', \quad (9.19)$$

which has the property

$$\mathcal{L} \left(C_0 + C_1 \cos \frac{\kappa \pi z}{a} + C_2 \sin \frac{\kappa \pi z}{a} \right) = 0$$

for arbitrary constants C_0, C_1, C_2 and positive integer $\kappa \geq 1$. In general, for the n th-order eigenvalue differential equation (9.1), we can choose

$$\mathcal{L}(u) = \left[\prod_{i=1}^m \left(\frac{d^2}{dx^2} + \omega_i^2 \right) \right] u, \quad \text{when } n = 2m, \quad (9.20)$$

or

$$\mathcal{L}(u) = \left[\prod_{i=1}^m \left(\frac{d^2}{dx^2} + \omega_i^2 \right) \right] u', \quad \text{when } n = 2m + 1. \quad (9.21)$$

where ω_i is a frequency, which may be different each other, such as

$$\omega_i = i \left(\frac{\kappa\pi}{a} \right), \quad (9.22)$$

or the same, like

$$\omega_i = \frac{\kappa\pi}{a}, \quad (9.23)$$

depending on the boundary conditions (9.2), where $\kappa \geq 1$ is a positive integer.

Note that the different κ of the above auxiliary linear operators correspond to a set of different base functions

$$\left\{ \sin \frac{j\kappa\pi z}{a}, \cos \frac{j\kappa\pi z}{a} \mid j = 0, 1, 2, 3, \dots \right\},$$

which provides us a convenient way to search for multiple eigenfunctions and eigenvalues of a 2nd-order nonlinear eigenvalue equation (9.1) in a finite interval $z \in [0, a]$, as shown in Sect. 9.3.1, Sect. 9.3.2 and Sect. 9.3.3.

It is well-known that a continuous function $u(z)$ in a finite interval $z \in [0, a]$ can be well approximated by a power series

$$u(z) = \sum_{k=0}^{+\infty} a_k z^k,$$

or by a Chebyshev series

$$u(z) = \sum_{k=0}^{+\infty} b_k T_k(z),$$

where $T_k(z)$ is the Chebyshev polynomial of the first kind. So, we sometimes express an eigenfunction in power or Chebyshev polynomial, as shown in Sect. 9.3.4 and Sect. 9.3.5. In this case, we simply choose the auxiliary linear operator

$$\mathcal{L}(u) = \frac{d^n u}{dz^n} \quad (9.24)$$

for a n th-order eigenvalue problem in a finite interval $z \in [0, a]$.

Note that $u_m(z)$ and λ_{m-1} contain the so-called convergence-control parameter c_0 , which has no physical meanings but provides us a convenient way to guarantee the convergence of the homotopy-series (9.10) and (9.11), as shown in Chapter 2 and proved in Chapter 4. Let

$$E_m(c_0) = \frac{1}{a} \int_0^a \left\{ \mathcal{F} \left[z, \sum_{i=0}^m u_i(z), \sum_{i=0}^{m-1} \lambda_i \right] \right\}^2 dz \quad (9.25)$$

denote the averaged squared residual of the governing equation (9.1) at the m th-order of approximation, where the integral is calculated numerically by means of many enough discrete points. The optimal value c_0^* of the convergence-control parameter c_0 is determined by

$$\frac{dE_m(c_0)}{dc_0} = 0.$$

In most cases, choosing a convergence-control parameter near its optimal value c_0^* , we can obtain quickly convergent homotopy-series solution of the eigenfunction $u(z)$ and eigenvalue λ , as illustrated later. In this way, we can solve highly nonlinear eigenvalue problems in $z \in [0, a]$ by means of the BVPh 1.0, as illustrated later.

For the choice of the auxiliary linear operator $\mathcal{L}$ and the initial guess $u_0(z)$, please also refer to Chapter 7.

9.3 Examples

All examples given below are solved by the BVPh 1.0, which is given in the appendix of Chapter 7 and free available at

<http://numericaltank.sjtu.edu.cn/BVPh.htm>

In addition, the input data files of all these examples for the BVPh 1.0 are given in the appendix of this chapter and free available at the above website.

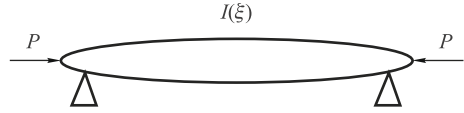
In the following examples, we set the number of truncated terms $N_t = 20$, and use 50 discrete points with equal space in the interval $z \in [0, a]$ to numerically calculate the related integrals mentioned above. Besides, if not mentioned, the 3rd-order HAM iteration approach is used, i.e. $M = 3$ in (7.51).

9.3.1 Non-uniform beam acted by axial load

Let us first consider a non-uniform beam with arbitrary cross-section (Boley, 1963; Chang and Popplewell, 1996; Katsikadelis and Tsiatas, 2004; Lee et al., 1993) on two supports under an axial load P , as shown in Fig. 9.1, where P is positive for compressive force and negative for tensile force. Let l and θ denote the length and slope of the beam, and u its deflection, respectively. Let s denote the arc-coordinate of the natural axis which passes through the centroid of each cross-section of the beam in its straight or unbuckled state, $I(s)$ the smallest moment of inertia of the cross-section about a line in its plane through the centroid, E the Young's modulus of the material, respectively. Assume that all of the principle axes of inertia are parallel so that the beam is not twisted. Mathematically, the problem is governed by

$$EI(s) \frac{d\theta}{ds} + Pu = 0, \quad u(0) = u(l) = 0,$$

Fig. 9.1 Beam acted by an axial load P with variable moment of inertia $I(z)$.



where $I(s)$, the moment of inertia, is non-negative. Assume that $I(s)$ and $I'(s)$ are bounded and continuous functions, where the prime denotes the differentiation with respect to s . Differentiating the original governing equation with respect to s and using the relationship $\sin \theta = du/ds$, we have the nonlinear buckling equation

$$(EI\theta')' + P \sin \theta = 0.$$

Substituting the boundary condition $u(0) = u(l) = 0$ into the original governing equation, we have the equivalent boundary conditions in the form

$$\theta'(0) = \theta'(l) = 0,$$

which means that the bending moment at two ends of the beam is zero.

Define the dimensionless arc-coordinate of the natural axis

$$z = \frac{\pi s}{l}$$

and write

$$I = I_0 \mu(z),$$

where $\mu(z)$ is a distribution function of I , and $I_0 > 0$ is a reference moment of inertia, respectively. For example, I_0 can be defined by

$$I_0 = \frac{1}{l} \int_0^l I(s) ds = \frac{1}{\pi} \int_0^\pi I(z) dz,$$

although this is not absolutely necessary. Then, the problem under consideration is governed by

$$\mu(z)\theta''(z) + \mu'(z)\theta'(z) + \lambda \sin \theta(z) = 0, \quad \theta'(0) = \theta'(\pi) = 0, \quad (9.26)$$

where the prime denotes the differentiation with respect to z , and

$$\lambda = \frac{P}{EI_0} \left(\frac{l}{\pi} \right)^2 \quad (9.27)$$

is called the axial load parameter. For given λ , it is straightforward to gain the corresponding axial load

$$P = \lambda (EI_0) \left(\frac{\pi}{l} \right)^2.$$

So, $\lambda > 0$ corresponds to a compressive force P , and $\lambda < 0$ to a tensile one, respectively.

Equation (9.26) with varying coefficients $\mu(z)$ and $\mu'(z)$ is only a special case of (9.1), i.e.

$$\mathcal{F}[z, \theta, \lambda] = \mu(z)\theta''(z) + \mu'(z)\theta'(z) + \lambda \sin \theta.$$

Obviously, for different non-zero eigenfunction $\theta(z)$, the value of $\theta(0)$ is different. This provides us an additional boundary condition

$$\theta(0) = \gamma \quad (9.28)$$

to distinguish different non-zero eigenfunctions.

The M th-order approximations of $u(z)$ and λ are given by (9.12), where $\theta_m(z)$ and λ_{m-1} are determined by the corresponding m th-order deformation equation

$$\mathcal{L}[\theta_m(z) - \chi_m \theta_{m-1}(z)] = c_0 \delta_{m-1}(z), \quad (9.29)$$

subject to the boundary conditions

$$\theta'_m(0) = 0, \quad \theta'_m(\pi) = 0, \quad \theta_m(0) = 0, \quad (9.30)$$

where

$$\delta_k(z) = \mu(z)\theta''_k(z) + \mu'(z)\theta'_k(z) + \sum_{i=0}^k \lambda_{k-i} \mathcal{D}_i \{\sin[\phi(z; q)]\} \quad (9.31)$$

is given by Theorem 4.1, and $\mathcal{D}_i \{\sin[\phi(z; q)]\}$ is gained by a recursion formula described by Theorem 4.8 and Theorem 4.10.

9.3.1.1 Uniform beam

First of all, let us consider the case of uniform beam, i.e. $\mu = 1$, corresponding to

$$\mathcal{F}[z, \theta, \lambda] = \theta'' + \lambda \sin \theta. \quad (9.32)$$

Since it is a 2nd-order differential equation, we use the auxiliary linear operator (9.18), which contains a positive integer κ . To satisfy the two original boundary conditions in (9.26) and the additional boundary condition (9.28), we choose such an initial guess $\theta_0(z)$ that

$$\theta'_0(0) = \theta'_0(\pi) = 0, \quad \theta_0(0) = \gamma.$$

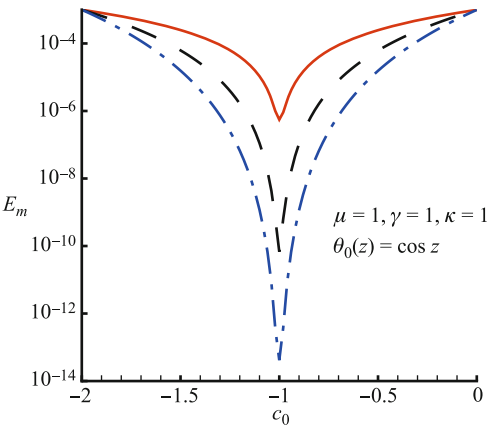
There are many functions satisfying the above conditions, such as

$$\theta_0(z) = \gamma \cos(\kappa z) \quad (9.33)$$

and

$$\theta_0(z) = \sigma - (\sigma - \gamma) \cos(\kappa z), \quad (9.34)$$

Fig. 9.2 Squared residual of governing equation versus c_0 when $\mu = 1, \gamma = 1, \kappa = 1$ and $\theta_0(z) = \cos z$. Solid line: 1st-order approximation; Dashed line: 2nd-order approximation; Dash-dotted line: 3rd-order approximation.



where σ is a real number, which provides us one additional degree of freedom in the initial guess. Note that the initial guess (9.33) is a special case of (9.34) when $\sigma = 0$. We call σ the *multiple-solution-control parameter*, because it provides us a convenient way to search for multiple eigenfunctions, as shown later.

This nonlinear eigenvalue problem is solved by means of the BVPh 1.0. Without loss of generality, we first consider the case of $\gamma = 1$ and $\kappa = 1$. Using the initial guess (9.33) with $\kappa = 1$, the curves of the squared residual of the governing equation versus the convergence-control parameter c_0 at up-to 3rd-order approximation are as shown in Fig. 9.2, which indicates that the optimal convergence-control parameter c_0^* is close to -1 . Using $c_0 = -1$, even the 3rd-order homotopy-approximation without iteration is quite accurate: the corresponding squared residual is only 4.1×10^{-14} . At the 2nd iteration by means of the 3rd-order iteration formula, the squared residual decreases to 1.3×10^{-26} , which is so small that more iterations are unnecessary. In this case, the eigenvalue λ converges to 1.137069 rather quickly, as shown in Table 9.1.

Table 9.1 First-type eigenvalue and squared residual given by the 3rd-order iteration approach with $c_0 = -1$ when $\gamma = 1, \kappa = 1$ and $\theta_0(z) = \cos z$.

Number of iteration m	λ	Squared residual E_m
1	1.137069	4.1×10^{-14}
2	1.137069	1.3×10^{-26}

Noticing that the initial guess (9.33) is a special case of (9.34) when $\sigma = 0$, we attempt the initial guess (9.34) with different values of σ , such as $\sigma = 2, 3, 4$ and so on. In a surprise, it is found that a new type of solution can be obtained when $\sigma = 2$ and $\sigma = 3$ by choosing proper convergence-control parameter c_0 . For example, in case of $\sigma = 3$, the curves of the squared residual versus c_0 at up-to the 3rd-order of homotopy-approximation are as shown in Fig. 9.3, which indicates that

Fig. 9.3 Squared residual of governing equation versus c_0 when $\mu = 1, \gamma = 1, \kappa = 1$ and $\theta_0(z) = 3 - 2\cos z$. Solid line: 1st-order approximation; Dashed line: 2nd-order approximation; Dash-dotted line: 3rd-order approximation.

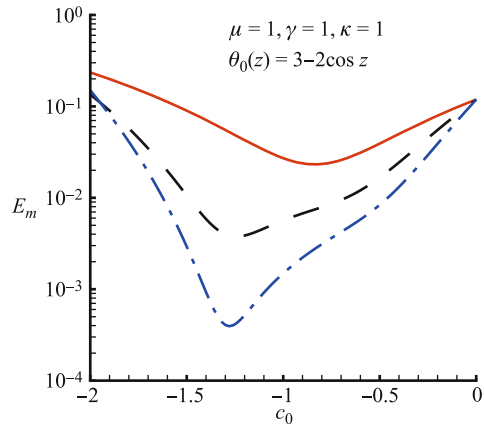
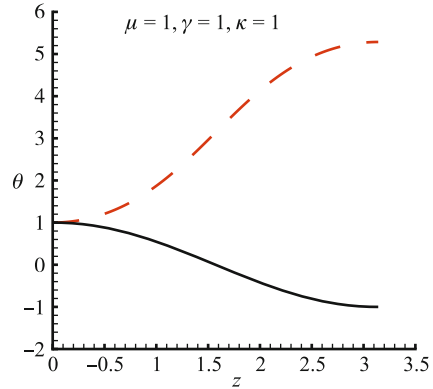


Fig. 9.4 Two different eigenfunctions when $\mu = 1, \gamma = 1$ and $\kappa = 1$. Solid line: the 1st eigenfunction related to $\lambda = 1.137069$ given by $\theta_0(z) = \cos z$; Dashed line: the 2nd eigenfunction related to $\lambda = -1.951368$ given by $\theta_0(z) = 3 - 2\cos z$.



the optimal convergence-control parameter c_0 is about -1.25 . Indeed, we obtain the convergent approximation of the eigenvalue $\lambda = -1.951368$ by means of the 3rd-order HAM iteration approach with $c_0 = -1$, and the corresponding squared residual decreases quickly, as shown in Table 9.2. Note that we introduce in Chapter 8 a similar parameter in initial guesses so as to find out multiple solutions of nonlinear ODEs in $z \in [0, a]$. This reveals the reasons why we call it *multiple-solution-control parameter*.

Therefore, in case of $\mu = 1, \gamma = 1$ and $\kappa = 1$, there exist two different types of eigenvalues: one positive eigenvalue $\lambda = 1.137069$ and one negative eigenvalue $\lambda = -1.951368$, corresponding to the two different types of eigenfunctions $\theta(z)$, as shown in Fig. 9.4. They give completely different displacements, as shown in Fig. 9.5, where x and u denote the horizontal and vertical displacement of the beam, defined by

$$x(z) = \int_0^z \cos \theta(s) ds, \quad u(z) = \int_0^z \sin \theta(s) ds,$$

Table 9.2 Second-type eigenvalue and squared residual given by the 3rd-order iteration approach with $c_0 = -1$ when $\gamma = 1$, $\kappa = 1$ and $\theta_0(z) = 3 - 2\cos z$.

Number of iteration m	λ	Squared residual E_m
1	-1.922288	1.5×10^{-3}
2	-1.954534	4.6×10^{-6}
3	-1.951247	3.6×10^{-8}
4	-1.951353	1.2×10^{-9}
5	-1.951371	9.7×10^{-12}
6	-1.951368	2.2×10^{-14}
7	-1.951368	5.0×10^{-16}
8	-1.951368	1.2×10^{-17}
9	-1.951368	2.0×10^{-18}
10	-1.951368	1.9×10^{-18}

Fig. 9.5 Displacement given by two different eigenfunctions when $\mu = 1, \gamma = 1$ and $\kappa = 1$. Solid line: displacement corresponding to $\lambda = 1.137069$ given by $\theta_0(z) = \cos z$; Dashed line: displacement corresponding to $\lambda = -1.951368$ given by $\theta_0(z) = 3 - 2\cos z$.

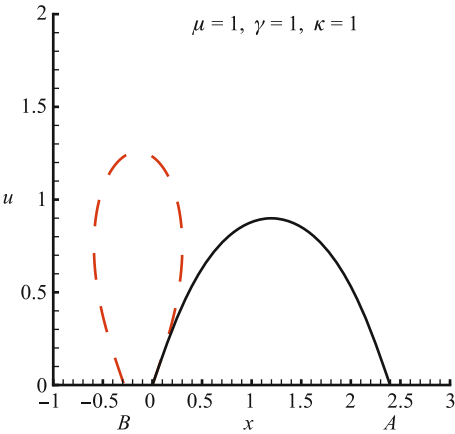
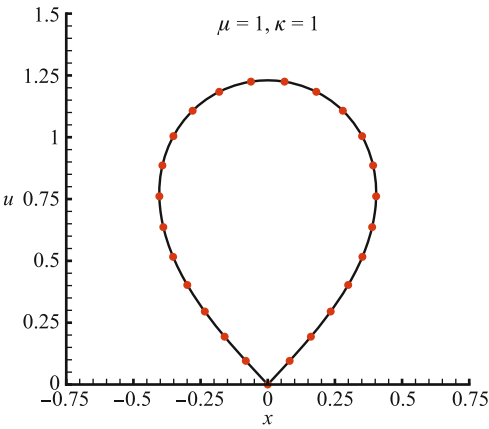


Fig. 9.6 Displacement given by two different types of eigenfunctions when $\mu = 1$ and $\kappa = 1$. Solid line: displacement corresponding to the 1st-type eigenfunction with the positive eigenvalue $\lambda^* = 2.183380$ given by $\gamma^* = 2.281319$; Symbols: displacement corresponding to the 2nd-type eigenfunction with the negative eigenvalue $\tilde{\lambda} = -2.183380$ given by $\tilde{\gamma} = 0.860274$.



respectively. For other different values of $\gamma = \theta(0)$, we can obtain the two different types of eigenfunctions and eigenvalues in a similar way by means of the BVPh 1.0.

In the above expressions about $x(z)$ and $u(z)$, we regard one end of the beam at $z = 0$ as a fixed point, i.e. $x(0) = 0$ and $u(0) = 0$. Let $(x_A, 0)$ and $(x_B, 0)$ denote the position of the other end of the beam, corresponding to the 1st and 2nd-type eigenfunction, respectively, as shown in Fig. 9.5. Obviously, x_A and x_B depend on the value of γ . It is found that, for the 1st-type eigenfunction, as γ increases from 0 to π , x_A decreases monotonously from $x_A = \pi$. For the 2nd-type eigenfunction, as γ decreases from π to 0, x_B increases monotonously from $x_B = -\pi$. Especially, it is found that, for the 1st-type eigenfunction, $x_A = -1.4 \times 10^{-6}$ when $\gamma = 2.281319 = \gamma^*$, with the corresponding eigenvalue $\lambda = 2.183380 = \lambda^*$. For the 2nd-type eigenfunction, $x_B = 7.2 \times 10^{-7}$ when $\gamma = 0.860274 = \tilde{\gamma}$, with the corresponding eigenvalue $\lambda = -2.183380 = \tilde{\lambda}$. It is interesting that

$$\lambda^* = -\tilde{\lambda}, \quad \gamma^* + \tilde{\gamma} \approx \pi, \quad (9.35)$$

and besides, the displacements given by these two different type eigenvalues are almost the same, as shown in Fig. 9.6.

In general case, let λ^* denote the eigenvalue of the 1st-type eigenfunction when $\theta(0) = \gamma^*$, and $\tilde{\lambda}$ the eigenvalue of the 2nd-type eigenfunction when $\theta(0) = \tilde{\gamma}$, respectively. It is found that, as long as

$$\gamma^* + \tilde{\gamma} = \pi,$$

it holds

$$\lambda^* = -\tilde{\lambda},$$

and the displacements corresponding to the two different eigenfunctions are symmetric, as shown in Fig. 9.7 and Fig. 9.8. So, the relationship (9.35) holds in general.

According to this kind of symmetry, we choose the initial guess

$$\theta_0(z) = \gamma \cos(\kappa z)$$

for the 1st-type eigenfunction, and the initial guess

$$\theta_0(z) = \pi - (\pi - \gamma) \cos(\kappa z)$$

for the 2nd-type eigenfunction, corresponding to $\sigma = 0$ and $\sigma = \pi$ in (9.34), respectively. By means of the above initial guess and the 3rd-order HAM iteration approach, we gain the curves of the eigenvalue versus γ (when $\mu = 1$ and $\kappa = 1$) by means of the BVPh 1.0, as shown in Fig. 9.9.

In addition, the above symmetry of the eigenfunctions and eigenvalues can be proved mathematically. Assume that the eigenfunction $\theta(z)$ and the eigenvalue λ satisfy

$$\theta'' + \lambda \sin \theta = 0, \quad \theta'(0) = \theta'(\pi) = 0.$$

Write $\tilde{\theta}(z) = \pi - \theta(z)$ and $\tilde{\lambda} = -\lambda$. Then, we have

Fig. 9.7 Displacement given by two different types of eigenfunctions when $\mu = 1$ and $\kappa = 1$. Solid line: displacement corresponding to the 1st-type eigenfunction with the eigenvalue $\lambda^* = 1.951368$ given by $\gamma^* = \pi - 1$; Dashed line: displacement corresponding to the 2nd-type eigenfunction with the eigenvalue $\tilde{\lambda} = -1.951368$ given by $\tilde{\gamma} = 1$.

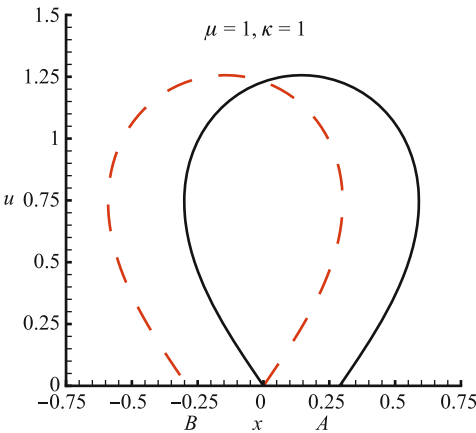


Fig. 9.8 Displacement given by two different types of eigenfunctions when $\mu = 1$ and $\kappa = 1$. Solid line: displacement corresponding to the 1st-type eigenfunction with the eigenvalue $\lambda^* = 3.202901$ given by $\gamma^* = \pi - 1/2$; Dashed line: displacement corresponding to the 2nd-type eigenfunction with the eigenvalue $\tilde{\lambda} = -3.202901$ given by $\tilde{\gamma} = 1/2$.

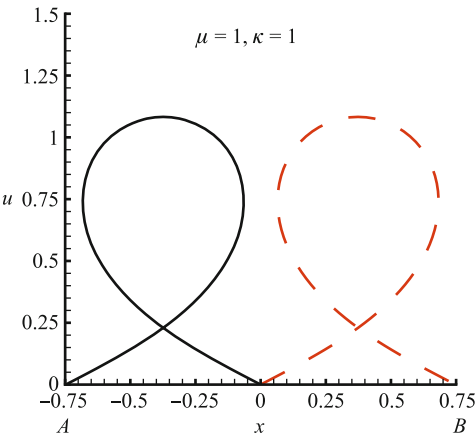
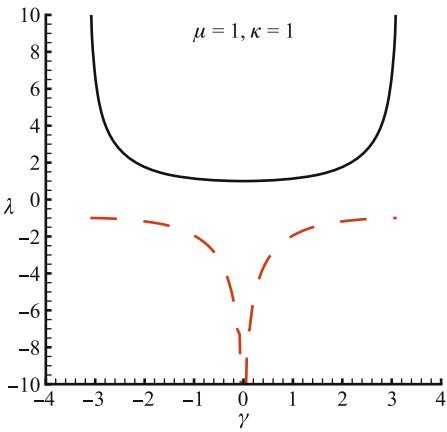


Fig. 9.9 Eigenvalue versus $\gamma = \theta(0)$ when $\mu = 1$ and $\kappa = 1$. Solid line: positive eigenvalue corresponding to the 1st-type eigenfunction; Dashed line: negative eigenvalue corresponding to the 2nd-type eigenfunction.



$$\tilde{\theta}'' + \tilde{\lambda} \sin \tilde{\theta} = -\theta'' - \lambda \sin(\pi - \theta) = -(\theta'' + \lambda \sin \theta) = 0$$

and

$$\tilde{\theta}'(0) = -\theta'(0) = 0, \quad \tilde{\theta}'(\pi) = -\theta'(\pi) = 0.$$

Besides, it holds

$$\tilde{x}(z) = \int_0^z \cos \tilde{\theta}(z) dz = \int_0^z \cos(\pi - \theta) dz = - \int_0^z \cos \theta = -x(z)$$

and

$$\tilde{u}(z) = \int_0^z \sin \tilde{\theta}(z) dz = \int_0^z \sin(\pi - \theta) dz = \int_0^z \sin \theta = u(z),$$

say, the displacements corresponding to the two different type eigenfunctions are symmetric. Therefore, we have the following theorem:

Theorem 9.1. *If the eigenfunction $\theta(z)$ and the eigenvalue λ satisfy*

$$\theta'' + \lambda \sin \theta = 0, \quad \theta'(0) = \theta'(\pi) = 0,$$

then the eigenfunction $\pi - \theta(z)$ and the eigenvalue $-\lambda$ also satisfy the same equation.

Such kind of symmetry holds even for a non-uniform beam with arbitrary $\mu(z)$, as described by the following theorem:

Theorem 9.2. *If the eigenfunction $\theta(z)$ and the eigenvalue λ satisfy*

$$\mu(z)\theta''(z) + \mu'(z)\theta'(z) + \lambda \sin \theta = 0, \quad \theta'(0) = \theta'(\pi) = 0,$$

then the eigenfunction $\pi - \theta(z)$ and the eigenvalue $-\lambda$ also satisfy the same equation.

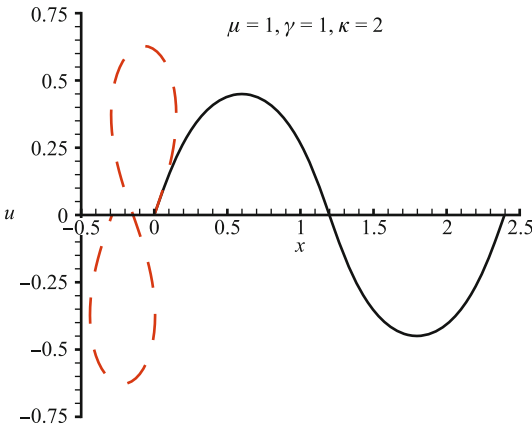
Mathematically, this symmetry is indeed true. However, the 2nd-type eigenfunction with the negative eigenvalue make us confused in physics, because the negative eigenvalue corresponds to a tensile force! It is a well-known knowledge in mechanics that a beam acted by a compressive force ($P > 0$) may have great deflection if the compressive force is greater than a critical value. But, it is traditionally believed that such kind of deflection never happens for a beam acted by a tensile force ($P < 0$). However, from the mathematical points of view, there indeed exists the 2nd-type eigenfunction with the negative eigenvalue, as proved above. What is the physical meaning of the 2nd-type eigenfunction with the negative eigenvalue?

Let us consider again the case of $\mu = 1$, $\gamma = 1$ and $\kappa = 1$. The two corresponding eigenfunctions are as shown in Fig. 9.4, and their displacements are as shown in Fig. 9.5, respectively. Assume that $\theta(z) = 0$ before the axial force P is suddenly acted at $t = 0$. Then, for a compressive force greater than the critical value, the beam deflects in such a “natural” way that the end of beam at $z = \pi$ begins to move

Table 9.3 Multiple eigenvalues of the uniform beam when $\gamma = 1$ and $\kappa > 1$.

κ	Positive eigenvalue	Negative eigenvalue
2	4.548275	-7.805465
3	10.233617	-17.562313

Fig. 9.10 Displacement given by different eigenfunctions when $\mu = 1, \gamma = 1$ and $\kappa = 2$. Solid line: displacement corresponding to the positive eigenvalue $\lambda = 4.548275$ given by $\theta_0(z) = \cos(2z)$; Dashed line: displacement corresponding to the negative eigenvalue $\lambda = -7.805465$ given by $\theta_0(z) = \pi - (\pi - 1)\cos(2z)$.



from the position $x = \pi$ to the left until $x = x_A$. This phenomena has been observed in physical experiments reported in textbooks. However, for a tensile force greater than the critical value, the end point of the beam at $z = \pi$ must be first *suddenly* moved to the left-hand side of the other end of the beam at $z = 0$ and then moves to the position $x = x_B$. In physics, this kind of sudden deformation is “unnatural” and needs much more energy than the “natural” ones, and therefore hardly happens in practice. Even so, the 2nd-type eigenfunction with the negative eigenvalue has still physical meanings. Thus, when a beam is acted by a large enough tensile force, the beam may also have a great deflection in theory, although this phenomena is hardly observed in practice without a sudden, hugh external disturbance.

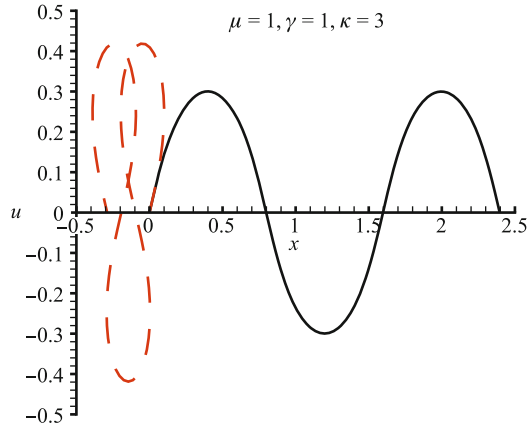
Note that $\sin \theta \approx \theta$ for small θ . However, the linearized equation

$$\theta'' + \lambda \theta = 0, \quad \theta'(0) = \theta'(\pi) = 0$$

has no such kind of symmetry, because its eigenvalues are always positive. This illustrates that we might lose a lots of solutions by linearizing a nonlinear equation. In other words, a nonlinear equation may have more interesting properties than its linearized one.

For other values of κ , we obtain the convergent eigenfunctions and eigenvalues in a similar way by means of the BVPh 1.0. For example, in case of $\gamma = 1$, we obtain the convergent positive and negative eigenvalues when $\kappa = 2$ and $\kappa = 3$, respectively, as listed in Table 9.3. The corresponding eigenfunctions are as shown in Fig. 9.10 and Fig. 9.11, respectively. In theory, given $\gamma = \theta(0)$, there exist one positive eigenvalue and one negative eigenvalue for each $\kappa \geq 1$. Therefore, there

Fig. 9.11 Displacement given by different eigenfunctions when $\mu = 1, \gamma = 1$ and $\kappa = 3$. Solid line: displacement corresponding to the positive eigenvalue $\lambda = 10.233617$ given by $\theta_0(z) = \cos(3z)$; Dashed line: displacement corresponding to the negative eigenvalue $\lambda = -17.562313$ given by $\theta_0(z) = \pi - (\pi - 1)\cos(3z)$.



exist an infinite number of eigenfunctions and eigenvalues for a given $\gamma = \theta(0)$. All of these eigenfunctions and eigenvalue can be found by means of the BVPh 1.0 in a similar way.

Therefore, using the BVPh 1.0, the multiple solutions of nonlinear eigenvalue equations in a finite interval $z \in [0, a]$ can be found out by means of different initial guesses and/or different auxiliary linear operators, as illustrated above. Note that, the so-called multiple-solution-control parameter σ provides us a convenient way to search for multiple eigenfunctions and eigenvalues. Similarly, we can also regard the positive integer κ in the auxiliary linear operator (9.18) as a kind of multiple-solution-control parameter, too.

It should be emphasized that, to the best of author's knowledge, the "unnatural" eigenfunctions of (9.32) corresponding to the negative eigenvalues have been never reported, although it happens hardly in practice. This shows the validity and potential of the BVPh 1.0 for nonlinear eigenvalue equations with multiple solutions in a finite interval $z \in [0, a]$.

9.3.1.2 Non-uniform beam

To show the general validity of the BVPh 1.0 for nonlinear eigenvalue problems, we further consider a beam with non-uniform distribution of the inertia moment

$$\mu(z) = 1 + \frac{\cos(4z)}{2\sqrt{1 + \exp(z^2) + \sin z^2}}. \quad (9.36)$$

Note that its averaged moment of inertia reads

$$\frac{1}{\pi} \int_0^\pi \left[1 + \frac{\cos(4z)}{2\sqrt{1 + \exp(z^2) + \sin z^2}} \right] dz \approx 1.002,$$

which is rather close to that of the uniform beam $\mu(z) = 1$.

In this case, the governing equation (9.26) contains varying coefficients $\mu(z)$, $\mu'(z)$ and thus becomes much more complicated than (9.32) for a uniform beam. Even so, by means of the BVPh 1.0, we still choose the same auxiliary linear operator (9.18) as that for the uniform beam. Besides, the same initial guess $\theta_0(z) = \gamma \cos(\kappa z)$ and $\theta_0(z) = \pi - (\pi - \gamma) \cos(\kappa z)$ are used to gain the two different types of eigenfunctions corresponding to positive and negative eigenvalues, respectively.

Without loss of generality, let us consider the case of $\gamma = 1$ and $\kappa = 1$. Two types of convergent eigenfunctions are obtained by $c_0 = -1$. The 1st-type eigenfunction corresponds to a positive eigenvalue $\lambda = 1.1061$, and the 2nd-type to a negative eigenvalue $\lambda = -1.8158$, respectively. As proved above, for arbitrary $\mu(z)$ and given $\gamma = \theta(0)$, there exist two types of eigenfunctions with positive and negative eigenvalue. All of them can be gained by means of the BVPh 1.0 in a similar way. This illustrates the validity and generality of the BVPh 1.0 for complicated eigenvalue equations with highly nonlinearity.

9.3.2 Gelfand equation

Let us further consider the Gelfand equation (Boyd, 1986; Jacobsen and Schmitt, 2002; McGough, 1998)

$$u'' + (K - 1) \frac{u'}{z} + \lambda e^u = 0, \quad u(0) = 0, \quad u(1) = 0, \quad (9.37)$$

where the prime denotes the differentiation with respect to z , $K \geq 1$ is a constant, $u(z)$ and λ denote eigenfunction and eigenvalue, respectively.

It is a special case of (9.1) when

$$\mathcal{F}[z, u, \lambda] = u'' + (K - 1) \frac{u'}{z} + \lambda e^u.$$

Note that (9.37) is highly nonlinear, since it contains the exponential term $\exp(u)$. Besides, it contains a singularity at $z = 0$ due to the term $u'(z)/z$. This kind of singularity results in difficulty to numerical techniques such as the shooting method used in BVPh 1.0, although the limit of $u'(z)/z$ as $z \rightarrow 0$ is a constant. However, this kind of singularity can be easily resolved by the BVPh 1.0, because the computer algebra system Mathematica provides us the ability to “compute with functions instead of numbers” (Trefethen, 2007).

Since different non-zero eigenfunctions have different values of $u'(0)$, we add an additional boundary condition

$$u'(0) = A, \quad (9.38)$$

where A is a given constant, so as to distinguish different eigenfunctions.

The M th-order approximations of $u(z)$ and λ are given by (9.12), where $u_m(z)$ and λ_{m-1} are determined by the m th-order deformation equation

$$\mathcal{L}[u_m(z) - \chi_m u_{m-1}(z)] = c_0 \delta_{m-1}(z), \quad (9.39)$$

subject to the boundary conditions

$$u_m(0) = 0, \quad u_m(1) = 0, \quad u'_m(0) = 0, \quad (9.40)$$

where

$$\delta_k(z) = u''_k(z) + (K-1) \frac{u'_k(z)}{z} + \sum_{i=0}^k \lambda_{k-i} \mathcal{D}_i \{\exp[\phi(x;q)]\}, \quad (9.41)$$

is given by Theorem 4.1, and the term $\mathcal{D}_i \{\exp[\phi(x;q)]\}$ is gained by a recursion formula described by Theorem 4.7 and Theorem 4.10.

The eigenfunction $u(z)$ is expressed by the hybrid-base of trigonometric functions and polynomial, described in Sect. 7.2.3. Although (9.37) is quite different from (9.26), we still choose the same auxiliary linear operator (9.18) with $\kappa = 1$. Besides, to satisfy the two original boundary conditions in (9.37) and the additional boundary condition (9.38), we choose the initial guess

$$u_0(z) = \frac{A}{2} [1 + \cos(\pi z)]. \quad (9.42)$$

This kind of nonlinear eigenvalue equation is solved by means of the BVPh 1.0. For given A , the optimal value of the convergence-control parameter c_0 is found by the minimum of the squared residual of the governing equation (9.37), defined by (9.25).

Without loss of generality, let us first consider the case of $K = 1$ and $A = 1$. The squared residual of governing equation at the up-to 3rd-order of homotopy-approximation versus c_0 are as shown in Fig. 9.12, which indicates that the optimal convergence-control parameter is near -0.55 . Indeed, we gain the fast convergent eigenfunction and eigenvalue $\lambda = 0.866215$ by means of $c_0 = -3/5$ and the 3rd-order iteration approach with $N_t = 20$, as shown in Table 9.4, which agrees well with the exact eigenvalue $\lambda = 0.866215$ given by the closed-form solution (Jacobsen and Schmitt, 2002)

$$u(z) = \frac{1}{2} e^{-A} \left\{ \ln \left[2e^A + 2\sqrt{e^A(e^A - 1)} - 1 \right] \right\}^2 \quad (9.43)$$

for $K = 1$. Given other values of A , we gain convergent eigenvalue and eigenfunction in a similar way by means of the BVPh 1.0, which agree well with the above exact formula, as shown in Fig. 9.13.

In case of $K = 2$, using the BVPh 1.0, we also gain convergent eigenfunctions and eigenvalues by the 3rd-order HAM iteration approach in a similar way, as shown in Table 9.5 and Fig. 9.13. Similarly, in case of $A = 1$ and $A = 2$, we also obtain convergent eigenfunctions and eigenvalues for different values of K , as shown in Fig. 9.14 and Table 9.6.

Fig. 9.12 Squared residual of Gelfand equation (9.37) versus c_0 when $K = 1$ and $A = 1$. Solid line: 1st-order approximation; Dashed line: 2nd-order approximation; Dash-dotted line: 3rd-order approximation.

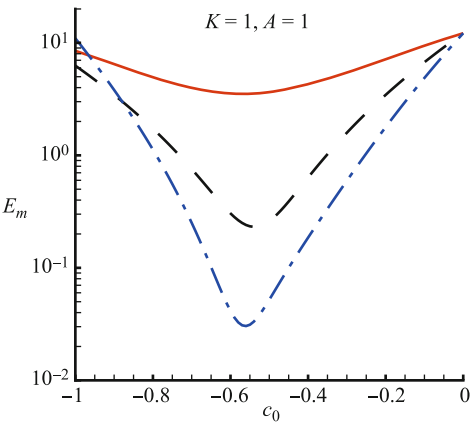


Fig. 9.13 Eigenvalue versus A of Gelfand equation (9.37) when $K = 1$ and $K = 2$. Solid line: $K = 1$; Dashed line: $K = 2$. Filled circles: exact solution (9.43).

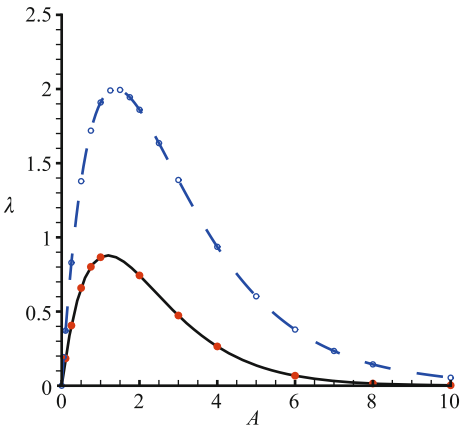


Fig. 9.14 Eigenvalue versus K of Gelfand equation (9.37) when $A = 1$ and $A = 2$. Solid line: $A = 1$; Dashed line: $A = 2$.

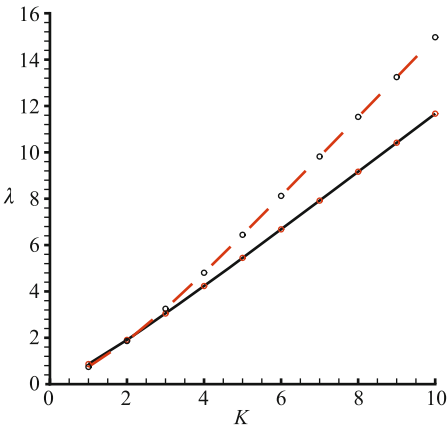


Table 9.4 Eigenvalue and squared residual of Gelfand equation (9.37) when $K = 1$ and $A = 1$ by means of $c_0 = -3/5$ and $N_l = 20$.

Number of iteration m	Eigenvalue λ	Squared residual E_m
1	0.779831	4.1×10^{-2}
2	0.866491	3.5×10^{-5}
3	0.866221	1.1×10^{-7}
4	0.866215	4.3×10^{-9}
5	0.866215	4.1×10^{-9}
6	0.866215	4.1×10^{-9}

Table 9.5 Eigenvalue of Gelfand equation (9.37) when $K = 2$.

A	Eigenvalue λ	A	Eigenvalue λ
0.05	0.192644	2	1.860353
0.10	0.371137	2.5	1.635358
0.25	0.829569	3	1.386745
0.50	1.378161	4	0.936157
0.75	1.719382	5	0.602776
1.00	1.909210	6	0.378467
1.25	1.990053	7	0.234285
1.50	1.993891	8	0.1439
1.75	1.944705	10	0.0535

Table 9.6 Eigenvalue of Gelfand equation (9.37) when $A = 1$ and $A = 2$.

K	Eigenvalue λ when $A = 1$	Eigenvalue λ when $A = 2$
1	0.8662	0.7436
2	1.9092	1.8604
3	3.0460	3.2522
4	4.2328	4.8059
5	5.4471	6.4428
6	6.6775	8.1209
7	7.9177	9.8198
8	9.1642	11.5299
9	10.4149	13.2462
10	11.6684	14.9663

The singularity term $u'(z)/z$ as $z \rightarrow 0$ is easy to resolve by means of computer algebra system like Mathematica, since it regards z as a function instead of a number, and besides the limit of $u'(z)/z$ as $z \rightarrow 0$ is a constant.

This example confirms the validity and generality of the BVPh 1.0 for highly nonlinear eigenvalue equations with singularity, since (9.37) contains the exponential term $\exp(u)$ and the singularity term $u'(z)/z$.

9.3.3 Equation with singularity and varying coefficient

Note that the BVP4c 1.0 is valid for the n th-order nonlinear eigenvalue equation (9.1) subject to the n linear boundary conditions (9.2), which are rather general in form. To show its general validity, we solve here a nonlinear eigenvalue equation with varying coefficients defined in a finite interval $0 < z < \pi$:

$$\sqrt{1+z^2} u'' + \frac{\cos(\pi z) u'}{z} + \lambda \left[\frac{e^u}{1+z^2} + (1+z) \sin u \right] = \sin(z^2 + e^{-z}), \quad (9.44)$$

subject to the two boundary conditions

$$u'(0) = 0, \quad u(\pi) - u'(\pi) = \frac{3}{5}, \quad (9.45)$$

where the prime denotes the differentiation with respect to z , $u(z)$ and λ are the unknown eigenfunction and eigenvalue, respectively.

The above problem is a special case of (9.1) when

$$\mathcal{F}[z, u, \lambda] = \sqrt{1+z^2} u'' + \frac{\cos(\pi z) u'}{z} + \lambda \left[\frac{e^u}{1+z^2} + (1+z) \sin u \right] - \sin(z^2 + e^{-z}).$$

Especially, it contains the varying coefficients

$$\sqrt{1+z^2}, \quad \frac{\cos(\pi z)}{z}, \quad \frac{1}{1+z^2}, \quad (1+z), \quad -\sin(z^2 + e^{-z})$$

and the highly nonlinear terms $\exp(u)$ and $\sin u$. In addition, it has a singularity at $z = 0$ due to the term $u'(z)/z$. Such kind of singularity leads to difficulty to numerical techniques such as the shooting method used by BVP4c. Thus, this equation is rather complicated. Fortunately, the limit of $u'(z)/z$ as $z \rightarrow 0$ is a constant and thus can be easily resolved by computer algebra system like Mathematica in the frame of the HAM, which regards z as a function instead of a number.

For a non-zero eigenfunction $u(z)$, it holds $u(0) \neq 0$. So, we use

$$u(0) = A \quad (9.46)$$

as the additional boundary condition to distinguish different non-zero eigenfunctions.

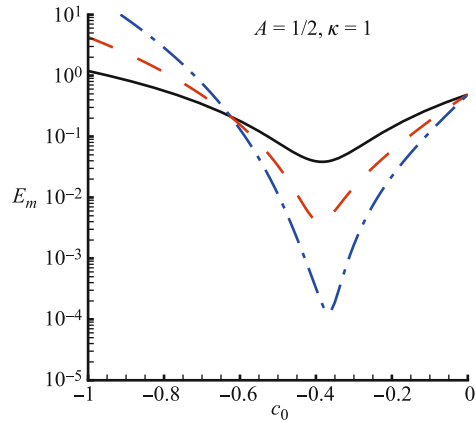
The M th-order approximations of $u(z)$ and λ are given by (9.12), where $u_m(z)$ and λ_{m-1} are determined by the m th-order deformation equation

$$\mathcal{L}[u_m(z) - \chi_m u_{m-1}(z)] = c_0 \delta_{m-1}(z) \quad (9.47)$$

subject to the boundary conditions

$$u'_m(0) = 0, \quad u_m(\pi) - u'_m(\pi) = 0, \quad (9.48)$$

Fig. 9.15 Squared residual of (9.44) when $A = 1/2$ versus c_0 . Solid line: 1st-order approximation; Dashed line: 2nd-order approximation; Dash-dotted line: 3rd-order approximation.



where

$$\begin{aligned} \delta_k(z) = & \sqrt{1+z^2} u_k'' + \frac{\cos(\pi z) u_k'}{z} - (1 - \chi_{k+1}) \sin(z^2 + e^{-z}) \\ & + \sum_{i=0}^k \lambda_{k-i} \left[\frac{\mathcal{D}_i \{\exp[\phi(z; q)]\}}{1+z^2} + (1+z) \mathcal{D}_i \{\sin[\phi(z; q)]\} \right] \end{aligned} \quad (9.49)$$

is given by Theorem 4.1, and the terms $\mathcal{D}_i \{\exp[\phi(z; q)]\}$ and $\mathcal{D}_i \{\sin[\phi(z; q)]\}$ are given by recursion formulas described by Theorem 4.7 and Theorem 4.8, or Theorem 4.10, respectively.

We successfully solved this highly nonlinear eigenvalue problem with singularity by means of the BVPh 1.0. Since it is defined in a finite interval $z \in [0, \pi]$, we express the eigenfunction by the hybrid-base mentioned in Sect. 7.2.3. Thus, although (9.44) is more complicated than (9.26) and (9.37), we still use the same auxiliary linear operator (9.18) with $\kappa = 1$, and besides choose the initial guess

$$u_0(z) = \frac{5A+3}{10} + \frac{5A-3}{10} \cos z, \quad (9.50)$$

which satisfies the two original boundary conditions (9.45) and the additional boundary condition (9.46).

Without loss of generality, let us first consider the case of $A = 1/2$. The squared residuals of the governing equation (9.44) at up-to the 3rd-order approximation are as shown in Fig. 9.15, which indicates that the optimal convergence-control parameter is about -0.4 . Indeed, choosing $c_0 = -2/5$, we gain fast convergent eigenvalue and eigenfunction by the 3rd-order HAM iteration approach with $N_t = 30$, as shown in Table 9.7 and Fig. 9.16. Note that the squared residual of the governing equation (9.44) stops decreasing at $E_m = 5.3 \times 10^{-8}$ for $m \geq 4$. However, if more truncated terms (i.e. larger value of N_t) are used, smaller squared residuals are gained.

Table 9.7 The eigenvalue and squared residual given by the 3rd-order HAM iteration approach with $c_0 = -2/5$, $N_t = 30$ and the initial guess (9.50) when $A = 1/2$ and $\kappa = 1$.

Number of iteration m	Eigenvalue λ	Squared residual E_m
1	0.373901	3.3×10^{-4}
2	0.380270	5.4×10^{-7}
3	0.379978	5.6×10^{-8}
4	0.379957	5.3×10^{-8}
5	0.379956	5.3×10^{-8}
6	0.379956	5.3×10^{-8}
7	0.379956	5.3×10^{-8}
8	0.379956	5.3×10^{-8}

Fig. 9.16 Eigenfunction of (9.44) and (9.45) by means of $\kappa = 1$. Solid line: 10th-iteration approximation when $A = 1/2$ by means of $c_0 = -2/5$; Dashed line: 10th-iteration approximation when $A = 0$ by means of $c_0 = -2/5$; Dash-dotted line: 15th-iteration approximation when $A = -1/2$ by means of $c_0 = -1/5$. Symbols: the 3rd-iteration approximations ($A = 1/2$ and $A = 0$) or the 8th-iteration approximation ($A = -1/2$).

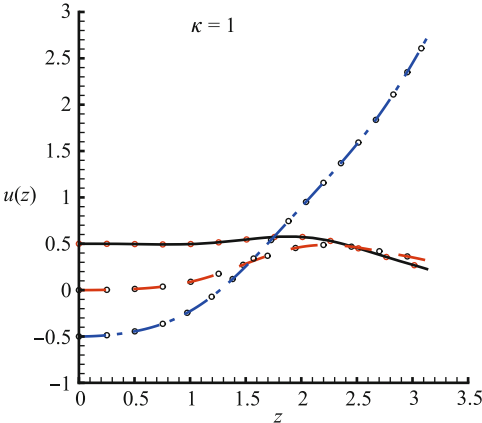
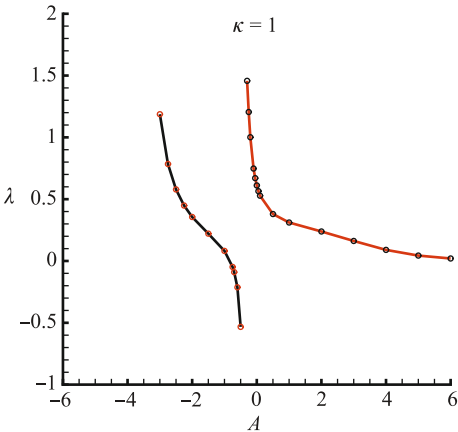


Fig. 9.17 Eigenvalue of (9.44) and (9.45) versus A by means of $\kappa = 1$.



Given other values of A , we gain convergent eigenvalues and eigenfunctions in a similar way by means of the BVPh 1.0. For example, the eigenfunctions when $A = 1/2, A = 0$ and $A = -1/2$ are as shown in Fig. 9.16. It is found that there is

no symmetry between the eigenfunctions for $A = 1/2$ and $A = -1/2$. Besides, the curves of the eigenvalue versus A by means of $\kappa = 1$ are as shown in Fig. 9.17. It is found that there are two branches of eigenvalues by means of $\kappa = 1$. The 1st branch of eigenvalues are positive, and tends to zero as $A \rightarrow +\infty$, as shown in Table 9.8. The 2nd branch of eigenvalues exists in a interval $c_0^L < c_0 < c_0^R$, where c_0^L is close to -3.2 , and $-0.49 < c_0^R < -0.33$. Different from the 1st branch of eigenvalues, the 2nd branch of eigenvalues are negative for some values of $u(0) = A$, as shown in Table 9.9. According to our computations, it seems that there exist no convergent eigenfunctions and eigenvalues when $c_0 < c_0^R$ by means of $\kappa = 1$. Figure 9.17 suggests that there might exist some kinds of singularity at $c_0 = c_0^L$ and $c_0 = c_0^R$.

All of the above results are gained by means of the auxiliary linear operator (9.18) with $\kappa = 1$ and the initial guess (9.50). As mentioned before, multiple solution can be found by means of different auxiliary linear operators and different initial guesses. Note that the initial guess (9.50) can be generalized by

$$u_0(z) = \frac{5A+3}{10} + \frac{5A-3}{10} \cos(\kappa z), \quad (9.51)$$

where $\kappa \geq 1$ is an odd integer. Using the above initial guess and the auxiliary linear operator (9.18) with $\kappa = 3$ and $\kappa = 5$, we gain the multiple eigenfunctions and eigenvalues $\lambda = 7.3500$ (when $\kappa = 3$) and $\lambda = 19.9043$ (when $\kappa = 5$) in case of $A = 1$ by means of $c_0 = -1/5$, as shown in Fig. 9.18 and Table 9.10. This sug-

Table 9.8 The 1st branch of eigenvalues of (9.44) by means of $\kappa = 1$.

A	Eigenvalue λ	A	Eigenvalue λ
-0.33	1.607740	0.05	0.565095
-0.32	1.559068	0.1	0.528119
-0.31	1.508751	0.5	0.379956
-0.30	1.457289	1	0.311333
-0.25	1.205111	2	0.238667
-0.20	1.001034	3	0.162146
-0.10	0.748010	4	0.089831
-0.05	0.670271	6	0.020306
0	0.611237	10	0.000797

Table 9.9 The 2nd branch of eigenvalues of (9.44) by means of $\kappa = 1$.

A	Eigenvalue λ	A	Eigenvalue λ
-3.2	2.356321	-2.0	0.355912
-3.1	1.517056	-1.5	0.221140
-3.0	1.187195	-1.0	0.081523
-2.9	0.984646	-0.7	-0.089925
-2.75	0.784556	-0.6	-0.212886
-2.5	0.578778	-0.5	-0.532902
-2.25	0.448467	-0.49	-0.634153

Fig. 9.18 Multiple eigenfunction of (9.44) and (9.45) when $A = 1$ gained by means of different values of κ . Solid line: 8th-iteration approximation by means of $\kappa = 1$ and $c_0 = -2/5$; Dashed line: 30th-iteration approximation by means of $\kappa = 3$ and $c_0 = -1/5$; Dash-dotted line: 30th-iteration approximation by means of $\kappa = 5$ and $c_0 = -1/5$. Symbols: 10th-iteration approximations ($\kappa = 3$ and $\kappa = 5$) or the 3rd-iteration approximation ($\kappa = 1$).

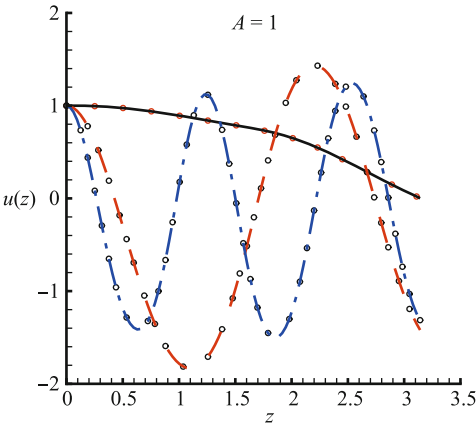


Table 9.10 Eigenvalues and the squared residual of (9.44) when $A = 1$ by means of $\kappa = 3$, $N_t = 40$, $c_0 = -1/5$ and $\kappa = 5$, $N_t = 50$, $c_0 = -1/5$.

m , Number of iteration	λ ($\kappa = 3$)	E_m ($\kappa = 3$)	λ ($\kappa = 5$)	E_m ($\kappa = 5$)
1	6.2210	18.96	13.4273	126.4
3	6.5234	1.57	18.7510	5.73
5	7.1834	0.25	19.6540	0.37
7	7.3451	2.1×10^{-2}	19.8553	2.1×10^{-2}
10	7.3555	1.6×10^{-4}	19.9021	1.7×10^{-4}
15	7.3500	7.1×10^{-7}	19.9044	9.1×10^{-7}
20	7.3500	6.3×10^{-7}	19.9043	8.4×10^{-7}
25	7.3500	6.3×10^{-7}	19.9043	8.4×10^{-7}

gests that the eigenvalue equation (9.44) with the boundary conditions (9.45) might have an infinite number of eigenfunctions and eigenvalues. Note that the multiple eigenfunctions and eigenvalues are gained simply by using different values of κ in the initial guess (9.51) and the auxiliary linear operator (9.18). Therefore, using trigonometric functions as base functions, we can find out multiple solutions of some nonlinear eigenvalue equations by means of different initial guesses and the different auxiliary linear operators in the frame of the HAM.

Note that the singularity term $u'(z)/z$ as $z \rightarrow 0$ is easily resolved by the BVPb 1.0, since the computer algebra system Mathematica regards z as a function instead of a number, and besides the limit of $u'(z)/z$ as $z \rightarrow 0$ is a constant. In addition, the multiple solutions are found out simply by means of different auxiliary linear operators (9.18) and different initial guesses (9.51) with different values of κ . Thus, we can also regard κ as a kind of multiple-solution-control parameter. Finally, the convergence-control parameter c_0 provides us a convenient way to guarantee the convergence of homotopy-solutions series for the highly nonlinear governing equation (9.44).

Therefore, this example confirms the general validity of the BVPh 1.0 for complicated eigenvalue problems with high nonlinearity and singularity, although (9.44) has no physical meanings.

9.3.4 Multipoint boundary-value problem with multiple solutions

Some nonlinear boundary-value problems with multiple solutions can be solved by transferring them into eigenvalue equations. To show the general validity of the BVPh 1.0, let us consider here a 4th-order nonlinear differential equation with multipoint boundary condition

$$u'''' = \beta z(1 + u^2), \quad u(0) = u'(1) = u''(1) = 0, \quad u''(0) - u''(\alpha) = 0, \quad (9.52)$$

where $\alpha \in (0, 1)$ and β are given constants. Graef et al. (2003, 2004) proved that the above equation has at least two positive solutions when $\alpha = 1/5$ and $\beta = 10$. Note that, different from the previous three examples, this boundary-value equation is 4th-order, and besides its boundary conditions are defined at three separate points, i.e. it is a so-called multipoint boundary-value problem.

Writing $u(z) = \lambda \theta(z)$ with the definition $\lambda = u(1)$, the original equation (9.52) becomes

$$\lambda \theta'''' = \beta z(1 + \lambda^2 \theta^2), \quad \theta(0) = \theta'(1) = \theta''(1) = 0, \quad \theta''(0) - \theta''(\alpha) = 0, \quad (9.53)$$

with one additional boundary condition

$$\theta(1) = 1. \quad (9.54)$$

Regarding λ as an unknown eigenvalue, (9.53) is a special case of (9.1) when

$$\mathcal{F}[z, \theta, \lambda] = \lambda \theta'''' - \beta z(1 + \lambda^2 \theta^2).$$

The M th-order approximations of $\theta(z)$ and λ are given by (9.12), where $\theta_m(z)$ and λ_{m-1} are determined by the m th-order deformation equation

$$\mathcal{L}[\theta_m(z) - \chi_m \theta_{m-1}(z)] = c_0 \delta_{m-1}(z), \quad (9.55)$$

subject to the multipoint boundary conditions

$$\theta_m(0) = \theta'_m(1) = \theta''_m(1) = 0, \quad \theta''_m(0) - \theta''_m(\alpha) = 0, \quad (9.56)$$

and the additional boundary condition

$$\theta_m(1) = 0, \quad (9.57)$$

where

$$\begin{aligned} \delta_k(z) = & \sum_{i=0}^k \lambda_{k-i} \theta_i'''' - (1 - \chi_{k+1}) \beta z \\ & - \beta z \sum_{i=0}^k \left(\sum_{j=0}^i \lambda_j \lambda_{i-j} \right) \left(\sum_{r=0}^{k-i} \theta_r \theta_{k-i-r} \right) \end{aligned} \quad (9.58)$$

is gained by Theorem 4.1 in Chapter 4.

This nonlinear eigenvalue problem with multipoint boundary conditions is solved by means of the BVPh 1.0. Since (9.53) contains the term βz , the polynomial of z is used to express the eigenfunction $\theta(z)$. Thus, we choose

$$\mathcal{L}(\theta) = \theta'''' \quad (9.59)$$

as the auxiliary linear operator and the polynomial

$$\theta_0(z) = \frac{1}{2\alpha - 3} [2(3\alpha - 4)z + 6(1 - \alpha)z^2 + 2\alpha z^3 - z^4], \quad (9.60)$$

as the initial guess, which satisfies the four original boundary conditions in (9.53) and the additional boundary condition (9.54).

Without loss of generality, let us first consider the case of $\alpha = 1/5$ and $\beta = 10$. It is found that, at the 1st-order of approximation, the additional boundary-condition (9.57), i.e. $\theta_1(1) = 0$, gives a nonlinear algebraic equation

$$\lambda_0^2 - 2.77904\lambda_0 + 1.35864 = 0, \quad (9.61)$$

which has two different solutions $\lambda_0 = 0.63313$ and $\lambda_0 = 2.14591$, respectively. Note that the above algebraic equation is independent of the convergence-control parameter c_0 . Choosing one solution of the above algebraic equation as the initial guess of the eigenvalue, we gain the m th-order homotopy-approximation of the eigenfunction $\theta(z)$ and the $(m - 1)$ th-order homotopy-approximation of the eigenvalue λ : both of them contain the convergence-control parameter c_0 . It is found that, when $\lambda_0 = 0.63313$, the squared residual E_m of the governing equation (9.53) decreases in an interval $c_0 \in (-3, 0)$ as m increases, and besides the optimal convergence-control parameter c_0 is about $-3/2$, as shown in Fig. 9.19. Indeed, by means of $\lambda_0 = 0.63313$ and $c_0 = -3/2$, the squared residual of (9.53) decreases monotonously to a rather small value 5.3×10^{-16} at the 10th-order homotopy-approximation, so that we gain the first eigenfunction $\theta(z)$ with the first eigenvalue $\lambda = 0.627315$, as shown in Table 9.11. Similarly, by means of $\lambda_0 = 2.14591$, it is found that the squared residual E_m decreases in an interval $c_0 \in (-0.7, 0)$ as m increases, and the optimal convergence-control parameter c_0 is about $-1/2$, as shown in Fig. 9.20. Indeed, by means of $c_0 = -1/2$ and $\lambda_0 = 2.14591$, the squared residual decreases to 1.7×10^{-16} at the 20th-order of approximation, and we obtain the 2nd eigenfunction with the 2nd eigenvalue $\lambda = 2.24118$, as shown in Table 9.12. It is found that, although the two eigenvalues are obviously different, the two corresponding eigenfunctions are

Fig. 9.19 Squared residual E_m of the governing equation (9.53) versus c_0 when $\alpha = 1/5$ and $\beta = 10$ by means of the first initial guess $\lambda_0 = 0.63313$ of the eigenvalue. Dashed line: 1st-order; Dash-dotted line: 3rd-order; Solid line: 5th-order.

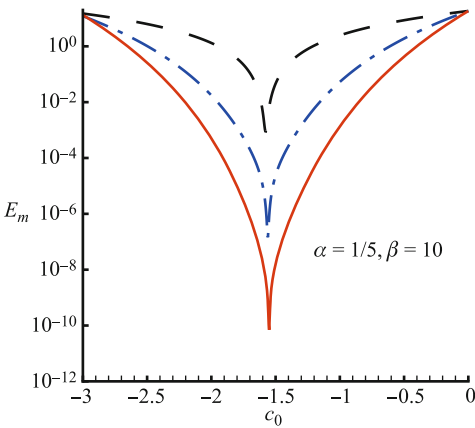


Fig. 9.20 Squared residual E_m of the governing equation (9.53) versus c_0 when $\alpha = 1/5$ and $\beta = 10$ by means of the 2nd initial guess $\lambda_0 = 2.14591$ of the eigenvalue. Dashed line: 1st-order; Dash-dotted line: 3rd-order; Solid line: 5th-order.

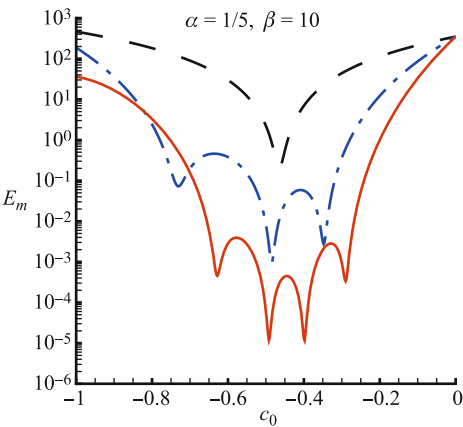


Table 9.11 The eigenvalue and squared residual E_m of (9.53) when $\alpha = 1/5$ and $\beta = 10$ by means of $\lambda_0 = 0.63313$ and $c_0 = -3/2$.

Order of approximation m	Eigenvalue λ	Squared residual E_m
2	0.627446	2.3×10^{-3}
4	0.627318	1.0×10^{-6}
6	0.627315	6.9×10^{-10}
8	0.627315	5.7×10^{-13}
10	0.627315	5.3×10^{-16}

rather close, as shown in Fig. 9.21. In general, it is rather difficult to distinguish such kind of very close eigenfunctions by means of numerical methods.

Note that, since $u(z) = \lambda \theta(z)$, we have two obviously different solutions $u(z)$ corresponding to the two different values of λ , as shown in Fig. 9.22.

Similarly, we can gain the two solutions of the nonlinear multipoint boundary-value equation (9.52) for different values of α and β . This illustrates that a nonlinear

Table 9.12 The eigenvalue and squared residual E_m of (9.53) when $\alpha = 1/5$ and $\beta = 10$ by means of $\lambda_0 = 2.14591$ and $c_0 = -1/2$.

Order of approximation m	Eigenvalue λ	Squared residual E_m
4	2.24105	6.9×10^{-3}
8	2.24118	1.6×10^{-6}
12	2.24118	6.3×10^{-10}
16	2.24118	3.2×10^{-13}
20	2.24118	1.7×10^{-16}

Fig. 9.21 Comparison of two eigenfunctions of (9.53) and (9.54) when $\alpha = 1/5$ and $\beta = 10$ by means of different initial guesses λ_0 of the eigenvalue. Solid line: the 1st eigenfunction $\theta(z)$ given by $\lambda_0 = 0.63313$ and $c_0 = -3/2$; Dashed line with open circles: the 2nd eigenfunction $\theta(z)$ given by $\lambda_0 = 2.14591$ and $c_0 = -1/2$.

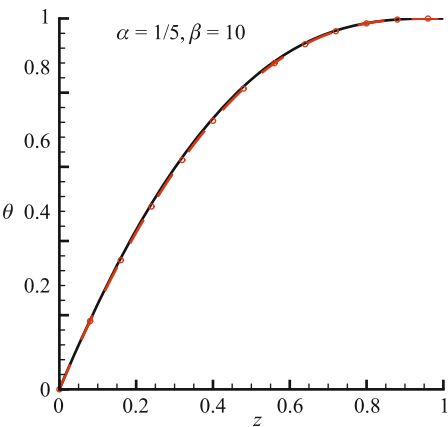
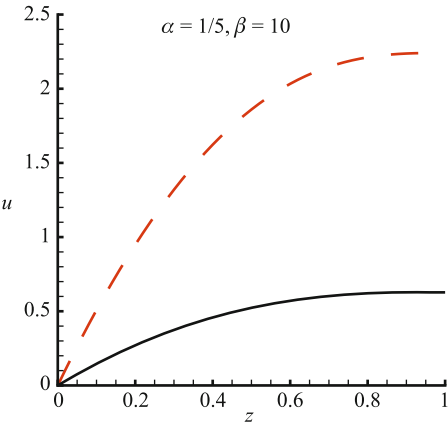


Fig. 9.22 Two solutions of the original equation (9.52) when $\alpha = 1/5$ and $\beta = 10$. Solid line: 1st solution $u(z)$ given by $\lambda_0 = 0.63313$ and $c_0 = -3/2$; Dashed line: 2nd solution $u(z)$ given by $\lambda_0 = 2.14591$ and $c_0 = -1/2$.



boundary-value problem with multiple solutions can be transferred into a nonlinear eigenvalue problem. Note that, as shown in Sect. 8.3.2, it can be directly solved by regarding (9.52) as a nonlinear boundary-value equation, too. Note also that (9.52) is a 4th-order nonlinear boundary-value equation, whose boundary conditions are satisfied at two endpoints and one seperated point in the interval $z \in (0, 1)$. Thus,

(9.52) can be solved by means of the BVPh 1.0, no matter we regard it either as a normal nonlinear boundary-value problem or a nonlinear eigenvalue problem.

Similarly as mentioned in Sect. 8.3.2, the multipoint boundary conditions can be easily resolved by the BVPh 1.0, since computer algebra system like Mathematica regards all of these boundary conditions in the same way. This is mainly because computer algebra system provides us the ability to “compute with functions instead of numbers” (Trefethen, 2007).

So, this example verifies the validity and generality of the BVPh 1.0 for high-order nonlinear eigenvalue problems with multipoint boundary conditions.

9.3.5 Orr-Sommerfeld stability equation with complex coefficient

As the final example, let us consider the famous Orr-Sommerfeld equation (Lin, 1955; Orszag, 1971) for the stability of plane Poiseuille flow

$$(D^2 - \alpha^2)^2 u - i(\alpha R) [(U_0 - \lambda)(D^2 - \alpha^2) - D^2 U_0] u = 0, \quad (9.62)$$

subject to the boundary conditions

$$u'(0) = u'''(0) = 0, \quad u(1) = 0, \quad u'(1) = 0, \quad (9.63)$$

where the prime denotes the differentiation with respect to z , $i = \sqrt{-1}$ is an imaginary number, the operator D is defined by $Du = u''$, R denotes the Reynolds number, λ is the complex eigenvalue, $U_0 = 1 - z^2$ is the exact solution of the plane Poiseuille flow, respectively. The two dimensional disturbance of velocity is proportional to

$$u(z) \exp[i\alpha(x - \lambda t)]$$

with α real and λ complex number. So, the flow becomes unstable when the imaginary part of λ is positive, i.e. $\text{Im}(\lambda) > 0$. For details, please refer to Lin (1955) and Orszag (1971).

This eigenvalue problem with complex coefficient is also a special case of (9.1) when

$$\mathcal{F}[z, u, \lambda] = (D^2 - \alpha^2)^2 u - i(\alpha R) [(U_0 - \lambda)(D^2 - \alpha^2) - D^2 U_0] u.$$

Note that, if $\bar{u}(z)$ is an eigenfunction, then $u(z) = \bar{u}(z)/\bar{u}(0)$ is also an eigenfunction for the same eigenvalue λ , since (9.62) is linear. So, we have the additional boundary condition

$$u(0) = 1.$$

The M th-order approximations of $u(z)$ and λ are given by (9.12), where $u_m(z)$ and λ_{m-1} are determined by the m th-order deformation equation

$$\mathcal{L}[u_m(z) - \chi_m u_{m-1}(z)] = c_0 \delta_{m-1}(z) \quad (9.64)$$

subject to the boundary conditions

$$u'_m(0) = u'''_m(0) = 0, \quad u_m(1) = 0, \quad u'_m(1) = 0, \quad (9.65)$$

and the additional boundary condition

$$u_m(0) = 0, \quad (9.66)$$

where c_0 is the convergence-control parameter, $\mathcal{L}$ is the auxiliary linear operator, $u_0(z)$ is the initial guess of $u(z)$, and

$$\begin{aligned} \delta_n(z) = & (D^2 - \alpha^2)^2 u_n(z) - i(\alpha R) [U_0 (D^2 - \alpha^2) - D^2 U_0] u_n(z) \\ & + i(\alpha R) \sum_{k=0}^n \lambda_k (D^2 - \alpha^2) u_{n-k}(z) \end{aligned} \quad (9.67)$$

is given by Theorem 4.1 in Chapter 4.

Let us consider here the eigenfunction $u(z)$ with the symmetry $u(z) = u(-z)$ only. Since $u(z)$ is defined in a finite interval $z \in [-1, 1]$, it can be expressed by polynomials of z . Therefore, we choose the auxiliary linear operator

$$\mathcal{L}(u) = u'''''. \quad (9.68)$$

Notice that the critical eigenvalue is often related to the simplest form of the eigenfunction, as shown in Sect. 9.3.1 for the first example. Thus, we choose the initial guess

$$u_0(z) = (1 - z^2)^2, \quad (9.69)$$

which is the simplest polynomial of z satisfying all boundary conditions (9.63) and the additional boundary condition $u(0) = 1$.

This problem is successfully solved by means of the BVPh 1.0. Without loss of generality, let us first consider the case of $\alpha = 1$ with different values of Reynolds number R . Note that (9.62) contains the imaginary number $i = \sqrt{-1}$. Fortunately, in the frame of the HAM, we have so great freedom to choose the convergence-control parameter c_0 that c_0 can be a complex number, as described in Theorem 5.3 and Theorem 5.4. Thus, we use here the 3rd-order HAM iteration formula with a complex convergence-control parameter c_0 . For example, in case of $R = 100$ and $\alpha = 1$, we gain the convergent eigenvalue

$$\lambda = 0.478494 - 0.162944i$$

by means of a complex convergence-control parameter $c_0 = (-1 + i)/2$, as shown in Table 9.13. This verifies that, in the frame of the HAM, the convergence-control parameter c_0 can be indeed a complex number. Furthermore, it is found that, for given R , one can always find a proper c_0 in such the form

$$c_0 = \frac{-1 + i}{\rho}, \quad \rho \geq 1,$$

Table 9.13 Eigenvalue and the squared residual of (9.62) in case of $R = 100$ and $\alpha = 1$ by means of the 3rd-order iteration formula with $c_0 = (-1 + i)/2$.

m , times of iteration	Eigenvalue λ	Squared residual E_m
1	$0.473522 - 0.158606 i$	2330
3	$0.478861 - 0.161875 i$	88.8
5	$0.478652 - 0.163091 i$	3.77
10	$0.478490 - 0.162945 i$	7.0×10^{-4}
15	$0.478494 - 0.162944 i$	9.3×10^{-8}
20	$0.478494 - 0.162944 i$	1.8×10^{-11}
25	$0.478494 - 0.162944 i$	6.2×10^{-15}
30	$0.478494 - 0.162944 i$	2.3×10^{-18}

Table 9.14 Convergent eigenvalues for different Reynolds number R when $\alpha = 1$ by means of the 3rd-order HAM iteration formula with the complex convergence-control parameter c_0 . The eigenfunction is expressed by a polynomial of z up-to $o(z^{N_i})$.

R	λ	c_0 with $i = \sqrt{-1}$	N_i
100	$0.478494 - 0.162944 i$	$(-1 + i)/2$	90
200	$0.430714 - 0.116810 i$	$(-1 + i)/4$	90
500	$0.380566 - 0.0704922 i$	$(-1 + i)/10$	90
1000	$0.346285 - 0.0421283 i$	$(-1 + i)/20$	150
2000	$0.312100 - 0.0197987 i$	$(-1 + i)/40$	200
3000	$0.292289 - 0.0101846 i$	$(-1 + i)/60$	300
5000	$0.268131 - 0.0017503 i$	$(-1 + i)/100$	300
5500	$0.263762 - 0.00060763 i$	$(-1 + i)/100$	300
5800	$0.261348 - 0.00002691 i$	$(-1 + i)/100$	300
5814	$0.261239 - 1.5004 \times 10^{-6} i$	$(-1 + i)/100$	300
5814.83	$0.261233 + 2.2495 \times 10^{-9} i$	$(-1 + i)/100$	300
5815	$0.261231 + 3.1000 \times 10^{-7} i$	$(-1 + i)/100$	300
5825	$0.261154 + 0.00001837 i$	$(-1 + i)/100$	300
6000	$0.259816 + 0.00032309 i$	$(-1 + i)/120$	300

that the corresponding 3rd-order HAM iteration approach converges, as shown in Table 9.14. The real and imaginary parts of the convergent eigenfunctions for different Reynolds numbers from $R = 100$ to the critical value $R = 5814.83$ are as shown in Fig. 9.23 and Fig. 9.24, respectively. It is found that, for small Reynolds number R , the imaginary part of the corresponding eigenvalue is negative, i.e. $\text{Im}(\lambda) < 0$, corresponding to a stable viscous flow. When $\alpha = 1$ and $R = 5814.83$, the imaginary part of the corresponding eigenvalue

$$0.261233 + 2.2495 \times 10^{-9} i$$

is rather close to zero. When $\alpha = 1$ and $R > 5814.83$, it holds $\text{Im}(\lambda) > 0$ so that the flow becomes unstable. So, the above eigenvalue and the corresponding eigenfunction correspond to the most unstable viscous flow. Note that, unlike Orszag (1971),

Fig. 9.23 The real part of the eigenfunctions of the Orr-Sommerfeld stability equation (9.62) when $\alpha = 1$. Dashed line: $R = 100$; Dash-dotted line: $R = 1000$; Solid line: $R = 5814.83$.

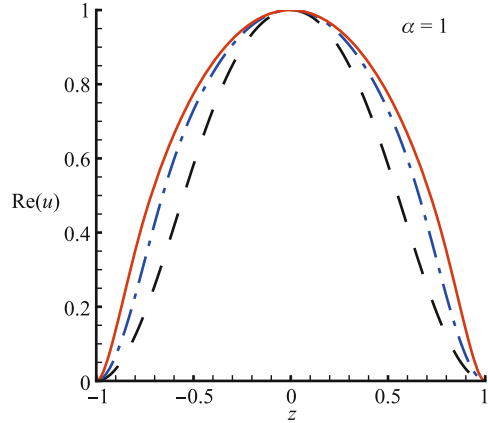
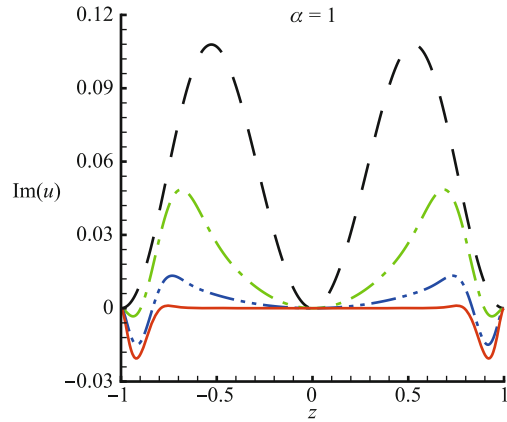


Fig. 9.24 The imaginary part of the eigenfunctions of the Orr-Sommerfeld stability equation (9.62) when $\alpha = 1$. Dashed line: $R = 100$; Dash-dotted line: $R = 1000$; Dash-double-dotted line: $R = 3000$; Solid line: $R = 5814.83$.



we need not calculate other higher modes of eigenfunctions and eigenvalues, which are not important to the stability of the flow.

Note that, as the Reynolds number R increases from 0 to the critical value $R_c \approx 5814.83$, the real part of the eigenfunctions changes very little, as shown in Fig. 9.23, but the imaginary part varies greatly, as shown in Fig. 9.24. Especially, when $\alpha = 1$ and $R = 5814.83$, the imaginary part of eigenfunction is very close to zero in a large interval $-0.7 \leq z \leq 0.7$. It would be valuable to reveal the physical meanings of this interesting result.

The critical Reynolds number R_c is defined by Orszag (1971) as the smallest value of R for which an unstable eigenmode exists. For the plane Poiseuille flow, Orszag (1971) reported the critical Reynolds number $R_c = 5772.22$ with $\alpha_c = 1.02056$. By means of the BVPh 1.0 with the 3rd-order HAM iteration approach and the complex convergence-control parameter $c_0 = (-1 + i)/100$, we obtain the corresponding eigenvalue

$$\lambda = 0.26943 - 3.085 \times 10^{-9} i$$

in case of $R_c = 5772.22$ and $\alpha_c = 1.02056$, which agrees well with the numerical ones given by Orszag (1971).

Thus, this example verifies the validity and generality of the BVPh 1.0 for stability equations with complex coefficients.

9.4 Concluding remarks

In this chapter, we illustrate the validity of the BVPh 1.0 for nonlinear eigenvalue equations $\mathcal{F}[z, u, \lambda] = 0$ in a finite interval $0 \leq z \leq a$, subject to the n linear boundary conditions $\mathcal{B}_k[z, u] = \gamma_k$ ($1 \leq k \leq n$), where $\mathcal{F}$ denotes a n th-order nonlinear ordinary differential operator, $\mathcal{B}_k$ is a linear differential operator, γ_k is a constant, $u(z)$ and λ denote eigenfunction and eigenvalue, respectively. Five different types of examples are used, such as a non-uniform beam acted by axial force, the Gelfand equation, an eigenvalue equation with varying coefficients, a multipoint boundary-value problem with multiple solutions, and the famous Orr-Sommerfeld stability equation with complex coefficients. These examples illustrate that, using the BVPh 1.0, multiple solutions of some highly nonlinear eigenvalue equations with singularity and multipoint boundary conditions can be found by means of different initial guesses and different types of base functions.

Using the first example, we illustrate that multiple solutions of nonlinear eigenvalue equations can be found out by means of different initial guesses. Especially, we successfully gain, maybe for the first time, the eigenfunctions with the *negative* eigenvalues of the beam equation (9.26), and even proved that such kind of “unnatural” eigenfunctions with the negative eigenvalues indeed exist for the non-uniform beam equation (9.26) in general. This kind of eigenfunctions with the negative eigenvalues indicate that a beam acted by a large-enough tensile force may also have a large deflection, just like a beam acted by a large enough compressive force. However, such kind of “unnatural” deflection needs a sudden and huge disturbance at $t = 0$, which requires much larger energy than the “natural” deflection, and thus hardly happens in practice. To the best of the author’s knowledge, such kind of “unnatural” eigenfunctions with negative eigenvalues of the non-uniform beam equation has *never* been reported. *Indeed, any a truly new method always gives something new and/or different.* This shows the great potential and validity of the BVPh 1.0 for highly nonlinear eigenvalue equations with multiple solutions and singularity.

In addition, we verifies the generality and validity of the BVPh 1.0 for eigenvalue problems with highly nonlinearity and singularity (Example 9.3.2), and/or separate multiple boundary conditions (Example 9.3.3), and/or varying coefficients and high-order of derivatives (Example 9.3.4), and/or the complex coefficients (Example 9.3.5), respectively.

In the frame of the HAM, we have extremely large freedom to choose different auxiliary linear operators and initial guesses so as to gain convergent eigenfunctions in different base functions, as illustrated by these examples. In the first three exam-

ples, the combination of trigonometric functions and polynomial are used as hybrid-base functions to express eigenfunctions, together with the auxiliary linear operator (9.18) and the initial guesses like (9.34) and (9.51), which contain a positive integer κ . The parameter κ can be also regarded as a multiple-solution-control parameter, too, since multiple eigenfunctions and eigenvalues can be obtained simply by using different values of κ .

The polynomial of z is used as base function to express the eigenfunctions of the last two 4th-order eigenvalue equations, together with the simple auxiliary linear operator

$$\mathcal{L}(u) = u''''.$$

Example 9.3.4 has only finite number of eigenfunctions, since it is originally a nonlinear boundary-value problem with multiple boundary conditions. Note the multiple solutions of Example 9.3.4 are gained by means of the multiple initial guess λ_0 , governed by a nonlinear algebraic equation (9.61). This shows another way to find out multiple solutions of nonlinear BVPs. The polynomial is also used in Example 9.3.5, mainly because we are only interested in the most important eigenfunction corresponding to the eigenvalue whose $\text{Im}(\lambda)$ is the maximum. Such kind of unique eigenfunction can be gained easily by means of the polynomial as the base function, as shown in Example 9.3.5.

All of these examples verify the validity and generality of the BVPh 1.0 for complicated, highly nonlinear BVPs with singularity, and/or multipoint boundary conditions, and/or complex coefficients.

Finally, although the BVPh 1.0 is developed for the general eigenvalue equation $\mathcal{F}[z, u, \lambda] = 0$ in a finite interval $z \in [0, a]$, it does *not* mean that *all* eigenvalue equations in such a form can be solved by it. As mentioned in Chapter 7, our aim is to develop a Mathematica package valid for as many nonlinear BVPs as possible. Certainly, further modifications (see the Problems of this chapter) and more applications are needed in future. Even so, the Mathematica package BVPh (version 1.0) provides us an useful and alternative tool to investigate many nonlinear eigenvalue problems in science and engineering.

Note that the Chebfun 4.0 also provides us the ability to “compute with functions instead of numbers” (Trefethen, 2007). So, it should be valuable to establish a similar HAM-based package for highly nonlinear multipoint BVPs with singularities by means of Chebfun, an open resource available (Accessed 15 April 2011) at

<http://www2.maths.ox.ac.uk/chebfun/>

Appendix 9.1 Input data of BVPh for Example 9.3.1

```
(* Input Mathematica package BVPh version 1.0 *)
<BVPh.txt;

(* Define the physical and control parameters *)
```

```

TypeEQ      = 2;
TypeL       = 2;
TypeBase    = 2;
ApproxQ     = 1;
Ntruncated  = 20;

(* Define the governing equation *)
mu[z_] := 1;
f[z_, u_, lambda_] := mu[z]*D[u, {z, 2}]
                    + D[mu[z], z]*D[u, z] + lambda*Sin[u];

(* Define Boundary conditions *)
zR      = Pi;
OrderEQ = 2;
BC[0, z_, u_, lambda_] := Limit[u - gamma, z -> 0];
BC[1, z_, u_, lambda_] := Limit[D[u, z], z -> 0];
BC[2, z_, u_, lambda_] := D[u, z] /. z -> Pi;

(* Define initial guess *)
u[0] = sigma - (sigma - gamma)*Cos[kappa*z];
kappa = 1;
sigma = Pi;
gamma = 1;

(* Define output term *)
output[z_, u_, k_] := Print["output          = ",
                             D[u[k], z] /. z -> 0/N];

(* Define the auxiliary linear operator *)
omega[1] = Pi/zR;
L[f_] := Module[{temp, numA, numB, i},
If[TypeL == 1,
    temp[1] = D[f, {z, OrderEQ}],
    numA = IntegerPart[OrderEQ/2];
    numB = OrderEQ - 2* numA//Expand;
    temp[0] = D[f, {z, numB}];
    For[i=1, i<=numA, i++,
        temp[1] = D[temp[0], {z, 2}]
                + (kappa*omega[i])^2*temp[0];
        temp[0] = temp[1];
    ];
];
temp[1]//Expand
];

(* Print input and control parameters *)
PrintInput[u[z]];

(* Use 3rd-order iteration approach *)
iter[1, 6, 3];

```

The Mathematica package BVPh (version 1.0) and the above input data are free available at

<http://numericaltank.sjtu.edu.cn/BVPh.htm>

Note that, for nonlinear boundary-value/eigenvalue problems defined in a finite interval, BVPh (version 1.0) has the default $zL = 0$.

Appendix 9.2 Input data of BVPh for Example 9.3.2

```
(* Input Mathematica package BVPh version 1.0 *)
<<BVPh.txt;

(* Define the physical and control parameters *)
TypeEQ      = 2;
TypeL       = 2;
TypeBase    = 2;
ApproxQ     = 1;
Ntruncated  = 20;

(* Define the governing equation *)
f[z_,u_,lambda_] := D[u,{z,2}]
                + (K-1)*D[u,z]/z + lambda * Exp[u] ;
K = 1;

(* Define Boundary conditions *)
zR      = 1;
OrderEQ = 2;
BC[0,z_,u_,lambda_] := Limit[D[u,z] - A, z -> 0];
BC[1,z_,u_,lambda_] := Limit[u, z -> 0];
BC[2,z_,u_,lambda_] := u /. z -> 1 ;
A = 1;

(* Define initial guess *)
u[0] = A/2*(1 + Cos[Pi*z]);

(* Define output term *)
output[z_,u_,k_] := Print["output          = ",
                        D[u[k],z] /. z->0/N];

(* Define the auxiliary linear operator *)
omega[1] = Pi/zR;
L[f_] := Module[{temp,numA,numB,i},
If[TypeL == 1,
  temp[1] = D[f,{z,OrderEQ}],
  numA = IntegerPart[OrderEQ/2];
  numB = OrderEQ - 2* numA/Expand;
  temp[0] = D[f,{z,numB}];
  For[i=1, i<=numA, i++,
    temp[1] = D[temp[0],{z,2}]
```

```

        + (kappa*omega[i])^2*temp[0];
    temp[0] = temp[1];
  ];
];
temp[1]//Expand
];

(* Print input and control parameters *)
PrintInput[u[z]];

(* Use 3rd-order iteration approach *)
iter[1,6,3];

```

The Mathematica package BVPh (version 1.0) and the above input data are free available at

<http://numericaltank.sjtu.edu.cn/BVPh.htm>

Note that, for nonlinear boundary-value/eigenvalue problems defined in a finite interval, BVPh (version 1.0) has the default $zL = 0$.

Appendix 9.3 Input data of BVPh for Example 9.3.3

```

(* Input Mathematica package BVPh version 1.0 *)
<<BVPh.txt;

(* Define the physical and control parameters *)
TypeEQ      = 2;
TypeL       = 2;
TypeBase    = 2;
ApproxQ     = 1;
Ntruncated  = 30;

(* Define the governing equation *)
L0[z_, u_] := Sqrt[1+z^2]*D[u,{z,2}]
             + Cos[Pi*z]*D[u,z]/z;
F[z_,u_]   := Exp[u]/(1+z^2) + (1+z)*Sin[u];
g[z_]      := Sin[z^2 + Exp[-z]];
f[z_,u_,lambda_] := L0[z,u] + lambda*F[z,u] - g[z];

(* Define Boundary conditions *)
zR      = Pi;
OrderEQ = 2;
BC[0,z_,u_,lambda_] := Limit[u - A, z->0];
BC[1,z_,u_,lambda_] := Limit[D[u,z], z->0];
BC[2,z_,u_,lambda_] := u-D[u,z] - 3/5 /. z->zR;
A      = 1/2;

```

```

(* Define initial guess *)
u[0] = (5*A+3)/10 + (5*A-3)/10*Cos[kappa*z];
kappa = 1;

(* Define output term *)
output[z_,u_,k_] := Print["output          = ",
                           u[k] /. z -> zR //N ];

(* Define the auxiliary linear operator *)
omega[1] = Pi/zR;
L[f_] := Module[{temp,numA,numB,i},
If[TypeL == 1,
  temp[1] = D[f,{z,OrderEQ}],
  numA = IntegerPart[OrderEQ/2];
  numB = OrderEQ - 2* numA//Expand;
  temp[0] = D[f,{z,numB}];
  For[i=1, i<=numA, i++,
    temp[1] = D[temp[0],{z,2}]
      + (kappa*omega[i])^2*temp[0];
    temp[0] = temp[1];
  ];
];
temp[1]//Expand
];

(* Print input and control parameters *)
PrintInput[u[z]];

(* Gain approximations by 3rd-order iteration *)
iter[1,6,3];

```

The Mathematica package BVPh (version 1.0) and the above input data are free available at

<http://numericaltank.sjtu.edu.cn/BVPh.htm>

Note that, for nonlinear boundary-value/eigenvalue problems defined in a finite interval, BVPh (version 1.0) has the default $z_L = 0$.

Appendix 9.4 Input data of BVPh for Example 9.3.4

```

(* Input Mathematica package BVPh version 1.0 *)
<<BVPh.txt;

(* Define the physical and control parameters *)
TypeEQ      = 2;

```

```

TypeL      = 1;
TypeBase   = 2;
ApproxQ    = 0;
Ntruncated = 20;

(* Define the governing equation *)
f[z_,u_,lambda_] := lambda*D[u,{z,4}]-beta*z
                  -beta*z*u^2*lambda^2;
beta = 10 ;

(* Define Boundary conditions *)
zR      = 1;
OrderEQ = 4;
BC[0,z_,u_,lambda_] := u - 1 /. z -> 1 ;
BC[1,z_,u_,lambda_] := Limit[u, z -> 0 ];
BC[2,z_,u_,lambda_] := D[u,z] /. z -> 1 ;
BC[3,z_,u_,lambda_] := D[u,{z,2}] /. z -> 1;
BC[4,z_,u_,lambda_] := Module[{temp},
    temp[1] = D[u,{z,2}] /. z -> 0;
    temp[2] = D[u,{z,2}] /. z -> alpha;
    temp[1]-temp[2]//Expand
];
alpha = 1/5;

(* Define initial guess *)
u[0] = sigma/(2*alpha-3)*((6*alpha-8)*z
    + 6*(1-alpha)*z^2+2*alpha*z^3-z^4);
sigma = 1;

(* Define output term *)
output[z_,u_,k_] := Print["output          = ",
    D[u[k],z] /. z->0//N];

(* Define the auxiliary linear operator *)
omega[1] = Pi/zR;
omega[2] = Pi/zR;
L[f_] := Module[{temp,numA,numB,i},
If[TypeL == 1,
    temp[1] = D[f,{z,OrderEQ}],
    numA = IntegerPart[OrderEQ/2];
    numB = OrderEQ - 2* numA//Expand;
    temp[0] = D[f,{z,numB}];
    For[i=1, i<=numA, i++,
        temp[1] = D[temp[0],{z,2}]
            + (kappa*omega[i])^2*temp[0];
        temp[0] = temp[1];
    ];
];
temp[1]//Expand
];

(* Print input and control parameters *)
PrintInput[u[z]];

```



```
(* Gain HAM approx. by 3rd-order iteration *)
iter[1,6,3];
```

The Mathematica package BVPh (version 1.0) and the above input data are free available at

<http://numericaltank.sjtu.edu.cn/BVPh.htm>

Note that, for nonlinear boundary-value/eigenvalue problems defined in a finite interval, BVPh (version 1.0) has the default $zL = 0$.

Appendix 9.5 Input data of BVPh for Example 9.3.5

```
(* Input Mathematica package BVPh version 1.0 *)
<<BVPh.txt;

(* Define the physical and control parameters *)
TypeEQ      = 2;
TypeL       = 1;
TypeBase    = 2;
ApproxQ     = 0;
ErrReq      = 10^(-20);
Nupdate     = 10000;
NtermMax    = 90;
ComplexQ    = 1;

(* Define the governing equation *)
L0[u_,z_] := D[u,{z,2}]-alpha^2*u;
L02[u_,z_] := L0[L0[u,z],z];
U0 = 1 - z^2;
U02 = D[U0,{z,2}];
f[z_,u_,lambda_] := L02[u,z]
                  - I*alpha*R*(U0*L0[u,z]-U02*u)
                  + I*lambda*R*L0[u,z];

alpha = 1;
R      = 100;

(* Define Boundary conditions *)
zR      = 1;
OrderEQ = 4;
BC[0,z_,u_,lambda_] := Limit[u - 1, z->0];
BC[1,z_,u_,lambda_] := Limit[D[u,z], z->0];
BC[2,z_,u_,lambda_] := Limit[D[u,{z,3}], z->0];
BC[3,z_,u_,lambda_] := u /. z->zR;
BC[4,z_,u_,lambda_] := D[u,z] /. z->zR;

(* Define initial guess *)
```

```

U[0] = u[0];
u[0] = ( 1 - z^2 )^2;

(* Define the auxiliary linear operator *)
L[f_] := D[f, {z, 4}];

(* Define output term *)
output[z_, u_, k_] := Print["output          = ",
                             D[u[k], {z, 2}] /. z -> 0 // N];

(* Print input and control parameters *)
PrintInput[u[z]];

(* Set convergence-control parameter *)
c0 = (-1 + I)/2;

(* Gain HAM approx. by 3rd-order iteration *)
iter[1, 31, 3];

```

The Mathematica package BVPh (version 1.0) and the above input data are free available at

<http://numericaltank.sjtu.edu.cn/BVPh.htm>

Note that, for nonlinear boundary-value/eigenvalue problems defined in a finite interval, BVPh (version 1.0) has the default $zL = 0$.

Problems

9.1. Eigenvalue problems with nonlinear boundary conditions

Develop a HAM-based analytic approach for the n th-order nonlinear eigenvalue equation

$$\mathcal{F}[z, u, \lambda] = 0, \quad 0 \leq z \leq a,$$

subject to the n nonlinear multipoint boundary conditions

$$\mathcal{B}_k[z, u, \lambda] = \gamma_k,$$

where $\mathcal{B}_k$ is a nonlinear operator and γ_k is a constant. Assume that the above equation has at least one smooth solution. Modify the Mathematica package BVPh (version 1.0) given in Chapter 7 for this kind of problems in general.

9.2. Coupled nonlinear eigenvalue problems

Develop a HAM-based analytic approach for n coupled nonlinear eigenvalue equations in a finite interval $z \in [0, a]$:

$$\mathcal{F}_k[z, u, \lambda_1, \lambda_2, \dots, \lambda_n] = 0, \quad 1 \leq k \leq n,$$

subject to some linear/nonlinear multipoint boundary conditions, where $n \geq 2$. Assume that the above equation has at least one smooth solution. Give a Mathematica package for this kind of problems in general.

9.3. Eigenvalue problems in an infinite interval

Develop a HAM-based analytic approach for the n th-order nonlinear eigenvalue equation in an infinite interval

$$\mathcal{F}[z, u, \lambda] = 0, \quad 0 \leq z < +\infty,$$

subject to the n multipoint nonlinear boundary conditions

$$\mathcal{B}_k[z, u, \lambda] = \gamma_k,$$

where $\mathcal{B}_k$ is a nonlinear operator and γ_k is a constant. Assume that the above equation has at least one smooth solution. Give a Mathematica package for this kind of problems in general.

9.4. Coupled nonlinear eigenvalue problems in an infinite interval

Develop a HAM-based analytic approach for n coupled nonlinear eigenvalue equations in an infinite interval $z \in [0, +\infty)$:

$$\mathcal{F}_k[z, u, \lambda_1, \lambda_2, \dots, \lambda_n] = 0, \quad 1 \leq k \leq n,$$

subject to some linear/nonlinear multipoint boundary conditions, where $n \geq 2$. Assume that the above equations have at least one smooth solution. Give a Mathematica package for this kind of problems in general.

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Chapter 10

A Boundary-layer Flow with an Infinite Number of Solutions

Abstract In this chapter, the Mathematica package BVPh (version 1.0) based on the homotopy analysis method (HAM) is used to gain exponentially and algebraically decaying solutions of a nonlinear boundary-value equation in an infinite interval. Especially, an infinite number of algebraically decaying solutions were found for the first time by means of the HAM, which illustrate the originality and validity of the HAM for nonlinear boundary-value problems.

10.1 Introduction

In this chapter, we illustrate the validity of the Mathematica package BVPh (version 1.0) for nonlinear boundary-value problems in an infinite interval, governed by a n th-order nonlinear ordinary differential equation (ODE)

$$\mathcal{F}[z, u] = 0, \quad 0 \leq z < +\infty, \quad (10.1)$$

subject to some linear boundary conditions, where $\mathcal{F}$ denotes a nonlinear differential operator, $u(z)$ is a smooth solution, respectively. Assume that $u(z)$ decays either exponentially or algebraically as $z \rightarrow +\infty$.

In Chapter 8 and Chapter 9, we illustrate that the BVPh 1.0 provides us an useful tool to gain multiple solutions of n th-order highly nonlinear boundary-value/eigenvalue problems with singularity and multipoint boundary conditions in a *finite* interval $z \in [0, a]$. In this chapter, we further illustrate the validity of the BVPh 1.0 for such kind of nonlinear ODEs in an *infinite* interval.

In 2005, Liao (2005) successfully applied the homotopy analysis method (HAM) (Liao, 1992, 1997, 1999a,b; Liao and Campo, 2002; Liao, 2003a,b, 2004, 2005, 2006, 2009, 2010a,b; Liao and Tan, 2007; Li et al., 2010; Xu et al., 2010) to solve the nonlinear boundary-value equation in an infinite interval

$$F''' + \frac{1}{2}F F'' - \beta F'^2 = 0, \quad F(0) = 0, \quad F'(0) = 1, \quad F'(+\infty) = 0,$$

where $-1 < \beta < +\infty$ is a constant. Introducing an unknown quantity $\delta = F(+\infty)$, Liao (2005) found one new branch of exponentially decaying solutions when $\beta > 1$, which had been *never* reported by other analytic methods and even neglected by numerical methods, mainly because the difference between the values of $F''(0)$ of the two branches of the exponentially decaying solutions is so small that it is hard to distinguish them. Besides, by means of the HAM, Liao and Magyari (2006) found a new branch of algebraically decaying solutions of a kind of boundary-layer flow. Indeed, *a truly new method always gives something new and/or different*. All of these illustrate the originality and validity of the HAM. In addition, they also illustrate that the HAM can be used to solve nonlinear boundary-value problems with either exponentially or algebraically decaying solutions in an infinite interval.

Without loss of generality, let us consider here a two-dimensional boundary-layer viscous flow in the region $x > 0$ and $y > 0$, where (x, y) denotes a Cartesian coordinate system and the flow results solely from the movement of an impermeable flat plate at $y = 0$ in its plane. Let $U_w(x) = a(x+b)^\kappa$ denote the speed of the flat plate, where $a > 0, b > 0$ are given constants. Assume that the boundary-layer equations are appropriate so that such kind of flow is described by the partial differential equations (PDEs)

$$\begin{aligned} u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= \nu \frac{\partial^2 u}{\partial y^2}, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \end{aligned}$$

subject to the boundary conditions

$$u = a(x+b)^\kappa, \quad v = 0, \quad \text{at } y = 0,$$

and

$$u = 0, \quad \text{at } y \rightarrow +\infty,$$

where ν is the kinematic viscosity and u, v are the velocity components in the directions of increasing x, y , respectively.

Let ψ denote the stream function. Using the similarity transformation

$$\psi = \sqrt{a\nu}(x+b)^{(\kappa+1)/2} f(\eta), \quad \eta = \sqrt{\frac{a}{\nu}} y (x+b)^{(\kappa-1)/2}, \quad (10.2)$$

the original PDEs become the following nonlinear ODE

$$f''' + \frac{(1+\kappa)}{2} f f'' - \kappa f'^2 = 0, \quad f(0) = 0, \quad f'(0) = 1, \quad f'(+\infty) = 0. \quad (10.3)$$

For details, please refer to Banks (1983). The above equation has the close-form solution

$$f(\eta) = 1 - \exp(-\eta), \quad \text{when } \kappa = 1, \quad (10.4)$$

and

$$f(\eta) = \sqrt{6} \tanh \frac{\eta}{\sqrt{6}}, \quad \text{when } \kappa = -1/3, \quad (10.5)$$

respectively. Besides, there exist solutions when $-1/2 < \kappa < +\infty$.

The nonlinear boundary-value equation (10.3) in the infinite interval $z \in [0, +\infty)$ has two types of solutions: one decays exponentially at infinity (Liao and Pop, 2004), the other decays algebraically (Liao and Magyari, 2006). These two types of solutions of (10.3) can be obtained by means of the BVPb 1.0, as shown below.

10.2 Exponentially decaying solutions

Physically, most of boundary-layer flows exponentially tend to a uniform flow at infinity. Mathematically, this is confirmed by the close-form solutions (10.4) and (10.5). Such kinds of exponentially decaying solutions can be gained by means of the HAM, as shown by Liao and Pop (2004). Here, it is solved by means of the BVPb 1.0, which is given in the Appendix 7.1 and free available (Accessed 25 Nov 2011, will be updated in the future) at

<http://numericaltank.sjtu.edu.cn/BVPb.htm>

The corresponding input data file is given in the Appendix 10.1 and free available at the above website.

For the sake of simplicity, define a nonlinear operator

$$\mathcal{N}[\phi(\eta; q)] = \phi''' + \frac{(1 + \kappa)}{2} \phi \phi'' - \kappa (\phi')^2. \quad (10.6)$$

In the frame of the HAM, we first construct the so-called zeroth-order deformation equation

$$(1 - q)\mathcal{L}[\phi(\eta; q) - f_0(\eta)] = q c_0 \mathcal{N}[\phi(\eta; q)], \quad (10.7)$$

subject to the boundary conditions

$$\phi(0; q) = 0, \quad \phi'(0; q) = 0, \quad \phi'(+\infty; q) = 0, \quad (10.8)$$

where the prime denotes the differentiation with respect to η , $f_0(\eta)$ is an initial guess, $\mathcal{L}$ is an auxiliary linear operator, and c_0 is the so-called convergence-control parameter, respectively. The above equation defines a kind of continuous variation $\phi(\eta; q)$ from the initial guess $f_0(\eta)$ (at $q = 0$) to the solution $f(\eta)$ of the original equation (10.3) (at $q = 1$). The homotopy-Maclaurin series of $\phi(\eta; q)$ reads

$$\phi(\eta; q) = f_0(\eta) + \sum_{n=1}^{+\infty} f_n(\eta) q^n,$$

where

$$f_n(\eta) = \mathcal{D}_n[\phi(\eta; q)] = \frac{1}{n!} \left. \frac{\partial^n \phi(\eta; q)}{\partial q^n} \right|_{q=0} \quad (10.9)$$

and $\mathcal{D}_n$ is called the n th-order homotopy-derivative operator.

If the initial guess $f_0(\eta)$, the auxiliary linear operator $\mathcal{L}$, and especially the convergence-control parameter c_0 are properly chosen so that the above homotopy-Maclaurin series absolutely converges at $q = 1$, we have the homotopy-series solution

$$f(\eta) = f_0(\eta) + \sum_{n=1}^{+\infty} f_n(\eta), \quad (10.10)$$

where $f_n(\eta)$ is governed by the n th-order deformation equation

$$\mathcal{L}[f_n(\eta) - \chi_n f_{n-1}(\eta)] = c_0 \delta_{n-1}(\eta), \quad (10.11)$$

subject to the boundary conditions

$$f_n(0) = 0, \quad f'_n(0) = 0, \quad f'_n(+\infty) = 0, \quad (10.12)$$

where

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1 \end{cases}$$

and

$$\delta_m(\eta) = \mathcal{D}_m \{ \mathcal{N}[\phi(\eta; q)] \} = f_m''' + \frac{(1 + \kappa)}{2} \sum_{i=0}^m f_{m-i} f_i'' - \kappa \sum_{i=0}^m f'_{m-i} f_i' \quad (10.13)$$

is gained by Theorem 4.1. Equation (10.11) is given by Theorem 4.15. For details, please refer to Chapter 4.

The exponentially decaying solution of (10.3) can be expressed in the form

$$f(\eta) = \sum_{n=0}^{+\infty} A_n(\eta) \exp(-n\eta), \quad (10.14)$$

where $A_n(\eta)$ is a polynomial to be determined. It provides us the so-called solution-expression of $f(\eta)$. To gain such kind of exponentially decaying solution by means of the BVPh 1.0, we choose the initial guess

$$f_0(\eta) = \sigma + (1 - 2\sigma) e^{-\eta} - (1 - \sigma) e^{-2\eta} \quad (10.15)$$

and the auxiliary linear operator

$$\mathcal{L}(f) = f''' - f', \quad (10.16)$$

respectively, where $\sigma = f_0(+\infty)$ is an unknown parameter. Note that the initial guess (10.15) satisfies all boundary-conditions (10.3) and besides $f_0(+\infty) = \sigma$. Obviously, a better value of σ corresponds to a better initial guess. So, the unknown σ provides us one additional degree of freedom in the initial guess $f_0(\eta)$. Note also that the auxiliary linear operator (10.16) has the property

$$\mathcal{L}[B_0 + B_1 \exp(-\eta) + B_2 \exp(+\eta)] = 0. \quad (10.17)$$

In addition, for any a given function $f^*(\eta)$ that decays exponentially at infinity, the unknown integral coefficients B_0, B_1, B_2 of the following function

$$f^*(\eta) + B_0 + B_1 \exp(-\eta) + B_2 \exp(\eta)$$

can be uniquely determined by the three boundary conditions (10.3). In other words, the auxiliary linear operator $\mathcal{L}$ is chosen in such a way that the solution $f(\eta)$ is expressed in the form of (10.14) for the exponentially decaying solution, and besides all integral coefficients are uniquely determined. For the choice of the initial guess and the auxiliary linear operator $\mathcal{L}$ in general, please refer to Sect. 7.1.3 and Sect. 7.1.4.

Let

$$f_n^*(\eta) = \chi_n f_{n-1}(\eta) + c_0 \mathcal{L}^{-1}[\delta_{n-1}(\eta)]$$

denote a special solution of (10.11). Its general solution reads

$$\begin{aligned} f_n(\eta) = & \chi_n f_{n-1}(\eta) + c_0 \mathcal{L}^{-1}[\delta_{n-1}(\eta)] \\ & + B_0 + B_1 \exp(-\eta) + B_2 \exp(\eta), \end{aligned} \quad (10.18)$$

where the integral coefficients B_0, B_1 and B_2 are uniquely determined by three boundary conditions (10.12). For details, please refer to Liao and Pop (2004).

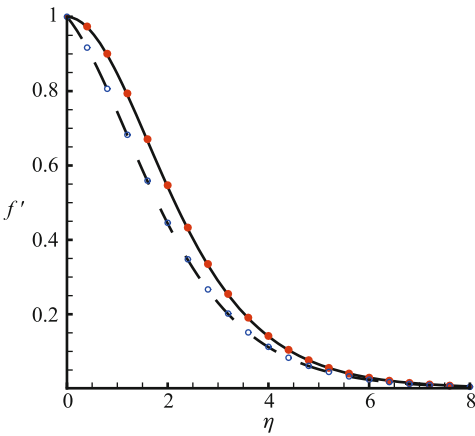
First, let us consider the case of $\kappa = -1/3$. The corresponding homotopy-approximations are obtained by means of the BVPh 1.0, which contain two unknown parameters: σ in the initial guess (10.15) and the convergence-control parameter c_0 . Using the Mathematica command `Minimize`, it is found that, at the 5th-order homotopy approximation, we have the minimum 6.8×10^{-6} of the averaged squared residual of (10.3) over the interval $\eta \in [0, 10]$, corresponding to the optimal convergence-control parameter $c_0^* = -1.2411$ and the optimal parameter $\sigma^* = 3.0435$. Indeed, by means of $c_0 = -5/4$ and $\sigma = 3$, the averaged squared residual of (10.3) over the interval $\eta \in [0, 10]$ decreases quickly, as shown in Table 10.1. Note that, $f''(0)$ quickly converges to the exact value $f''(0) = 0$, and the 10th-order homotopy-approximation of $f(\eta)$ agrees well with the exact solution, as shown in Fig. 10.1.

Note that the unknown parameter σ in the initial guess (10.15) is used here to search for the optimal initial guess, which can be regarded as a kind of convergence-control parameter. So, we have here two convergence-control parameters: c_0 in the zeroth-order deformation equation (10.7) and σ in the initial guess (10.15). Note that, in Chapter 8 and Chapter 9, the unknown parameter σ in initial guesses are used as the so-called multiple-solution-control parameter to find out multiple solutions of nonlinear ODEs in a finite interval $z \in [0, a]$. So, an unknown parameter in initial guesses can be used either as *the multiple-solution-control parameter* to find out multiple solutions, or as *the convergence-control parameter* to control the convergence of homotopy-series solutions. This is mainly because, based on the HAM, the BVPh 1.0 provides us large freedom to choose initial guesses.

Table 10.1 The averaged squared residual of (10.3) at the m th-order (exponentially decaying) homotopy approximation over the interval $\eta \in [0, 10]$ when $\kappa = -1/3$ by means of $c_0 = -5/4$ and $\sigma = 3$.

Order of approximation, m	$f''(0)$	Squared residual E_m
2	7.0×10^{-2}	5.2×10^{-3}
4	-2.1×10^{-2}	3.7×10^{-5}
6	-1.4×10^{-2}	3.8×10^{-6}
8	-8.9×10^{-3}	1.8×10^{-6}
10	-4.8×10^{-3}	6.8×10^{-7}
15	-6.6×10^{-4}	3.9×10^{-8}
20	8.2×10^{-5}	1.4×10^{-9}

Fig. 10.1 The exponentially decaying solutions $f'(\eta)$ of (10.3). Filled circle: the 10th-order homotopy approximation when $\kappa = -1/3$ by means of $c_0 = -5/4$ and $\sigma = 3$; Solid line: the exact solution when $\kappa = -1/3$; Open circle: the 10th-order homotopy approximation when $\kappa = -1/4$ by means of $c_0 = -5/4$ and $\sigma = 11/4$; Dashed line: the 30th-order homotopy approximation when $\kappa = -1/4$.



Similarly, we can gain accurate homotopy-approximation $f(\eta)$ of (10.3) for $-1/2 < \kappa < +\infty$ by means of the BVPh 1.0. For example, when $\kappa = -1/4$, at the 5th-order homotopy approximation, we have the minimum 6.5×10^{-6} of the averaged squared residual over the interval $\eta \in [0, 10]$, corresponding to the two optimal convergence-control parameters $c_0^* = -1.2133$ and $\sigma^* = 2.7369$. By means of $c_0 = -5/4$ and $\sigma = 11/4$, we indeed gain the accurate homotopy-series solution whose $f''(0)$ converges to -0.1620 , as shown in Table 10.2. Besides, the 10th-order approximation of $f'(\eta)$ is rather accurate, as shown in Fig. 10.1. Therefore, by means of the BVPh 1.0, we successfully gain the exponentially decaying solutions of (10.3). For simplicity, we denote this kind of exponentially decaying solution by $f_{\text{exp}}(\eta)$ in the following section.

Note that, using the transformation

$$f(\eta) = \sqrt{\frac{2}{\kappa + 1}} F(z), \quad z = \sqrt{\frac{\kappa + 1}{2}} \eta,$$

Equation (10.3) becomes

Table 10.2 The averaged squared residual of (10.3) at the m th-order (exponentially decaying) homotopy approximation over the interval $\eta \in [0, 10]$ when $\kappa = -1/4$ by means of $q_0 = -5/4$ and $\sigma = 11/4$.

Order of approximation, m	$f''(0)$	Squared residual E_m
5	-0.1692	2.9×10^{-6}
10	-0.1639	1.4×10^{-7}
15	-0.1623	4.5×10^{-9}
20	-0.1620	8.7×10^{-11}
25	-0.1620	7.8×10^{-12}
30	-0.1620	1.1×10^{-12}

$$F''' + F F'' - \varepsilon F'^2 = 0, \quad F(0) = 0, \quad F'(0) = 1, \quad F'(+\infty) = 0,$$

where $\varepsilon = 2\kappa/(1 + \kappa)$, and the prime denotes the differentiation with respect to z . By means of the HAM, Liao and Pop (2004) obtained a 3rd-order approximation

$$F''(0) = -\frac{145293 + 231153\varepsilon + 94999\varepsilon^2 + 12395\varepsilon^3}{15120(3 + \varepsilon)^{5/2}}, \quad (10.19)$$

which agrees well with numerical results in the whole interval $0 \leq \varepsilon < +\infty$. For details, please refer to Liao and Pop (2004). Thus, for the exponentially decaying solutions $f_{\text{exp}}(\eta)$ of (10.3), we have the 3rd-order homotopy-approximation

$$\begin{aligned} f''_{\text{exp}}(0) &= \sqrt{\frac{\kappa + 1}{2}} F''(0) \\ &= -\frac{145293 + 898185\kappa + 1740487\kappa^2 + 1086755\kappa^3}{15120\sqrt{2}(3 + 5\kappa)^{5/2}}, \end{aligned} \quad (10.20)$$

which is valid in an infinite interval $\kappa \in (0, +\infty)$.

This example verifies the validity of the BVPh 1.0 for nonlinear boundary-value ODEs in an infinite interval, whose solutions decay exponentially at infinity.

10.3 Algebraically decaying solutions

As pointed out by Magyari et al. (2003), when $\kappa = -1/3$, (10.3) has an *infinite* number of solutions in the closed-form

$$f(\eta) = (36\mu)^{1/3} \left[\frac{Bi'(t_0)Ai'(t) - Ai'(t_0)Bi'(t)}{Bi'(t_0)Ai(t) - Ai'(t_0)Bi(t)} \right], \quad (10.21)$$

where $Ai(t)$ and $Bi(t)$ are two kinds of Airy functions,

$$t_0 = \left(\sqrt{6} \mu \right)^{-2/3}, \quad t = t_0 (1 + \mu \eta),$$

and

$$\mu = f''(0) \geq f''_{\text{exp}}(0), \quad (10.22)$$

respectively. Here, $f_{\text{exp}}(\eta)$ denotes the solution of (10.3) exponentially decaying at infinity, mentioned in the above section. Since $f''_{\text{exp}}(0) = 0$ when $\kappa = -1/3$, then the solution (10.21) satisfies (10.3) (when $\kappa = -1/3$) for *arbitrary* values of $\mu = f''(0) \geq 0$. It was proved (Magyari et al., 2003) that the solution (10.21) is equal to the close-form solution (10.5) as $\mu = f''(0)$ tends to zero, but for any other values of $\mu > 0$, the solution (10.21) decays *algebraically* at infinity. In other words, when $\kappa = -1/3$, the exponentially decaying solution

$$f_{\text{exp}}(\eta) = \sqrt{6} \tanh(\eta/\sqrt{6})$$

is the limit of the algebraically decaying solutions (10.21) as $f''(0) \rightarrow f''_{\text{exp}}(0) = 0$.

The BVPh 1.0 is also valid for such kind of algebraically decaying boundary-layer flows, too. By means of BVPh 1.0 as a tool, it is found that (10.3) has indeed an infinite number of algebraically decaying solutions not only at $\kappa = -1/3$ but also in the *whole* interval $-1/2 < \kappa < 0$, as shown below.

Let us first consider the asymptotic property of the algebraically decaying solution $f(\eta)$ at infinity. Write the asymptotic expression

$$f' \sim \eta^b, \text{ i.e. } f \sim \frac{1}{(b+1)} \eta^{b+1}, \quad \text{as } \eta \rightarrow +\infty,$$

where b is a constant to be determined. Substituting these asymptotic expressions into (10.3) and balancing the dominant terms, we have

$$b = \frac{2\kappa}{1-\kappa}, \text{ i.e. } 1+b = \frac{1+\kappa}{1-\kappa} = \beta.$$

Obviously, to satisfy the boundary condition $f'(+\infty) = 0$, b must be negative, corresponding to $-1/2 < \kappa < 0$, which gives

$$\frac{1}{3} < \beta < 1.$$

Then, the algebraically decaying solution has the asymptotic property

$$f \sim \eta^\beta, \quad \frac{1}{3} < \beta < 1, \quad \text{as } \eta \rightarrow +\infty.$$

To avoid the singularity of the above asymptotic expression at $\eta = 0$, we use the transformation

$$\xi = 1 + \alpha \eta, \quad f(\eta) = \alpha^{-1} g(\xi),$$

where $\alpha > 0$ is a constant. Then, the original equation (10.3) becomes

$$\alpha^2 g'''(\xi) + \frac{1+\kappa}{2} g g'' - \kappa (g')^2 = 0, \quad g(1) = 0, g'(1) = 1, g'(+\infty) = 0, \quad (10.23)$$

where the prime denotes the differentiation with respect to ξ . Since (10.3) has an *infinite* number of solutions dependent upon $\mu = f''(0) > f''_{\text{exp}}(0)$, we should add an additional boundary condition $f''(0) = \mu$, i.e.

$$g''(0) = \frac{\mu}{\alpha} \quad (10.24)$$

so as to distinguish these different algebraically decaying solutions. Considering the asymptotic property mentioned above, our aim is to gain the algebraically decaying solutions in the form

$$g(\xi) = a_{0,0} \xi^\beta + \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} a_{m,n} \xi^{-(1-\beta)m-n}, \quad (10.25)$$

which provides us the so-called *solution-expression* of the algebraically decaying solution $g(\xi)$.

Such kind of algebraically decaying solutions of the nonlinear boundary-value ODE (10.23), with the additional boundary condition $g''(0) = \mu/\alpha$, can be gained by means of the BVPh 1.0, as shown below. The corresponding input data file is given in the Appendix 10.2 and free available at

<http://numericaltank.sjtu.edu.cn/BVPh.htm>

Define a nonlinear operator

$$\check{\mathcal{N}}(u) = \alpha^2 u''' + \frac{1+\kappa}{2} u u'' - \kappa (u')^2,$$

where the prime denotes the differentiation with respect to ξ . Let $q \in [0, 1]$ denote the embedding-parameter. In the frame of the HAM, we should first construct such a continuous variation $\check{\phi}(\xi; q)$ (or deformation) that $\check{\phi}(\xi; q) = g_0(\xi)$ at $q = 0$ and $\check{\phi}(\xi; q) = g(\xi)$ at $q = 1$, respectively. Such a kind of continuous variation is defined by the zeroth-order deformation equation

$$(1-q)\check{\mathcal{L}}[\check{\phi}(\xi; q) - g_0(\xi)] = c_0 q \check{H}(\xi) \check{\mathcal{N}}[\check{\phi}(\xi; q)], \quad (10.26)$$

subject to the boundary conditions

$$\check{\phi}(\xi; q) = 0, \quad \check{\phi}'(\xi; q) = 1, \quad \check{\phi}''(\xi; q) = \mu/\alpha, \quad \text{at } \xi = 1, \quad (10.27)$$

and

$$\check{\phi}'(\xi; q) = 0, \quad \text{as } \xi \rightarrow +\infty, \quad (10.28)$$

where $\check{\mathcal{L}}$ is the auxiliary linear operator, $g_0(\xi)$ is the initial guess of $g(\xi)$, c_0 is the convergence-control parameter, $\check{H}(\xi)$ is an auxiliary function, the prime denotes the differentiation with respect to ξ , respectively. Assuming that the initial guess $g_0(\xi)$, the auxiliary function $\check{H}(\xi)$, and especially the convergence-control parameter c_0

are so properly chosen that the homotopy-Maclaurin series

$$\check{\phi}(\xi; q) = g_0(\xi) + \sum_{m=1}^{+\infty} g_m(\xi) q^m,$$

absolutely converges at $q = 1$, we have the homotopy-series solution

$$g(\xi) = g_0(\xi) + \sum_{m=1}^{+\infty} g_m(\xi), \quad (10.29)$$

where $g_m(\xi)$ is governed by the m th-order deformation equation

$$\mathcal{L}[g_m(\xi) - \chi_m g_{m-1}(\xi)] = c_0 \check{H}(\xi) \check{\delta}_{m-1}(\xi) \quad (10.30)$$

subject to the boundary conditions

$$g_m(1) = 0, \quad g'_m(1) = 0, \quad g''_m(1) = 0, \quad g'_m(+\infty) = 0, \quad (10.31)$$

where

$$\check{\delta}_n(\xi) = \alpha^2 g'''_m + \frac{(1 + \kappa)}{2} \sum_{i=0}^m g_{m-i} g''_i - \kappa \sum_{i=0}^m g'_{m-i} g'_i \quad (10.32)$$

is gained by Theorem 4.1. Equation (10.30) is given by Theorem 4.15. For details, please refer to Chapter 4.

The solution-expression (10.25) plays an important role in choosing the initial guess $g_0(\xi)$ and the auxiliary linear operator (10.34). To satisfy the solution-expression (10.25), we choose the initial guess in the form

$$g_0(\xi) = a_0 \xi^\beta + a_1 \xi^{\beta-1} + a_2 \xi^{2(\beta-1)},$$

where the unknown coefficients a_0, a_1 and a_2 are determined by the two boundary conditions $g(1) = 0, g'(1) = 1$ and the additional boundary condition $g''(1) = \mu/\alpha$. Thus, we have

$$\begin{aligned} g_0(\xi) = & \frac{(4 - 3\beta + \alpha^{-1}\mu)}{2 - \beta} \xi^\beta - \frac{(3 - 3\beta + \alpha^{-1}\mu)}{1 - \beta} \xi^{\beta-1} \\ & + \frac{(2 - 2\beta + \alpha^{-1}\mu)}{(1 - \beta)(2 - \beta)} \xi^{2\beta-2}, \end{aligned} \quad (10.33)$$

which automatically satisfies the boundary condition $g'(+\infty) = 0$, and besides decays *algebraically* at infinity.

To satisfy the solution-expression (10.25), we choose such an auxiliary linear operator $\mathcal{L}$ in the form

$$\mathcal{L}(u) = u''' + B_1(\xi)u'' + B_2(\xi)u' + B_3(\xi)u$$

that the 3rd-order linear differential equation

$$\mathcal{L}(u) = 0$$

has the general solutions algebraically decaying at infinity, i.e.

$$\mathcal{L}\left(C_1 \xi^\beta + C_2 \xi^{\beta-1} + C_3 \xi^{2(\beta-1)}\right) = 0.$$

Substituting $u = \xi^\beta$, $u = \xi^{\beta-1}$ and $u = \xi^{2(\beta-1)}$ into $\mathcal{L}(u) = 0$ gives three linear algebraic equations of $B_1(\xi)$, $B_2(\xi)$ and $B_3(\xi)$, which uniquely determine them. In this way, we gain the auxiliary linear operator

$$\mathcal{L}(u) = \xi^3 u''' - 2(2\beta - 3)\xi^2 u'' + (\beta - 1)(5\beta - 6)\xi u' - 2\beta(\beta - 1)^2 u. \quad (10.34)$$

Let

$$g_m^*(\xi) = \chi_m g_{m-1}(\xi) + c_0 \mathcal{L}^{-1} \left[\check{H}(\xi) \check{\delta}_{m-1}(\xi) \right]$$

denote a special solution of (10.30), where $\mathcal{L}^{-1}$ is the inverse operator of $\mathcal{L}$. Then, its general solution reads

$$g_m(\xi) = g_m^*(\xi) + C_{m,1} \xi^\beta + C_{m,2} \xi^{\beta-1} + C_{m,3} \xi^{2(\beta-1)},$$

where the integral coefficients $C_{m,1}$, $C_{m,2}$ and $C_{m,3}$ are determined by the three boundary conditions (10.31) at $\xi = 1$. Note that the boundary condition at infinity, i.e. $g'_m(+\infty) = 0$, is *automatically* satisfied. Thus, by means of proper base functions, it is very easy to satisfy the boundary conditions at infinity. This is mainly because the computer algebra system like Mathematica provides us the ability to “compute with functions instead of numbers” (Trefethen, 2007).

Besides, to satisfy the so-called solution-expression (10.25), the term $\check{H}(\xi) \check{\delta}_{m-1}(\xi)$ on the right-hand side of (10.30) should not contain the following terms

$$\xi^\beta, \xi^{\beta-1}, \xi^{2(\beta-1)}.$$

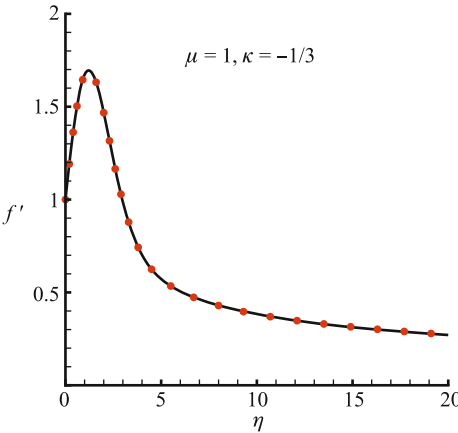
For this reason, we must choose the auxiliary function $\check{H}(\xi) = \xi$. For details, please refer to Liao and Magyari (2006).

To show the validity of the BVPh 1.0 for algebraically decaying solutions of non-linear ODEs in an infinite interval, let us first consider the case $\kappa = -1/3$, whose closed-form solution (10.21) is known. According to (10.23), we have the exact relationship $\alpha^2 g'''(1) = \kappa$, which is used to check the accuracy of the homotopy-approximation. Note that both of c_0 and α are unknown, whose optimal values are determined by the minimum of the squared residual of the governing equation (10.23). For example, when $\mu = f''(0) = 1$, the averaged squared residual of (10.23) at the 5th-order homotopy-approximation over the interval $\xi \in [1, 20]$ has the minimum 4.2×10^{-4} by means of the optimal convergence-control parameter $c_0^* = -8.1789$ and the optimal parameter $\alpha^* = 0.4185$. Indeed, by means of $c_0 = -8$ and $\alpha = 2/5$, the corresponding homotopy-approximations converge quickly to the exact solution (10.21), as shown in Table 10.3 and Fig. 10.2. This illustrates the va-

Table 10.3 The averaged squared residual of (10.23) at the m th-order (algebraically decaying) homotopy-approximation over the interval $\xi \in [1, 20]$ when $\kappa = -1/3$ and $\mu = 1$ by means of $c_0 = -8, \alpha = 2/5$ and $\tilde{H}(\xi) = \xi$.

Order of approximation, m	$\alpha^2 g'''(1)$	Squared residual E_m
2	-0.4546	9.8×10^{-3}
4	-0.3428	1.6×10^{-3}
6	-0.3341	1.7×10^{-4}
8	-0.3334	2.8×10^{-5}
10	-0.3333	1.1×10^{-5}
20	-0.3333	6.1×10^{-7}
30	-0.3333	6.2×10^{-8}
50	-0.3333	1.4×10^{-9}

Fig. 10.2 The algebraically decaying solutions $f'(\eta)$ of (10.3) when $\kappa = -1/3$ and $\mu = 1$. Solid line: the exact solution (10.21); Filled circles: the 20th-order homotopy-approximation by means of $c_0 = -8, \alpha = 2/5$ and $\tilde{H}(\xi) = \xi$.



lidity of the BVPh 1.0 for algebraically decaying solutions of nonlinear boundary-value ODEs in an infinite interval $\eta \in [0, +\infty)$.

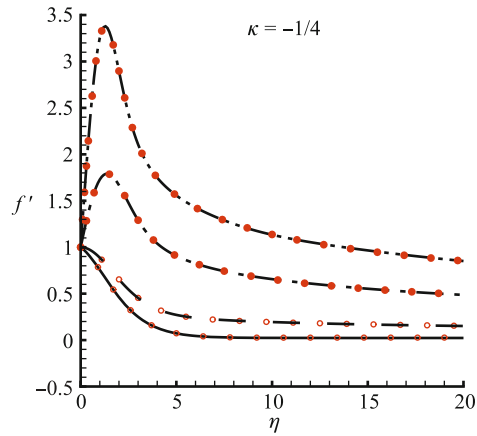
Note that the unknown parameter α in (10.33) provides us one additional degree of freedom in initial guess, which supplies a convenient way to find out the optimal initial guess by means of the minimum of the squared residual of the governing equation. Therefore, we can also regard α as a convergence-control parameter, too. Obviously, the use of the two convergence-control parameters c_0 and α improves our ability to guarantee the convergence of homotopy-series solution in the frame of the HAM, as shown above.

Similarly, the BVPh 1.0 can be used to gain the algebraically decaying solutions for other values of κ in the whole interval $\kappa \in (-1/2, 0)$. For example, when $\kappa = -1/4$ and $\mu = 1$, it is found that the averaged squared residual of (10.23) at the 5th-order homotopy-approximation over the interval $\xi \in [1, 20]$ has the minimum 1.7×10^{-3} by means of the two optimal convergence-control parameter $c_0^* = -6.8702$ and $\alpha^* = 0.4133$. Using $c_0 = -7$ and $\alpha = 2/5$, the corresponding homotopy-approximations converge quickly, as shown in Table 10.4 and Fig. 10.3.

Table 10.4 The averaged squared residual of (10.23) at the m th-order (algebraically decaying) homotopy approximation over the interval $\xi \in [1, 20]$ when $\kappa = -1/4$ and $\mu = 1$ by means of $c_0 = -7, \alpha = 2/5$ and $\dot{H}(\xi) = \xi$.

Order of approximation, m	$\alpha^2 g'''(1)$	Squared residual E_m
2	-0.2699	2.6×10^{-2}
4	-0.2503	5.1×10^{-3}
6	-0.2500	9.0×10^{-4}
8	-0.2500	2.4×10^{-4}
10	-0.2500	1.0×10^{-4}
20	-0.2500	9.9×10^{-6}
30	-0.2500	1.1×10^{-6}
40	-0.2500	1.6×10^{-7}
50	-0.2500	3.7×10^{-8}

Fig. 10.3 The algebraically decaying solutions $f'(\eta)$ of (10.3) when $\kappa = -1/4$. Line: 50th-order homotopy-approximation; Open circles: 20th-order homotopy-approximations; Filled circles: 30th-order homotopy-approximations; Solid line: $\mu = -0.15$; Dashed-line: $\mu = 0$; Dash-dotted line: $\mu = 1$; Dash-double-dotted line: $\mu = 3$.

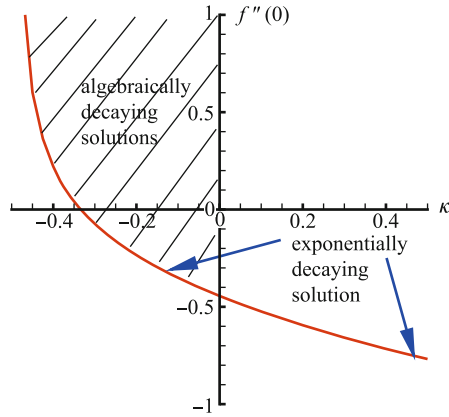


Similarly, when $\kappa = -1/4$, we gain the algebraically decaying solutions for other values $\mu = f''(0) > f''_{\text{exp}}(0) = -0.1620$, as shown in Fig. 10.3. It is found that, when $\kappa = -1/4$, there exist an infinite number of algebraically decaying solutions $f(\eta)$ of the boundary-layer equation (10.3) with the property $f''(0) \geq f''_{\text{exp}}(0) = -0.1620$, where $f_{\text{exp}}(\eta)$ is the corresponding exponentially decaying solution.

The above conclusion has general meanings: as reported by Liao and Magyari (2006), when $-1/2 < \kappa < 0$, there exist an infinite number of algebraically decaying solutions $f(\eta)$ of the boundary-layer equation (10.3) with the property $f''(0) \geq f''_{\text{exp}}(0)$, where $f_{\text{exp}}(\eta)$ is the corresponding exponentially decaying solution. In other words, not only at $\kappa = -1/3$ but in the *whole* interval $-1/2 < \kappa < 0$, the exponentially decaying solution $f_{\text{exp}}(\eta)$ of the boundary-layer equation (10.3) is the limiting case of an infinite number of algebraically decaying solutions $f(\eta)$, as shown in Fig. 10.4.

Note that these algebraically decaying solutions have closed-form expression only when $\kappa = -1/3$. It should be emphasized that, the new algebraically decaying

Fig. 10.4 $f''(0)$ versus κ for the exponentially and algebraically decaying solutions of (10.3). Solid line: exponentially decaying solutions; Dashed area: algebraically decaying solutions.



solutions of the boundary-layer equation (10.3) in the whole interval $-1/2 < \kappa < 0$ (except $\kappa = -1/3$) are found (Liao and Magyari, 2006), for the first time, by means of the HAM. This shows the originality and general validity of the HAM.

All of above results are obtained by means of the BVPh 1.0, which is free available at

<http://numericaltank.sjtu.edu.cn/BVPh.htm>

In addition, the corresponding input data is given in the Appendix 10.2 and free available at the same website.

10.4 Concluding remarks

In this chapter, the BVPh 1.0 is applied to gain the exponentially and algebraically decaying solutions of boundary-layer flows, governed by a nonlinear ODE in an infinite interval. Thus, a lots of boundary-layer flows can be solved by means of the BVPh 1.0 in a similar way.

Note that it is *impossible* for numerical techniques to resolve the infinite interval with boundary conditions at infinity exactly. Many numerical packages like BVP4 regard the infinite interval as a kind of singularity and replace it by a finite one in practice. However, based on computer algebra system, the BVPh 1.0 can solve nonlinear ODEs in an infinite interval exactly. This is mainly because the computer algebra system provides us the ability to “compute with functions instead of numbers” (Trefethen, 2007), so that boundary conditions at infinity can be easily resolved by choosing proper base functions, as shown in this chapter.

Note that the BVPh 1.0 can be used to gain both exponentially and algebraically decaying solutions of nonlinear boundary-value ODEs in an infinite interval. This is mainly because, based on the HAM, the BVPh 1.0 provides us extremely large freedom and great flexibility to choose different types of auxiliary linear operators and

different base functions. Besides, the convergence-control parameter c_0 provides us a convenient way to guarantee the convergence of homotopy-series solution.

In addition, the unknown parameter σ in (10.15) and α in (10.33) also provide us one additional degree of freedom in initial guess, which supplies a convenient way to find out the optimal initial guess. Like c_0 , both of σ and α can be regarded as a convergence-control parameter: combined with c_0 together, each of them further improves our ability to guarantee the convergence of homotopy-series solution in the frame of the HAM, as illustrated in this chapter.

It should be emphasized that the infinite number of algebraically decaying solutions of (10.3) have closed-form solution only when $\kappa = -1/3$. It is by means of the HAM that new algebraically decaying solutions of (10.3) were found, for the first time, by Liao and Magyari (2006) in the whole interval $-1/2 < \kappa < 0$ (except $\kappa = -1/3$). In fact, by means of the HAM, new solutions of some other boundary-layer flows have been found by Liao (2005), which had been never reported and even neglected by numerical methods. Indeed, *a truly new method always gives something new and/or different*. All of these show the originality of the HAM.

The BVPh 1.0 can be used as an analytic tool to solve many nonlinear ODEs in an infinite interval, especially those related to boundary-layer flows. It can be even applied to solve some types of nonlinear PDEs in an infinite interval, such as the non-similarity boundary-layer flow as shown in Chapter 11, and the unsteady boundary-layer flow as shown in Chapter 12. However, these do not mean that the BVPh 1.0 is valid for *all* boundary-value problems in an infinite interval. As mentioned in Chapter 7, our aim is to develop a package for as many nonlinear boundary-value problems as possible. Thus, further modifications and more applications are needed in future.

Note that the Chebfun 4.0 also provides us the ability to “compute with functions instead of numbers” (Trefethen, 2007). So, it should be valuable to establish a similar HAM-based package for highly nonlinear multipoint BVPs with singularities by means of Chebfun, an open resource available (Accessed 15 April 2011) at

<http://www2.maths.ox.ac.uk/chebfun/>

Appendix 10.1 Input data of BVPh for exponentially decaying solution

```
(* Input Mathematica package BVPh version 1.0 *)
<BVPh.txt;

(* Define the physical and control parameters *)
TypeEQ      = 1;
ApproxQ     = 0;
ErrReq      = 10^(-10);
zRintegral  = 10;
```

```

(* Define the governing equation *)
kappa = -1/4;
f[z_,u_,lambda_] := D[u,{z,3}]
                    + (1+kappa)/2*u*D[u,{z,2}] - kappa*D[u,z]^2 ;

(* Define Boundary conditions *)
zR = infinity;
OrderEQ = 3;
BC[1,z_,u_,lambda_] := Limit[u, z -> 0 ];
BC[2,z_,u_,lambda_] := Limit[D[u,z] - 1, z -> 0 ];
BC[3,z_,u_,lambda_] := Limit[D[u,z], z -> zR ];

(* Define initial guess *)
temp[1] = (1-2*sigma);
temp[2] = (1-sigma);
u[0] = sigma+temp[1]*Exp[-z]-temp[2]*Exp[-2*z];

(* Define output term *)
output[z_,u_,k_] := Print["output          = ",
                          D[u[k],{z,2}] /. z->0//N];

(* Defines the auxiliary linear operator *)
L[u_] := D[u,{z,3}] - D[u,z];

(* Print input and control parameters *)
PrintInput[u[z]];

(* Set optimal c0 and sigma *)
c0 = -5/4 ;
sigma = 3;
Print[" c0 = ",c0, "      sigma = ",sigma];

(* Gain up to 10th-order approximation *)
BVPh[1,10];

```

The above input data of the BVPh (version 1.0) for the exponentially decaying solution of (10.3) is free available at

<http://numericaltank.sjtu.edu.cn/BVPh.htm>

Appendix 10.2 Input data of BVPh for algebraically decaying solution

```

(* Input Mathematica package BVPh version 1.0 *)
<<BVPh.txt;

(* Define the physical and control parameters *)
TypeEQ = 1;

```

```

ApproxQ      = 0;
ErrReq       = 10^(-10);
zRintegral   = 10;
H[z_]       := z;

(* Define the governing equation *)
kappa = -1/3;
alpha = 2/5;
f[z_,u_,lambda_] := alpha^2*D[u,{z,3}]
                  + (1+kappa)/2*u*D[u,{z,2}] - kappa*D[u,z]^2;

(* Define Boundary conditions *)
zL = 1;
zR = infinity;
OrderEQ = 3;
BC[1,z_,u_,lambda_] := Limit[u, z -> 1];
BC[2,z_,u_,lambda_] := Limit[D[u,z] - 1, z -> 1];
BC[3,z_,u_,lambda_] := Limit[D[u,{z,2}] - mu/alpha,
                              z -> 1];

(* Define initial guess *)
mu = 1;
beta = (1+kappa)/(1-kappa);
u[0] = Module[{temp},
temp[1] = (4-3*beta+mu/alpha)/(2-beta);
temp[2] = (3-3*beta+mu/alpha)/(1-beta);
temp[3] = (2-2*beta+mu/alpha)/(1-beta)/(2-beta);
temp[1]*z^beta - temp[2]*z^(beta-1)
      + temp[3]*z^(2*beta-2)
];

(* Define output term *)
output[z_,u_,k_] := Print["output" = ",
                        alpha^2*D[u[k],{z,3}] /. z->1//N];

(* Define the auxiliary linear operator *)
(L[u_] := Module[{temp},
temp[1] = 2*(2*beta-3);
temp[2] = (beta-1)*(5*beta-6);
temp[3] = 2*beta*(beta-1)^2;
z^3*D[u,{z,3}] - temp[1]*z^2*D[u,{z,2}]
      + temp[2]*z*D[u,z] - temp[3]*u
];

(* Print input and control parameters *)
PrintInput[u[z]];

(* Exact solution when kappa = -1/3 *)
Uexact = Module[{temp,t,t0,Ai,Bi,Ait,Bit},
Ai = AiryAi[t];
Bi = AiryBi[t];
Ait = D[Ai,t];
Bit = D[Bi,t];
Ait0 = Ait /. t -> t0;

```

```

Bit0 = Bit /. t -> t0;
temp[1] = Bit0*Ait - Ait0*Bit;
temp[2] = Bit0*Ai - Ait0*Bi ;
t0 = (Sqrt[6]*mu)^(-2/3);
t = t0*(1 + mu*x);
(36*mu)^(1/3)*temp[1]/temp[2]//Expand
];
Uzexact = D[Uexact,x];

(* Coordinate transform *)
Wz[k_] := Uz[k] /. z -> 1 + alpha*x;

(* Set optimal c0 *)
c0 = -8;
Print[" c0 = ",c0];

(* Gain up to 10th-order approximation *)
BVPh[1,30];

```

The above input data of the BVPh (version 1.0) for the algebraically decaying solution of (10.3) is free available at

<http://numericaltank.sjtu.edu.cn/BVPh.htm>

Note that, for nonlinear boundary-value problems defined in an infinite interval, we must set $zR = \text{infinity}$ in the input data of BVPh (version 1.1).

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Chapter 11

Non-similarity Boundary-layer Flows

Abstract In this chapter, we illustrate the validity of the HAM-based Mathematica package BVPh (version 1.0) for nonlinear partial differential equations (PDEs) related to non-similarity boundary-layer flows. We show that, using BVPh 1.0, a non-similarity boundary-layer flow can be solved in a rather similar way to that for similarity ones governed by nonlinear ODEs. In other words, in the frame of the HAM, solving non-similarity boundary-layer flows is as easy as similarity ones. This shows the validity of the BVPh 1.0 for some nonlinear PDEs, especially for those related to boundary-layer flows.

11.1 Introduction

In the previous chapters of Part II, we illustrate that the HAM-based Mathematica package BVPh (version 1.0) provides us an analytic tool to solve boundary-value problems governed by a nonlinear ordinary differential equation (ODE) either in a finite interval $z \in [0, a]$ or in an infinite interval $z \in [b, +\infty)$, where $a > 0$ and $b \geq 0$ are bounded constants. In this and the next chapter, we illustrate that the BVPh (version 1.0) can be even applied to solve some nonlinear partial differential equations (PDEs), especially those related to boundary-layer flows.

Without loss of generality, let us consider here a nonlinear PDE arising from a kind of non-similarity boundary-layer flow. Since Prandtl (1904) proposed the revolutionary concept of boundary-layer flows of viscous fluid in 1904, the boundary-layer theory (Blasius, 1908; Howarth, 1938; Van Dyke, 1962; Schlichting and Gersten, 2000; Sobey, 2000; Magyari and Keller, 2000) has been developing greatly and applied in nearly all regions of fluid mechanics (Tani, 1977). When similarity solutions exist, boundary-layer flows are governed by nonlinear ODEs. However, when there exist no such kind of similarity, one had to solve nonlinear PDEs, which are much more difficult to solve than ODEs. Mainly due to this reason, most of researchers in this field focused on similarity boundary-layer flows: thousands of articles related to similarity boundary-layer flows have been published, but in con-

trast to the large number of publications in similarity boundary-layer flows (Blasius, 1908; Howarth, 1938; Van Dyke, 1962; Schlichting and Gersten, 2000; Sobey, 2000; Magyari and Keller, 2000), articles on non-similarity flows (Stewartson and Williams, 1965; Gorla and Kumari, 1998; Duck et al., 1999; Massoudi, 2001; Cheng and Lin, 2002; Cimpean et al., 2006) are much less.

For example, let us consider here a non-similarity boundary-layer flow of Newtonian fluid over a stretching flat sheet (Crane, 1970; Banks, 1983). Two forces in opposite directions with same magnitude are added along the sheet. Thus, there is a rest point on the sheet, which is defined as the origin of the coordinate system. The x and y axes are along and perpendicular to the sheet, respectively. The fluid is at rest far from the sheet (i.e. as $y \rightarrow +\infty$). Due to the symmetry of flows, we can only consider the flows in the upper quarter plane $x \geq 0$ and $y \geq 0$. Let $U_w(x)$ denote the stretching velocity of the sheet, (u, v) the velocity components, ν the kinematic viscosity of the fluid, respectively. As mentioned by Prandtl (1904), the velocity variation across the flow direction is much larger than that in the flow direction, so that there exists a thin boundary-layer near the sheet. In the frame of the boundary-layer theory, this kind of viscous flow is governed by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (11.1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}, \quad (11.2)$$

subject to the boundary conditions

$$u = U_w(x), \quad v = 0 \quad \text{at } y = 0 \quad (11.3)$$

and

$$u = 0, \quad \frac{\partial v}{\partial x} = 0 \quad \text{at } x = 0, \quad (11.4)$$

$$u \rightarrow 0 \quad \text{as } y \rightarrow +\infty. \quad (11.5)$$

The similarity solutions exist only in some special cases of $U_w(x)$. Following Görtler (1952), we define the so-called principal function

$$\Delta(x) = \frac{U'_w(x)}{U_w^2(x)} \int_0^x U_w(\xi) d\xi. \quad (11.6)$$

When $\Delta(x)$ equals to a constant β , i.e.

$$\frac{U'_w(x)}{U_w^2(x)} \int_0^x U_w(\xi) d\xi = \beta, \quad (11.7)$$

there exists the similarity solution

$$\psi = \sqrt{\nu \int_0^x U_w(\xi) d\xi} g(\eta), \quad \eta = \frac{U_w(x)}{\sqrt{\nu \int_0^x U_w(\xi) d\xi}} y, \quad (11.8)$$

where ψ is a stream function, and $g(\eta)$ is governed by a nonlinear ODE

$$g''' + \frac{1}{2}gg'' - \beta g'^2 = 0, \quad g(0) = 0, \quad g'(0) = 1, \quad g'(+\infty) = 0. \quad (11.9)$$

The corresponding local coefficient of skin friction of the similarity boundary-layer flows reads

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U_w^2(x)} = 2g''(0) \sqrt{\frac{v}{\int_0^x U_w(\xi) d\xi}}. \quad (11.10)$$

For example, in case of $U_w(x) = ax^\lambda$, where $a(1+\lambda) > 0$, it holds

$$\beta = \lambda / (1 + \lambda),$$

so that the similarity criteria (11.7) is satisfied and therefore there exists the similarity solution

$$\psi = \sqrt{\frac{av}{(1+\lambda)}} x^{\frac{\lambda+1}{2}} g(\eta), \quad \eta = \sqrt{\frac{a(1+\lambda)}{v}} x^{\frac{\lambda-1}{2}} y. \quad (11.11)$$

In this case, the system of the two coupled PDEs (11.1) and (11.2) becomes one ODE (11.9) that is much easier to solve than the original ones. So, from the viewpoint of mathematics, the problem is greatly simplified when similarity solutions exist.

For similarity boundary-layer flows, all velocity profiles at different x are similar. However, such kind of similarity is lost for non-similarity flows (Wanous and Sparrow, 1965; Stewartson and Williams, 1965; Sparrow and Quack, 1970; Sparrow and Yu, 1971; Gorla and Kumari, 1998; Duck et al., 1999; Massoudi, 2001; Cheng and Lin, 2002; Cimpean et al., 2006). Physically speaking, the non-similarity boundary-layer flows are more general in practice, and thus are more important than similarity ones.

When similarity solutions do not exist, one had to solve a nonlinear PDE. Traditionally, there are two different approaches: analytic and numerical ones. Numerical methods are widely applied to investigate non-similarity boundary-layer flows. As shown by Sahu et al. (2000) and Roy et al. (2007), one can use numerical methods to obtain approximate results at a large number of discretized points. However, one had to replace an infinite interval by a finite ones, and this results in some additional inaccuracy and uncertainty into numerical results. On the other hand, by means of analytical methods, one can solve nonlinear PDEs in an infinite interval. However, it is a pity that, using the traditional analytic techniques such as perturbation techniques, it is hard to get analytic approximations that are valid and accurate for *all* physical variables in the whole interval. This is mainly because perturbation methods are often dependent on small physical variables or parameters, and thus perturbation results are often invalid for *all* physical parameters/variables. Currently, Cimpean et al. (2006) applied the perturbation techniques, combined with numerical techniques, to solve a free convection non-similarity boundary-layer problem over a vertical flat sheet in a porous medium. Like most of perturbation solutions,

their results are valid only for small or large x , which are regarded as perturbation quantities.

In addition, the so-called “method of local similarity” (Sparrow and Yu, 1971; Massoudi, 2001) for non-similarity boundary-layer flows is based on such an assumption that non-similarity terms in governing equations are so small that they can be regarded as zero and thus the original PDEs become an ODE. However, the results given by “the method of local similarity” are of “uncertain accuracy”, as pointed out by Sparrow and Yu (1971), and valid only for small variables in general, as pointed out by Massoudi (2001), respectively. This is easy to understand, because non-similarity terms are certainly *not* zero and must be considered. Sparrow et al. (Wanous and Sparrow, 1965; Sparrow and Quack, 1970; Sparrow and Yu, 1971) introduced the so-called “method of local non-similarity”, which was applied by Massoudi (2001) to solve a non-similarity flow of non-Newtonian fluid over a wedge. Differentiating the original governing equations by a dimensionless variable ξ along the free stream velocity, Massoudi (2001) gave two additional auxiliary nonlinear PDEs for both momentum and energy equations, then regarded the variable ξ in these two PDEs to be a *constant* so as to reduce them as a system of ODEs, and finally used numerical techniques to solve the more complicated system of four equations. It is a pity that Massoudi (2001) only gave numerical results for small ξ , although it was reported (Wanous and Sparrow, 1965; Stewartson and Williams, 1965) that the results given by “the method of local non-similarity” agree well with numerical or series solutions in some cases.

The above-mentioned attempts reveal the mathematical difficulties for non-similarity boundary-layer flows. This might be the reason why the publications about non-similarity boundary-layer flows are much less than those for similarity ones, although the former is more important than the latter, not only in theory but also in applications.

By means of the stream function ψ , (11.1) is automatically satisfied, and then (11.2) becomes such a nonlinear PDE

$$v \frac{\partial^3 \psi}{\partial y^3} + \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} = 0, \quad (11.12)$$

subject to the boundary conditions

$$\psi = 0, \quad \frac{\partial \psi}{\partial y} = U_w(x) \quad \text{at } y = 0, \quad \frac{\partial \psi}{\partial y} \rightarrow 0 \quad \text{as } y \rightarrow +\infty. \quad (11.13)$$

Using the transformation

$$\eta = \frac{y}{v^{1/2} \sigma(x)}, \quad \psi = v^{1/2} \sigma(x) f(\eta, x), \quad (11.14)$$

where $\sigma(x) > 0$ is a real function to be chosen later, we have the velocities

$$u = \frac{\partial f}{\partial \eta}, \quad v = v^{1/2} \left[\sigma'(x) \left(\eta \frac{\partial f}{\partial \eta} - f \right) - \sigma(x) \frac{\partial f}{\partial x} \right].$$

Then, the governing equation (11.12) becomes

$$\frac{\partial^3 f}{\partial \eta^3} + \frac{1}{2}[\sigma^2(x)]' f \frac{\partial^2 f}{\partial \eta^2} + \sigma^2(x) \left(\frac{\partial f}{\partial x} \frac{\partial^2 f}{\partial \eta^2} - \frac{\partial f}{\partial \eta} \frac{\partial^2 f}{\partial x \partial \eta} \right) = 0, \quad (11.15)$$

subject to the boundary conditions

$$f(0, x) = 0, \quad f_\eta(0, x) = U_w(x), \quad f_\eta(+\infty, x) = 0, \quad (11.16)$$

where f_η denotes the partial differentiation of $f(\eta, x)$ with respect to η . Note that (11.15) is a nonlinear PDE with variable coefficients $[\sigma^2(x)]'/2$ and $\sigma^2(x)$.

There exist an infinite number of sheet stretching velocity $U_w(x)$ which do not satisfy the similarity-criteria (11.7). Here, without loss of generality, let us consider the case $U_w(x) = \check{U}_w(\xi)$, where $\xi = \Gamma(x)$ defines a kind of transform. Then, (11.15) becomes

$$\frac{\partial^3 f}{\partial \eta^3} + \sigma_1(\xi) f \frac{\partial^2 f}{\partial \eta^2} + \sigma_2(\xi) \left(\frac{\partial f}{\partial \xi} \frac{\partial^2 f}{\partial \eta^2} - \frac{\partial f}{\partial \eta} \frac{\partial^2 f}{\partial \xi \partial \eta} \right) = 0, \quad (11.17)$$

subject to the boundary conditions

$$f(0, \xi) = 0, \quad f_\eta(0, \xi) = \check{U}_w(\xi), \quad f_\eta(+\infty, \xi) = 0, \quad (11.18)$$

where

$$\sigma_1(\xi) = \frac{1}{2}[\sigma^2(x)]', \quad \sigma_2(\xi) = \Gamma'(x) \sigma^2(x), \quad (11.19)$$

in which x is expressed by ξ , i.e. $x = \Gamma^{-1}(\xi)$. For example, when $\sigma(x) = \sqrt{1+x}$ and $\xi = \Gamma(x) = x/(1+x)$, we have $\sigma_1(\xi) = 1/2$ and $\sigma_2(\xi) = 1 - \xi$. For details, please refer to Liao (2009b).

The corresponding local coefficient of skin friction of the non-similarity boundary-layer flows is given by

$$C_f(x) = \frac{\tau(x)}{\frac{1}{2}\rho U_w^2(x)} = \frac{2\nu^{1/2}}{\sigma(x) U_w^2(x)} \left. \frac{\partial^2 f}{\partial \eta^2} \right|_{\eta \rightarrow 0}. \quad (11.20)$$

So, it is important to get accurate results of $f_{\eta\eta}(0, \xi)$. The replacement boundary-layer thickness $\bar{\delta}(x)$ is given by

$$\bar{\delta}(x) = \frac{1}{U_w(x)} \int_0^{+\infty} u(x, y) dy. \quad (11.21)$$

11.2 Brief mathematical formulas

In the frame of the homotopy analysis method (HAM) (Liao, 1992, 1997, 1999a,b; Liao and Campo, 2002; Liao, 2003a,b, 2004, 2005, 2006a,b, 2009a,b, 2010a,b; Liao

and Pop, 2004; Liao et al., 2006; Liao and Magyari, 2006; Liao and Tan, 2007), the non-similarity flow governed by the nonlinear PDE (11.17) can be solved *without* any additional assumptions, as shown by Liao (2009b). This is completely different from all analytic methods mentioned above. As shown below, this kind of non-similarity boundary-layer flow can be solved even by means of the BVPb 1.0 in a rather similar way to the similarity flows mentioned in Chapter 10.

According to (11.17), we define a nonlinear operator

$$\mathcal{N}(f) = \frac{\partial^3 f}{\partial \eta^3} + \sigma_1(\xi) f \frac{\partial^2 f}{\partial \eta^2} + \sigma_2(\xi) \left(\frac{\partial f}{\partial \xi} \frac{\partial^2 f}{\partial \eta^2} - \frac{\partial f}{\partial \eta} \frac{\partial^2 f}{\partial \xi \partial \eta} \right). \quad (11.22)$$

Let $q \in [0, 1]$ denote the homotopy-parameter, $c_0 \neq 0$ the convergence-control parameter, $H(\eta) \neq 0$ an auxiliary function, $\mathcal{L}$ an auxiliary linear operator with the property

$$\mathcal{L}(0) = 0, \quad (11.23)$$

and $f_0(\eta, \xi)$ an initial guess that satisfies the boundary conditions (11.18), respectively. Note that the HAM provides us extremely large freedom to choose the auxiliary linear operator $\mathcal{L}$ and the initial guess $f_0(\eta, \xi)$. In the frame of the HAM, we first construct such a continuous variation (or deformation) $\phi(\eta, \xi; q)$ that, as q increases from 0 to 1, $\phi(\eta, \xi; q)$ varies from the initial guess $f_0(\eta, \xi)$ to the solution $f(\eta, \xi)$ of (11.17) and (11.18). Such kind of continuous variation (or mapping) is governed by the so-called zeroth-order deformation equation

$$(1 - q)\mathcal{L}[\phi(\eta, \xi; q) - f_0(\eta, \xi)] = q c_0 H(\eta) \mathcal{N}[\phi(\eta, \xi; q)], \quad (11.24)$$

subject to the boundary conditions on the sheet

$$\phi(0, \xi; q) = 0, \quad \phi_\eta(0, \xi; q) = \check{U}_w(\xi), \quad (11.25)$$

and the boundary condition at infinity

$$\phi_\eta(+\infty, \xi; q) \rightarrow 0. \quad (11.26)$$

Note that the initial guess $f_0(\eta, \xi)$ satisfies the boundary conditions (11.18), and besides $\mathcal{L}$ has the property (11.23). Thus, when $q = 0$, we have the initial guess

$$\phi(\eta, \xi; 0) = f_0(\eta, \xi). \quad (11.27)$$

When $q = 1$, since $c_0 \neq 0$, the zeroth-order deformation equations (11.24) to (11.26) are equivalent to the original equations (11.17) and (11.18), provided

$$\phi(\eta, \xi; 1) = f(\eta, \xi). \quad (11.28)$$

Thus, as the embedding parameter q increases from 0 to 1, $\phi(\eta, \xi; q)$ indeed varies *continuously* from the initial guess $f_0(\eta, \xi)$ to the exact solution $f(\eta, \xi)$ of the original equations (11.17) and (11.18).

Then, expanding $\phi(\eta, \xi; q)$ in Maclaurin series with respect to q and using (11.27), we have the homotopy-Maclaurin series

$$\phi(\eta, \xi; q) = f_0(\eta, \xi) + \sum_{m=1}^{+\infty} f_m(\eta, \xi) q^m, \quad (11.29)$$

where

$$f_m(\eta, \xi) = \mathcal{D}_m [\phi(\eta, \xi; q)] = \frac{1}{m!} \frac{\partial^m \phi(\eta, \xi; q)}{\partial q^m} \Big|_{q=0}$$

is the m th-order homotopy-derivative of $\phi(\eta, \xi; q)$, and $\mathcal{D}_m$ is the m th-order homotopy-derivative operator, respectively. Note that the convergence of the homotopy-Maclaurin series (11.29) depends on the initial guess $f_0(\eta, \xi)$, the auxiliary linear operator $\mathcal{L}$, the auxiliary function $H(\eta)$, and the convergence-control parameter c_0 . Assuming that all of them are so properly chosen that the homotopy-Maclaurin series (11.29) absolutely converges at $q = 1$, we have due to (11.28) the homotopy-series solution

$$f(\eta, \xi) = f_0(\eta, \xi) + \sum_{m=1}^{+\infty} f_m(\eta, \xi). \quad (11.30)$$

According to Theorem 4.15, we have the m th-order deformation equation

$$\mathcal{L} [f_m(\eta, \xi) - \chi_m f_{m-1}(\eta, \xi)] = c_0 H(\eta) \delta_{m-1}(\eta, \xi), \quad (11.31)$$

subject to the boundary conditions on the sheet

$$f_m = 0, \quad \frac{\partial f_m}{\partial \eta} = 0, \quad \text{at } \eta = 0 \quad (11.32)$$

and the boundary condition at infinity

$$\frac{\partial f_m}{\partial \eta} \rightarrow 0, \quad \text{as } \eta \rightarrow +\infty, \quad (11.33)$$

where

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1, \end{cases} \quad (11.34)$$

and

$$\begin{aligned} \delta_n(\eta, \xi) = & \frac{\partial^3 f_n}{\partial \eta^3} + \sigma_1(\xi) \sum_{k=0}^n f_{n-k} \frac{\partial^2 f_k}{\partial \eta^2} \\ & + \sigma_2(\xi) \sum_{k=0}^n \left(\frac{\partial f_k}{\partial \xi} \frac{\partial^2 f_{n-k}}{\partial \eta^2} - \frac{\partial f_k}{\partial \eta} \frac{\partial^2 f_{n-k}}{\partial \xi \partial \eta} \right) \end{aligned} \quad (11.35)$$

is gained by Theorem 4.1. For details, please refer to Chapter 4.

It should be emphasized here that the high-order deformation equations (11.31) to (11.33) are *linear*. Besides, unlike perturbation techniques, we do *not* need any

small/large physical parameters to obtain these linear differential equations. Furthermore, different from “the method of local similarity” (Sparrow and Yu, 1971; Massoudi, 2001) and “the method of local non-similarity” (Wanous and Sparrow, 1965; Sparrow and Quack, 1970; Sparrow and Yu, 1971), we *neither* enforce the non-similarity terms to be zero, *nor* regard the variable ξ as a constant. In a word, different from all other previous analytic methods for the non-similarity flows, our approach does *not* need any additional assumptions. More importantly, as mentioned before, we have extremely large freedom to choose $\mathcal{L}$: this freedom is so large that, in the frame of the HAM, a nonlinear PDE can be (although sometimes) transferred into an infinite number of linear ODEs, as shown below.

Mathematically, the essence to approximate a nonlinear differential equation is to find a set of *proper* base functions to fit its solutions. Physically, it is well-known that most of viscous flows decay exponentially at infinity (i.e. $\eta \rightarrow +\infty$). So, for the non-similarity boundary-layer flows over a stretching flat sheet, the velocities u and v should decay exponentially as $\eta \rightarrow +\infty$. Therefore, $f(\eta, \xi)$ should be in the form

$$f(\eta, \xi) = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} a_{m,n}(\xi) \eta^n \exp(-m\eta), \quad (11.36)$$

where $a_{m,n}(\xi)$ is a polynomial of ξ . This expression is called *the solution-expression* of $f(\eta, \xi)$, which plays an important role in the frame of the HAM, as shown below.

To satisfy the solution-expression (11.36) and the boundary conditions (11.18), we choose the initial guess

$$f_0(\eta, \xi) = \check{U}_w(\xi) (1 - e^{-\eta}), \quad (11.37)$$

which contains the simplest but leading terms of (11.36) as $\eta \rightarrow +\infty$. Note that $f_0(\eta, \xi)$ satisfies the boundary conditions (11.18) and decays exponentially at infinity.

As mentioned before, we have extremely large freedom to choose the auxiliary linear operator $\mathcal{L}$. This freedom is however restricted by the solution-expression (11.36) and the boundary conditions (11.18). Note that the original governing equation (11.17) is a nonlinear PDE with variable coefficients. So, if we choose a partial differential operator as $\mathcal{L}$, the high-order deformation equation (11.31) is a PDE. It is well-known that a PDE with variable coefficients is more difficult to solve than an ODE with constant coefficients. So, mathematically, it is much easier to solve (11.31) if $\mathcal{L}$ is a linear differential operator which contains derivatives with respect to either η or ξ only, and besides without any variable coefficients. Physically, for boundary-layer flows, the velocity variation across the flow direction is much larger than that in the flow direction. Therefore, the derivatives

$$\frac{\partial f}{\partial \eta}, \frac{\partial^2 f}{\partial \eta^2}, \frac{\partial^3 f}{\partial \eta^3}$$

across the flow direction are considerably larger and thus physically more important than the derivatives

$$\frac{\partial f}{\partial \xi}, \frac{\partial^2 f}{\partial \xi \partial \eta}$$

in the flow direction. Considering all of these mentioned above, we choose the auxiliary linear operator

$$\mathcal{L}(f) = \frac{\partial^3 f}{\partial \eta^3} - \frac{\partial f}{\partial \eta}, \quad (11.38)$$

which is independent of ξ , and besides does not contain any variable coefficients. For details, please refer to Liao (2009b). Notice that the auxiliary linear operator (11.38) is exactly the *same* as the auxiliary linear operator (10.16) used in Sect. 10.2 to solve a kind of *similarity* boundary-layer flow!

Using the initial guess (11.37) and the auxiliary linear operator (11.38), it is easy to solve the *linear* ODEs (11.31) to (11.33). The special solution of (11.31) reads

$$f_m^*(\eta, \xi) = \chi_m f_{m-1}(\eta, \xi) + c_0 \mathcal{L}^{-1}[H(\eta) \delta_{m-1}(\eta, \xi)], \quad (11.39)$$

where $\mathcal{L}^{-1}$ denotes the inverse operator of $\mathcal{L}$. Then, the solution of the high-order deformation equations (11.31) to (11.33) is

$$f_m(\eta, \xi) = f_m^*(\eta, \xi) + C_0(\xi) + C_1(\xi) \exp(-\eta),$$

where

$$C_0(\xi) = -f_m^*(0, \xi) - \left. \frac{\partial f_m^*}{\partial \eta} \right|_{\eta=0}, \quad C_1(\xi) = \left. \frac{\partial f_m^*}{\partial \eta} \right|_{\eta=0}$$

are determined by the boundary conditions (11.32) and (11.33). In this way, it is easy to solve the high-order deformation equations (11.31) to (11.33), especially by means of computer algebra system such as Mathematica. For details about mathematical formulas, please refer to Liao (2009b).

In this way, the original *nonlinear* PDE with *variable* coefficients is transferred into an infinite number of *linear* ODEs with *constant* coefficients. It should be emphasized that the high-order deformation equation (11.31) for the non-similarity boundary-layer flow is quite similar to the high-order deformation equations for similarity boundary-layer flows with exponentially decaying solutions in Chapter 10: both of them use the *same* auxiliary linear operator. In other words, in the frame of the HAM, the non-similarity boundary-layer flow can be solved in the similar way as similarity ones. This greatly simplifies solving the nonlinear PDEs related to non-similarity boundary-layer flows.

Finally, it should be emphasized that $f_m(\eta, \xi)$ contains the convergence-control parameter c_0 . As pointed out before, it is the convergence-control parameter c_0 which provides us a simple way to guarantee the convergence of the homotopy-series solution for *all* physical variables/parameters, as shown below.

11.3 Homotopy-series solution

Without loss of generality, let us consider such a sheet stretching velocity

$$U_w(x) = \frac{x}{1+x}$$

that the corresponding flow is a non-similarity ones, because it does not satisfy the similarity criteria (11.7).

In this case, the stretching velocity $U_w(x)$ increases monotonously from 0 to 1 along the sheet. In addition, $U_w \rightarrow x$ as $x \rightarrow 0$, and $U_w \rightarrow 1$ as $x \rightarrow +\infty$, respectively. Physically, the flow near $x = 0$ should be close to the similarity ones with $U_w = x$, and the flow at $x \rightarrow +\infty$ should be close to the similarity ones with $U_w = 1$, respectively. It is well-known that, for similarity boundary-layer flows related to $U_w = x$ and $U_w = 1$, the corresponding similarity variables are $y/\sqrt{v}$ and $y/\sqrt{vx}$, respectively. Therefore, according to the definition (11.14) of η , we choose

$$\sigma(x) = \sqrt{1+x}$$

so that η tends to the corresponding similarity variable $y/\sqrt{v}$ as $x \rightarrow 0$ and to the corresponding similarity variable $y/\sqrt{vx}$ as $x \rightarrow +\infty$, respectively. Besides, it is natural for us to define

$$\xi = \Gamma(x) = \frac{x}{1+x},$$

which gives, according to (11.19), that

$$\check{U}_w(\xi) = \xi, \quad \sigma_1(\xi) = \frac{1}{2}, \quad \sigma_2(\xi) = 1 - \xi.$$

Like similarity boundary-layer flows, the high-order deformation equations (11.31) to (11.33) of the non-similarity flow are linear ODEs. Therefore, we can solve it even by means of the BVPh 1.0 directly, which is given in the Appendix 7.1 and free available (Accessed 25 Nov 2011, will be updated in the future) at

<http://numericaltank.sjtu.edu.cn/BVPh.htm>

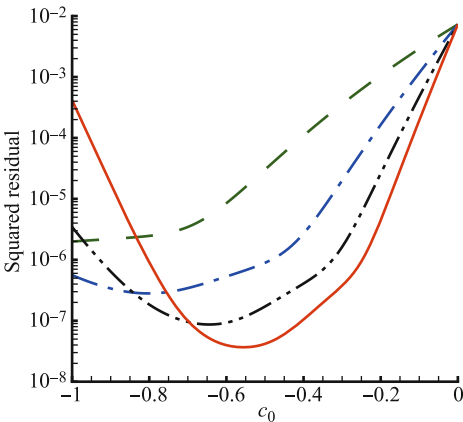
The corresponding input data for the BVPh 1.0 is given in the appendix of this chapter and free available at the above-mentioned website.

Up to now, only the auxiliary function $H(\eta)$ and the convergence-control parameter c_0 are not determined. For the sake of simplicity, we choose

$$H(\eta) = 1.$$

Then, the solution $f_m(\eta, \xi)$ of the high-order deformation equations (11.31) to (11.33) is dependent upon the convergence-control parameter c_0 only, which provides us a convenient way to guarantee the convergence of the homotopy-series solution, as mentioned in Chapter 2. The averaged squared residual of (11.17) over $\eta \in [0, 10]$ at the k th-order approximation is gained by means of the module

Fig. 11.1 Squared residual of (11.17) versus c_0 . Dashed line: 5th-order approximation; Dash-dotted line: 10th-order approximation; Dash-double-dotted line: 15th-order approximation; Solid line: 20th-order approximation.



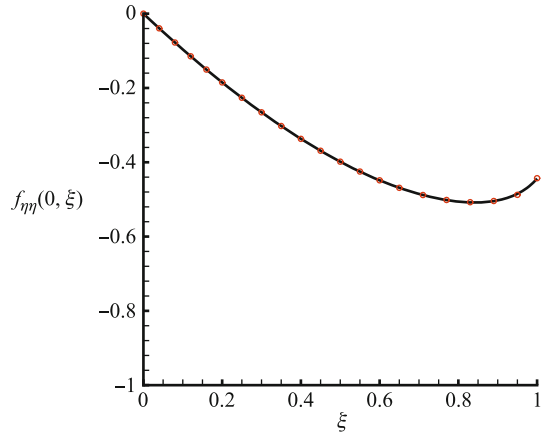
GetErr [k] of the BVPh 1.0, which is then integrated in the interval $\xi \in [0, 1]$. The corresponding squared residuals of (11.17) versus c_0 are as shown in Fig. 11.1. Note that, at the 10th and 15th-order approximation, the optimal convergence-control parameter c_0 is about -0.8 and -0.6, respectively. At the 20th-order of approximation, the minimum of the squared residual is 3.6×10^{-8} , corresponding to the optimal convergence-control parameter $c_0^* = -0.5554$. It is found that, when $c_0 = -1/2$, the homotopy-series solution (11.30) converges in the *whole* spatial interval $0 \leq \xi \leq 1$ and $0 \leq \eta < +\infty$, corresponding to $0 \leq x < +\infty$ and $0 \leq y < +\infty$, as shown in Table 11.1 and Fig. 11.2.

According to Fig. 11.1, the homotopy-series (11.30) diverges when $c_0 = -1$. As proved by Liao (2003b), some non-perturbation techniques, such as Lyapunov’s artificial small parameter method (Lyapunov, 1992), Adomian’s decomposition method (Adomian, 1976, 1994) and the δ -expansion method (Karmishin et al., 1990), are special cases of the HAM in case of $c_0 = -1$. Besides, it is proved (Sajid and Hayat, 2008; Liang and Jeffrey, 2009) that the so-called “homotopy perturbation method”

Table 11.1 Averaged squared residual of the governing equation (11.17) over $\eta \in [0, 10]$ and $\xi \in [0, 1]$ by means of $c_0 = -1/2$.

Order of approximation	Averaged squared residual of (11.17)
1	2.3×10^{-3}
2	7.4×10^{-4}
4	8.5×10^{-5}
6	1.1×10^{-5}
8	2.3×10^{-6}
10	8.9×10^{-7}
15	1.7×10^{-7}
20	4.2×10^{-8}
25	1.8×10^{-8}
30	1.0×10^{-8}

Fig. 11.2 Homotopy-approximation of $f_{\eta\eta}(0, \xi)$ by means of $c_0 = -1/2$. Solid line: 30th-order approximation; Symbols: 20th-order approximation.



(He, 1999) is also a special case of the HAM when $c_0 = -1$. So, if any of the above-mentioned methods are used, one can not gain convergent results. This illustrates once again the importance of the convergence-control parameter c_0 , and also the validity of the HAM for highly nonlinear problems.

It is important to give accurate local coefficient of skin friction of the non-similarity boundary-layer flow, which is related to $f_{\eta\eta}(0, \xi)$ via (11.20). When $c_0 = -1/2$, the 15th-order HAM approximation reads

$$\begin{aligned} f_{\eta\eta}(0, \xi) = & -\xi + 0.357142 \xi^2 + 0.0745784 \xi^3 + 0.0329563 \xi^4 + 0.0187480 \xi^5 \\ & + 0.0121641 \xi^6 + 0.00856336 \xi^7 + 0.00638095 \xi^8 + 0.00468866 \xi^9 \\ & + 0.0125286 \xi^{10} - 0.108975 \xi^{11} + 0.729854 \xi^{12} - 2.509780 \xi^{13} \\ & + 4.702786 \xi^{14} - 4.475992 \xi^{15} + 1.690259 \xi^{16}, \end{aligned} \quad (11.40)$$

which agrees well with the 20th-order approximations and is accurate in the whole region $0 \leq \xi \leq 1$, as shown in Fig. 11.2. Then, it is straightforward to calculate the local coefficient C_f of the skin friction by (11.20). It is found that $C_f \rightarrow -2\sqrt{v}/x$ as $x \rightarrow 0$, and $C_f \rightarrow -0.8875\sqrt{v}/x$ as $x \rightarrow +\infty$, respectively, as shown in Fig. 11.3. Besides, the boundary-layer thickness $\bar{\delta}(x)$ of the non-similarity flow tends to $\sqrt{v}$ as $x \rightarrow 0$ and $1.61613\sqrt{v}x$ as $x \rightarrow +\infty$, respectively, as shown in Fig. 11.4. Note that the boundary-layer thickness of the corresponding similarity flow is just $\sqrt{v}$ in case of $U_w = x$ (near $x = 0$) and $1.61613\sqrt{v}x$ in case of $U_w = 1$ (at $x \rightarrow +\infty$), respectively. Thus, the homotopy-series solution of the non-similarity flow gives

$$C_f(x) \rightarrow -2\sqrt{v}/x, \quad \delta(x) \rightarrow \sqrt{v}, \quad \text{as } x \rightarrow 0$$

and

$$C_f(x) \rightarrow -0.8875\sqrt{v}/x, \quad \delta(x) \rightarrow 1.61613\sqrt{v}x, \quad \text{as } x \rightarrow +\infty,$$

respectively. Therefore, physically speaking, the non-similarity boundary-layer flow in the region $x \rightarrow 0$ and $x \rightarrow +\infty$ is very close to the corresponding similarity ones

Fig. 11.3 The local coefficient of skin friction $C_f(x)/\sqrt{v}$ of the non-similarity flow in case of $U_w = x/(1+x)$. Solid-line: 30th-order HAM result; Symbols: 20th-order HAM result; Dashed-line: $C_f(x) = -0.8875\sqrt{v}/x$; Dash-dotted line: $C_f(x) = -2\sqrt{v}/x$.

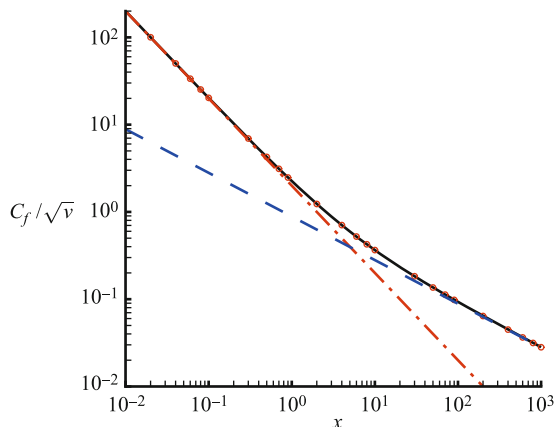
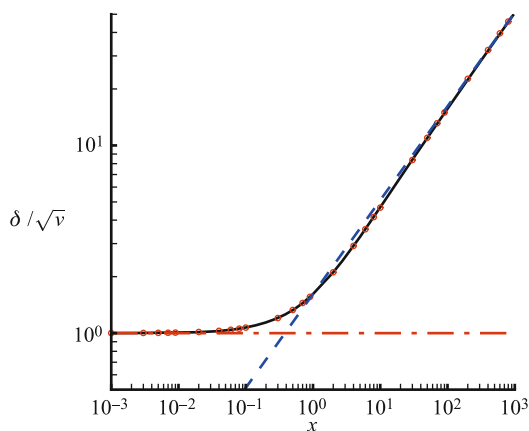


Fig. 11.4 The boundary-layer thickness $\delta(x)/\sqrt{v}$ of the non-similarity flow in case of $U_w = x/(1+x)$. Solid-line: 30th-order HAM result; Symbols: 20th-order HAM result; Dashed-line: $\delta(x) = 1.61613\sqrt{v}x$; Dash-dotted line: $\delta(x) = \sqrt{v}$.



in case of $U_w = x$ and $U_w = 1$, respectively. However, the flows in other regions are non-similarity ones, as shown in Fig. 11.3 and Fig. 11.4, respectively. The velocity profile $u \sim y/\sqrt{v}$ of the non-similarity boundary-layer flow at different x are given in Fig. 11.5. For more discussions on the physical meanings of the homotopy-series solution, please refer to Liao (2009b).

Therefore, by means of the BVPh 1.0, we successfully gain the convergent solution of the non-similarity boundary-layer flow governed by the nonlinear PDE (11.17), which is valid in the whole area $0 \leq x < +\infty$ and $0 \leq y < +\infty$.

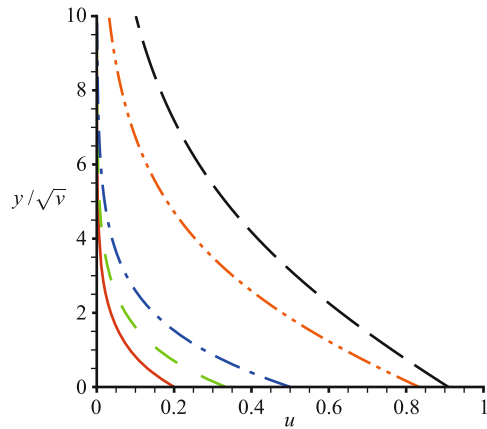
All of above results are given by means of $H(\eta) = 1$. Note that we have great freedom to choose the auxiliary function $H(\eta)$. For example, using

$$H(\eta) = \exp(-\eta),$$

we gain the series solution in the form

$$f(\eta, \xi) = \sum_{m=0}^{+\infty} b_m(\xi) \exp(-m\eta),$$

Fig. 11.5 The velocity profile $u \sim y/\sqrt{v}$ of the non-similarity flow at different x . Solid line: $x = 1/4$; Dashed line: $x = 1/2$; Dash-dotted line: $x = 1$; Dash-double-dotted line: $x = 5$; Long-dashed line: $x = 10$.



where $b_m(\xi)$ is a polynomial of ξ . In this case, we can gain convergent series solution by means of $c_0 = -3/2$ in a similar way. This illustrates once again the flexibility of the HAM.

11.4 Concluding remarks

We illustrate in this chapter that, in the frame of the HAM, the nonlinear PDE describing a kind of non-similarity boundary-layer flow can be solved directly by means of the BVPh 1.0 in a rather similar way to that for similarity ones. This shows the validity of the BVPh 1.0 for some nonlinear PDEs, especially for those related to boundary-layer flows.

Note that the original nonlinear PDE (11.17) contains the derivatives of $f(\eta, \xi)$ with respect to both η and ξ . However, we choose here the auxiliary linear operator (11.38), which only contains the derivatives with respect to η . Mathematically, this is mainly because the HAM provides us extremely large freedom to choose the auxiliary linear operator, as mentioned in Chapter 2. More importantly, the HAM also provides us a convenient way to guarantee the convergence of the homotopy-series solution by means of choosing a proper convergence-control parameter c_0 : the freedom on the choice of the auxiliary linear operator $\mathcal{L}$ has no meanings if one can not guarantee the convergence of homotopy-series solutions. It is interesting that the *same* auxiliary linear operator (11.38) is used by Liao and Pop (2004) to solve the *similarity* boundary-layer flows, too. In addition, the auxiliary linear operator (11.38) is exactly the *same* as the auxiliary linear operator (10.16) used in Sect. 10.2 to solve a kind of *similarity* boundary-layer flow. Thus, in the frame of the HAM, non-similarity boundary-layer flows can be solved in a similar way like similarity ones. Physically, it is mainly due to the existence of the boundary-layer: the velocity variation across the flow direction is much larger than that in the flow direction.

Thus, the BVPh 1.0 can be applied to solve other non-similarity boundary-layer flows (Kousar and Liao, 2010, 2011; You et al., 2010) in a rather similar way.

It is well-known that a nonlinear PDE is much more difficult to solve than a linear ODE. So, it is a good idea to replace a nonlinear PDE by a sequence of linear ODEs, if possible. Thus, this approach has general meanings. For example, in the frame of the HAM, a nonlinear PDE describing unsteady boundary-layer flows can be transferred into an infinite number of linear ODEs similar to (11.31), as shown by Liao (2006a,b). Besides, a nonlinear PDE describing an unsteady heat transfer can be replaced by an infinite number of linear ODEs, as shown by Liao et al. (2006). All of these kinds of nonlinear PDEs can be solved by means of the BVPh 1.0 in a similar way. This shows the general validity of the BVPh 1.0.

Finally, it should be pointed out that, although this approach has some general meanings, it is however *not* valid for all types of nonlinear PDEs, especially for those related to waves. As mentioned in Chapter 7, our aim is to develop a package valid for as many nonlinear boundary-value problems as possible. Even so, the HAM-based Mathematica package BVPh (version 1.0) can be used as a tool to solve many nonlinear PDEs, especially those related to non-similarity and/or unsteady boundary-layer flows and heat transfer.

Appendix 11.1 Input data of BVPh

```
(* Input Mathematica package BVPh version 1.0 *)
<<BVPh.txt;

(* Define the physical and control parameters *)
TypeEQ      = 1;
ApproxQ     = 0;
ErrReq      = 10^(-10);
NgetErr     = 100;
zRintegral  = 10;

(* Define the governing equation *)
f[z_,u_,lambda_] := Module[{temp},
temp[1] = D[u,{z,3}] + u*D[u,{z,2}]/2;
temp[2] = D[u,t]*D[u,{z,2}] - D[u,z]*D[u,z,t];
temp[1] + (1-t)*temp[2] // Expand
];

(* Define boundary conditions *)
zR = infinity;
OrderEQ = 3;
BC[1,z_,u_,lambda_] := u /. z-> 0 ;
BC[2,z_,u_,lambda_] := D[u,z] - t /. z -> 0 ;
BC[3,z_,u_,lambda_] := Limit[D[u,z], z -> zR ];

(* Define initial guess *)
```

```

u[0] = t*(1 - Exp[-z]);

(* Define the auxiliary linear operator *)
L[u_] := D[u,{z,3}] - D[u,z];

(* Define output term *)
output[z_,u_,k_] := Print["output" = ",
                          D[u[k] , {z,2}]/. {z->0,t->1}]/N];

(* Define Getdelta[k] *)
Getdelta[k_] := Module[{temp,i},
uz[k] = D[u[k],z]//Expand;
uzz[k] = D[uz[k],z]//Expand;
uzzz[k] = D[uzz[k],z]//Expand;
ut[k] = D[u[k],t]//Expand;
uzt[k] = D[uz[k],t]//Expand;
uzzu[k] = Sum[uzz[i]*u[k-i],{i,0,k}]/Expand;
uzuzt[k] = Sum[uz[i]*uzt[k-i],{i,0,k}]/Expand;
uzzut[k] = Sum[uzz[i]*ut[k-i],{i,0,k}]/Expand;
temp[1] = uzzz[k] + uzzu[k]/2 //Expand;
temp[2] = (1-t)*(uzzut[k] - uzuzt[k]);
delta[k] = temp[1] + temp[2]//Expand;
];

(* Print input and control parameters *)
PrintInput[u[z,t]];

(* Set convergence-control parameter c0 *)
c0 = -1/2;
Print["c0 = ",c0];

(* Gain 10th-order HAM approximation *)
BVPh[1,10];

(* Calculate the squared residual *)
For[k=2, k<=10, k=k+2,
GetErr[k];
err[k] = Integrate[Err[k],{t,0,1}];
Print[" k = ", k, " Squared residual = ",
      err[k]/N];
];

```

The Mathematica package BVPh (version 1.0) and the above input data are free available at

<http://numericaltank.sjtu.edu.cn/BVPh.htm>

Note that, for nonlinear boundary-value problems defined in an infinite interval, we must set $z_R = \text{infinity}$ in the input data file of BVPh (version 1.0). Besides, we must define the module `Getdelta` in the input data file so as to calculate the

term defined by (11.35), when a nonlinear PDE is solved by means of the BVPh (version 1.0).

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Chapter 12

Unsteady Boundary-layer Flows

Abstract In this chapter, we illustrate the validity of the HAM-based Mathematica package BVPh (version 1.0) for nonlinear partial differential equations (PDEs) related to unsteady boundary-layer flows. We show that, using BVPh 1.0, an unsteady boundary-layer flow can be solved in a rather similar way to that for steady-state similarity ones governed by nonlinear ODEs. In other words, in the frame of the HAM, solving unsteady boundary-layer flows is as easy as steady-state ones. This shows the validity of the BVPh 1.0 for some nonlinear PDEs, especially for those related to boundary-layer flows.

12.1 Introduction

In Chapter 11, we illustrate that the HAM-based Mathematica package BVPh (version 1.0) can be used to solve a kind of *non-similarity* boundary-layer flow governed by a nonlinear partial differential equation (PDE) in an infinite area. In this chapter, we further illustrate that the BVPh 1.0 can be used to gain accurate approximations of an *unsteady* similarity boundary-layer flow governed by a nonlinear PDE in the infinite spatial and temporal interval.

Without loss of generality, let us consider here an unsteady boundary-layer flow of viscous fluid developed by an impulsively stretching plate (Stewartson, 1951; Hall, 1969; Dennis, 1972; Watkins, 1975; Stewartson, 1973; Pop and Na, 1996; Wang et al., 1997; Seshadri et al., 2002; Nazar et al., 2004), governed by the PDE

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}, \quad (12.1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (12.2)$$

subject to the boundary conditions

$$t > 0 : u = ax, v = 0 \text{ at } y = 0, \quad (12.3)$$

$$t > 0 : u \rightarrow 0 \text{ as } y \rightarrow +\infty \quad (12.4)$$

and the initial conditions

$$t = 0 : u = v = 0 \text{ at all points } (x, y), \quad (12.5)$$

where ν is the kinematic viscosity, t denotes the time, (u, v) are the velocity components in the directions of increasing x, y , respectively. Here, we consider the case $a > 0$ only, corresponding to a stretching plate.

There exist similarity variables for this kind of flow. Following Seshadri et al. (2002) and Nazar et al. (2004), we use Williams and Rhyne's similarity transformation (Williams and Rhyne, 1980):

$$\psi = \sqrt{av\xi} x f(\eta, \xi), \quad \eta = \sqrt{\frac{a}{v\xi}} y, \quad \xi = 1 - \exp(-\tau), \quad \tau = at, \quad (12.6)$$

where ψ denotes the stream-function. Note that the new dimensionless time ξ is bounded in a finite interval

$$0 \leq \xi \leq 1,$$

corresponding to $0 \leq \tau < +\infty$. Besides, the similarity variable η is dependent upon not only the spatial variable y but also the time t so that the initial condition (12.5) is automatically satisfied. Thus, at any time $\tau \in [0, +\infty)$, i.e. $\xi \in [0, 1]$, it is still a similarity boundary-layer flow.

Using the above similarity transformation, the original PDEs become

$$\frac{\partial^3 f}{\partial \eta^3} + \frac{1}{2} (1 - \xi) \eta \frac{\partial^2 f}{\partial \eta^2} + \xi \left[f \frac{\partial^2 f}{\partial \eta^2} - \left(\frac{\partial f}{\partial \eta} \right)^2 \right] = \xi (1 - \xi) \frac{\partial^2 f}{\partial \eta \partial \xi}, \quad (12.7)$$

subject to the boundary conditions

$$f(0, \xi) = 0, \quad \frac{\partial f}{\partial \eta} \Big|_{\eta=0} = 1, \quad \frac{\partial f}{\partial \eta} \Big|_{\eta=+\infty} = 0. \quad (12.8)$$

This is still a nonlinear PDE. However, since the initial conditions (12.5) are automatically satisfied so that there exist only boundary conditions (12.8), the above nonlinear PDE is easier to solve than the original two coupled PDEs.

When $\xi = 0$, corresponding to $\tau = 0$, (12.7) becomes the Rayleigh type of equation

$$\frac{\partial^3 f}{\partial \eta^3} + \frac{1}{2} \eta \frac{\partial^2 f}{\partial \eta^2} = 0, \quad (12.9)$$

subject to

$$f(0, 0) = 0, \quad \frac{\partial f}{\partial \eta} \Big|_{\eta=0, \xi=0} = 1, \quad \frac{\partial f}{\partial \eta} \Big|_{\eta=+\infty, \xi=0} = 0. \quad (12.10)$$

The above equation has the closed-form solution

$$f(\eta, 0) = \eta \operatorname{erfc}(\eta/2) + \frac{2}{\sqrt{\pi}} [1 - \exp(-\eta^2/4)], \quad (12.11)$$

where $\operatorname{erfc}(\eta)$ is the complementary error function defined by

$$\operatorname{erfc}(\eta) = \frac{2}{\sqrt{\pi}} \int_{\eta}^{+\infty} \exp(-z^2) dz.$$

When $\xi = 1$, corresponding to $\tau \rightarrow +\infty$, we have from (12.7) that

$$\frac{\partial^3 f}{\partial \eta^3} + f \frac{\partial^2 f}{\partial \eta^2} - \left(\frac{\partial f}{\partial \eta} \right)^2 = 0, \quad (12.12)$$

subject to

$$f(0, 1) = 0, \quad \frac{\partial f}{\partial \eta} \Big|_{\eta=0, \xi=1} = 1, \quad \frac{\partial f}{\partial \eta} \Big|_{\eta=+\infty, \xi=1} = 0. \quad (12.13)$$

The above equation has the closed-form solution

$$f(\eta, 1) = 1 - \exp(-\eta). \quad (12.14)$$

So, as the time variable ξ increases from 0 to 1, $f(\eta, \xi)$ varies from the initial solution (12.11) to the steady-state solution (12.14). Note that, although $f'(+\infty, \xi) \rightarrow 0$ exponentially for all ξ , where the prime denotes the differentiation with respect to η , the initial solution (12.11) decays much more quickly as $\eta \rightarrow +\infty$ than the steady-state solution (12.14). So, mathematically, the initial solution (12.11) is different from the steady-state solution (12.14) in essence. This might be the reason why it is so hard to give an accurate analytic solution uniformly valid for all time $0 \leq \tau < +\infty$.

When $\xi = 0$ and $\xi = 1$, we have

$$\frac{\partial^2 f}{\partial \eta^2} \Big|_{\eta=0, \xi=0} = -\frac{1}{\sqrt{\pi}} \quad (12.15)$$

and

$$\frac{\partial^2 f}{\partial \eta^2} \Big|_{\eta=0, \xi=1} = -1, \quad (12.16)$$

respectively. The skin friction coefficient is given by

$$c_f^x(x, \xi) = (\xi Re_x)^{-1/2} f''(0, \xi), \quad 0 \leq \xi \leq 1, \quad (12.17)$$

where $Re_x = ax^2/\nu$ is the local Reynolds number.

12.2 Perturbation approximation

Like Seshadri et al. (2002) and Nazar et al. (2004), one can regard ξ as a small parameter to search for the perturbation approximation in the form

$$f(\eta, \xi) = g_0(\eta) + g_1(\eta) \xi + g_2(\eta) \xi^2 + \cdots = \sum_{m=0}^{+\infty} g_m(\eta) \xi^m. \quad (12.18)$$

Substituting it into (12.7) and (12.8), and balancing the coefficients of the like-power of ξ , we have the zeroth-order perturbation equation

$$g_0'''(\eta) + \frac{\eta}{2} g_0''(\eta) = 0, \quad g_0(0) = 0, \quad g_0'(0) = 1, \quad g_0'(+\infty) = 0, \quad (12.19)$$

and the m th-order ($k \geq 1$) perturbation equation

$$g_m'''(\eta) + \frac{\eta}{2} g_m''(\eta) - m g_m'(\eta) = \frac{\eta}{2} g_{m-1}''(\eta) - (m-1) g_{m-1}'(\eta) - \sum_{i=0}^{m-1} [g_i(\eta) g_{m-1-i}''(\eta) - g_i'(\eta) g_{m-1-i}'(\eta)], \quad (12.20)$$

subject to the boundary conditions

$$g_m(0) = 0, \quad g_m'(0) = 0, \quad g_m'(+\infty) = 0. \quad (12.21)$$

All of the above perturbation equations are linear ODEs with respect to η only.

The solution of the zeroth-order perturbation equation (12.19) reads

$$g_0(\eta) = f(\eta, 0) = \eta \operatorname{erfc}(\eta/2) + \frac{2}{\sqrt{\pi}} [1 - \exp(-\eta^2/4)], \quad (12.22)$$

where $\operatorname{erfc}(\eta/2)$ is a complementary error function and $\exp(-\eta^2/4)$ is a Gaussian distribution function, respectively. Substituting the above expression into (12.20) and (12.21) gives the 1st-order perturbation equation

$$g_1'''(\eta) + \frac{\eta}{2} g_1''(\eta) - g_1'(\eta) = \left[\frac{2}{\pi} - \frac{\eta}{2\sqrt{\pi}} + \frac{\eta}{\sqrt{\pi}} \operatorname{erfc}\left(\frac{\eta}{2}\right) \right] \exp\left(-\frac{\eta^2}{4}\right) - \frac{2}{\pi} \exp\left(-\frac{\eta^2}{2}\right) + \left[\operatorname{erfc}\left(\frac{\eta}{2}\right) \right]^2, \quad (12.23)$$

subject to the boundary condition

$$g_1(0) = 0, \quad g_1'(0) = 0, \quad g_1'(+\infty) = 0. \quad (12.24)$$

Although the above equation is a linear ODE, it is however difficult to solve. First, even the homogeneous equation

$$g_1'''(\eta) + \frac{\eta}{2} g_1''(\eta) - g_1'(\eta) = 0$$

has a rather complicated special solution

$$g_1^*(\eta) = \left(1 + \frac{\eta^2}{4}\right) \exp\left(-\frac{\eta^2}{4}\right) - \frac{3\sqrt{\pi}}{4} \left(\eta + \frac{\eta^3}{6}\right) \operatorname{erfc}\left(\frac{\eta}{2}\right),$$

although two other special solutions

$$g_1^*(\eta) = 1, \quad g_1^*(\eta) = \eta + \frac{\eta^3}{6}$$

are simple. Therefore, it seems impossible for the perturbation approximations to avoid the complementary error function $\operatorname{erfc}(\eta/2)$ and the Gaussian distribution function $\exp(-\eta^2/4)$. Secondly, the right-hand side of the first-order perturbation equation (12.23) contains the complementary error function $\operatorname{erfc}(\eta/2)$, the Gaussian distribution function $\exp(-\eta^2/4)$ and their combinations such as

$$\eta \operatorname{erfc}\left(\frac{\eta}{2}\right) \exp\left(-\frac{\eta^2}{4}\right).$$

These might be the reasons why Seshadri et al. (2002) and Nazar et al. (2004) reported only the 1st-order perturbation approximation

$$\begin{aligned} g_1(\eta) = & \left(\frac{1}{2} - \frac{2}{3\pi}\right) \left[\left(1 + \frac{\eta^2}{2}\right) \operatorname{erfc}\left(\frac{\eta}{2}\right) - \frac{\eta}{\sqrt{\pi}} e^{-\eta^2/4} \right] \\ & - \frac{1}{2} \left(1 - \frac{\eta^2}{2}\right) \operatorname{erfc}^2\left(\frac{\eta}{2}\right) - \frac{3\eta}{2\sqrt{\pi}} e^{-\eta^2/4} \operatorname{erfc}\left(\frac{\eta}{2}\right) \\ & - \frac{1}{\sqrt{\pi}} \left(\frac{4}{3\sqrt{\pi}} - \frac{\eta}{4}\right) e^{-\eta^2/4} + \frac{2}{\pi} e^{-\eta^2/2}. \end{aligned} \quad (12.25)$$

Notice that the right-hand side of the high-order perturbation equations (12.20) becomes more and more complicated as the order of approximation increases. So, although we gain, very luckily, the solution (12.25) of the first-order perturbation equations (12.23) and (12.24), it becomes more and more difficult to solve the high-order ones. More importantly, even if we could solve all high-order perturbation equations efficiently, we still can not guarantee the convergence of the perturbation series (12.18).

Therefore, although the original nonlinear PDE (12.7) can be transferred into an infinite number of linear ODEs (12.19) and (12.20) by regarding ξ as a small physical parameter and expanding $f(\eta, \xi)$ into the perturbation series (12.18), we still can not gain high-order perturbation approximations efficiently, because the corresponding linear perturbation equations (12.20) become more and more difficult to solve. Note that the perturbation equations (12.19) and (12.20) are completely determined by the perturbation quantity ξ and the original nonlinear PDE (12.7) so that we have no freedom to choose either an initial solution better than (12.22), or a

linear operator better than

$$\mathcal{L}_p(f) = f''' + \frac{\eta}{2} f'' - m f', \quad (12.26)$$

where the prime denotes the differentiation with respect to η .

The skin friction coefficient given by the 1st-order of perturbation approximation reads

$$C_f^x(x, \xi) \approx -\frac{1}{\sqrt{\pi \xi Re_x}} \left[1 + \left(\frac{5}{4} - \frac{4}{3\pi} \right) \xi \right], \quad (12.27)$$

which is not a good approximation in the whole time interval, as shown in Fig. 12.5. So, by means of perturbation approach, one can not gain accurate enough approximation valid in the whole temporal and spatial interval for this unsteady boundary-layer flow.

12.3 Homotopy-series solution

12.3.1 Brief mathematical formulas

As shown by Liao (2006a,b), some unsteady similarity boundary-layer flows can be solved by means of the homotopy analysis method (HAM) (Liao, 1992, 1997, 1999a,b; Liao and Campo, 2002; Liao, 2003a,b, 2004, 2005, 2006a,b, 2009, 2010a,b; Liao and Tan, 2007; Li et al., 2010; Xu et al., 2010). Here, we illustrate that the unsteady boundary-layer flow, governed by the nonlinear PDE (12.7) with the boundary conditions (12.8), can be solved by means of the BVPh 1.0 in a similar way for steady similarity boundary-layer flows.

According to (12.7), we define a nonlinear operator

$$\begin{aligned} \mathcal{N}(\phi) = & \frac{\partial^3 \phi}{\partial \eta^3} + \frac{1}{2} (1 - \xi) \eta \frac{\partial^2 \phi}{\partial \eta^2} + \xi \left[\phi \frac{\partial^2 \phi}{\partial \eta^2} - \left(\frac{\partial \phi}{\partial \eta} \right)^2 \right] \\ & - \xi (1 - \xi) \frac{\partial^2 \phi}{\partial \eta \partial \xi}. \end{aligned} \quad (12.28)$$

Let $f_0(\eta, \xi)$ denote the initial guess of $f(\eta, \xi)$, $q \in [0, 1]$ the embedding parameter, respectively. In the frame of the HAM, we should first construct such a kind of continuous variation (deformation) $\phi(\eta, \xi; q)$ that, as the embedding parameter q increases from 0 to 1, $\phi(\eta, \xi; q)$ varies (or deforms) continuously from the initial guess $f_0(\eta, \xi)$ to the solution $f(\eta, \xi)$. Such kind of continuous variation $\phi(\eta, \xi; q)$ is governed by the so-called zeroth-order deformation equation

$$(1 - q) \mathcal{L}[\phi(\eta, \xi; q) - f_0(\eta, \xi)] = q c_0 H(\eta) \mathcal{N}[\phi(\eta, \xi; q)], \quad (12.29)$$

subject to the boundary conditions

$$\phi(0, \xi; q) = 0, \quad \left. \frac{\partial \phi(\eta, \xi; q)}{\partial \eta} \right|_{\eta=0} = 1, \quad \left. \frac{\partial \phi(\eta, \xi; q)}{\partial \eta} \right|_{\eta=+\infty} = 0, \quad (12.30)$$

where $\mathcal{L}$ is an auxiliary linear operator with the property $\mathcal{L}(0) = 0$, $c_0 \neq 0$ is the convergence-control parameter, $H(\eta) \neq 0$ is an auxiliary function, respectively. Note that, unlike perturbation techniques, we have extremely large freedom to choose $\mathcal{L}$, c_0 and $H(\eta)$. Obviously, when $q = 0$ and $q = 1$, we have

$$\phi(\eta, \xi; 0) = f_0(\eta, \xi) \quad (12.31)$$

and

$$\phi(\eta, \xi; 1) = f(\eta, \xi), \quad (12.32)$$

respectively. Thus, as q increases from 0 to 1, $\phi(\eta, \xi; q)$ indeed varies from the initial approximation $f_0(\eta, \xi)$ to the solution $f(\eta, \xi)$ of the original equations (12.7) and (12.8).

Expanding $\phi(\eta, \xi; q)$ into the Maclaurin series with respect to q and then using (12.31), we have the homotopy-Maclaurin series

$$\phi(\eta, \xi; q) = f_0(\eta, \xi) + \sum_{n=1}^{+\infty} f_n(\eta, \xi) q^n, \quad (12.33)$$

where

$$f_n(\eta, \xi) = \mathcal{D}_n [\phi(\eta, \xi; q)] = \frac{1}{n!} \left. \frac{\partial^n \phi(\eta, \xi; q)}{\partial q^n} \right|_{q=0} \quad (12.34)$$

and $\mathcal{D}_n$ is the n th-order homotopy-derivative operator.

As mentioned above, we have extremely large freedom to choose the initial guess $f_0(\eta, \xi)$, the auxiliary linear operator $\mathcal{L}$, and especially the convergence-control parameter c_0 . Assuming that all of them are so properly chosen that the homotopy-Maclaurin series (12.33) absolutely converges at $q = 1$, we have from (12.32) the homotopy-series solution

$$f(\eta, \xi) = f_0(\eta, \xi) + \sum_{n=1}^{+\infty} f_n(\eta, \xi). \quad (12.35)$$

According to Theorem 4.15, we have the m th-order deformation equation

$$\mathcal{L}[f_m(\eta, \xi) - \chi_m f_{m-1}(\eta, \xi)] = c_0 H(\eta) \delta_{m-1}(\eta, \xi), \quad (12.36)$$

subject to the boundary conditions

$$f_m(0, \xi) = 0, \quad \left. \frac{\partial f_m(\eta, \xi)}{\partial \eta} \right|_{\eta=0} = 0, \quad \left. \frac{\partial f_m(\eta, \xi)}{\partial \eta} \right|_{\eta=+\infty} = 0, \quad (12.37)$$

where

$$\chi_n = \begin{cases} 1, & n > 1, \\ 0, & n = 1, \end{cases} \quad (12.38)$$

and

$$\begin{aligned} \delta_k(\eta, \xi) &= \mathcal{D}_k \{ \mathcal{N} [\phi(\eta, \xi; q)] \} \\ &= \frac{\partial^3 f_k}{\partial \eta^3} + \frac{1}{2}(1 - \xi)\eta \frac{\partial^2 f_k}{\partial \eta^2} - \xi(1 - \xi) \frac{\partial^2 f_k}{\partial \eta \partial \xi} \\ &\quad + \xi \sum_{n=0}^k \left[f_{k-n} \frac{\partial^2 f_n}{\partial \eta^2} - \frac{\partial f_{k-n}}{\partial \eta} \frac{\partial f_n}{\partial \eta} \right] \end{aligned} \quad (12.39)$$

is given by Theorem 4.1. For details, please refer to Chapter 4 and Liao (2006b).

Since the high-order deformation equations (12.36) and (12.37) are linear, the original nonlinear problem is transferred into an infinite number of linear sub-problems. However, unlike perturbation approach mentioned above, such kind of transformation does *not* need any small perturbation quantities. More importantly, we have extremely large freedom to choose the initial guess $f_0(\eta, \xi)$, the auxiliary linear operator $\mathcal{L}$, the auxiliary function $H(\eta)$, and the convergence-control parameter c_0 : it is due to such kind of freedom that we can gain accurate analytic approximations valid in the whole spatial and temporal interval, as shown below.

In general, a continuous function can be approximated by different base functions. Mathematically, according to the above discussions about the failure of the perturbation approach, $f(\eta, \xi)$ should not contain the complementary error function $\text{erfc}(\eta/2)$ and the Gaussian distribution function $\exp(-\eta^2/4)$, otherwise it is very difficult to solve the high-order deformation equations (12.36) and (12.37). Physically, at arbitrary time $\tau \in [0, +\infty)$, corresponding to $\xi \in [0, 1]$, (12.7) and (12.8) describe a similarity boundary-layer flow, which decays exponentially as $\eta \rightarrow +\infty$. Therefore, it is reasonable to assume that $f(\eta, \xi)$ could be expressed by the base functions

$$\left\{ \xi^k \eta^m \exp(-n\eta) \mid k \geq 0, m \geq 0, n \geq 1 \right\} \quad (12.40)$$

in the form

$$f(\eta, \xi) = a_{0,0}(\xi) + \sum_{m=0}^{+\infty} \sum_{n=1}^{+\infty} a_{m,n}(\xi) \eta^m \exp(-n\eta), \quad (12.41)$$

where $a_{m,n}(\xi)$ is a polynomial of ξ to be determined. It provides us the so-called *solution-expression* of $f(\eta, \xi)$, which plays an important role in the frame of the HAM, as shown below.

Unlike perturbation methods, we have now extremely large freedom to choose the initial guess $f_0(\eta, \xi)$, the auxiliary linear operator $\mathcal{L}$ and the auxiliary function $H(\eta)$ in the zeroth-order deformation equation: all of them should be chosen in such a way that the high-order deformation equations (12.36) and (12.37) are easy to solve, and besides that the homotopy-series (12.35) converges in the whole spatial and temporal interval.

The initial guess $f_0(\eta, \xi)$ should satisfy the boundary conditions (12.8) and obey the solution-expression (12.41). Therefore, we choose the initial guess

$$f_0(\eta, \xi) = 1 - \exp(-\eta). \quad (12.42)$$

Note that, this initial guess exactly satisfies the governing equation (12.7) at $\xi = 1$, corresponding to $\tau \rightarrow +\infty$, although it is not a good approximation of $f(\eta, \xi)$ at $\xi = 0$.

Note that the governing equation (12.7) contains a linear operator

$$\mathcal{L}_0(f) = \frac{\partial^3 f}{\partial \eta^3} + \frac{1}{2} (1 - \xi) \eta \frac{\partial^2 f}{\partial \eta^2} - \xi (1 - \xi) \frac{\partial^2 f}{\partial \eta \partial \xi}.$$

However, if we choose the above linear operator as the auxiliary linear operator $\mathcal{L}$, the high-order deformation equation (12.36) becomes a PDE with the variable coefficients

$$\frac{1}{2} (1 - \xi) \eta, \quad -\xi (1 - \xi),$$

and thus is rather difficult to solve. If we choose $\mathcal{L}_p$ defined by (12.26) as the auxiliary linear operator, the high-order deformation equation is indeed an ODE, but its solution contains the complementary error function $\text{erfc}(\eta/2)$ and the Gaussian distribution function $\exp(-\eta^2/4)$, which disobeys the solution expression (12.41) of $f(\eta, \xi)$. So, neither $\mathcal{L}_0$ nor $\mathcal{L}_p$ is a good choice for us. Fortunately, the HAM provides us extremely large freedom to choose the auxiliary linear operator $\mathcal{L}$. Note that, when $\xi = 1$, the corresponding steady-state similarity boundary-layer flow is solved in Sect. 10.2 by means of the auxiliary linear operator

$$\mathcal{L}(u) = \frac{\partial^3 u}{\partial \eta^3} - \frac{\partial u}{\partial \eta}, \quad (12.43)$$

which has the property

$$\mathcal{L}[C_1 + C_2 \exp(-\eta) + C_3 \exp(\eta)] = 0. \quad (12.44)$$

Besides, for an arbitrary polynomial $b(\xi)$ of ξ , the linear ODE

$$\frac{\partial^3 u}{\partial \eta^3} - \frac{\partial u}{\partial \eta} = b(\xi) \eta^m \exp(-n\eta)$$

can be solved quickly by computer algebra system like Mathematica. So, if we choose (12.43) as the auxiliary linear operator, the high-order deformation equation (12.36) can be easily solved: it is quite interesting that the accurate approximations valid in the whole temporal and spatial interval can be obtained even by such a simple auxiliary linear operator, as shown below.

Therefore, thanks to such kind of freedom provided by the HAM, we simply choose (12.43) as the auxiliary linear operator $\mathcal{L}$. Let

$$f_m^*(\eta, \xi) = \chi_m f_{m-1}(\eta, \xi) + c_0 \mathcal{L}^{-1} [H(\eta) \delta_{m-1}(\eta, \xi)]$$

denote a special solution of (12.36), where $\mathcal{L}^{-1}$ is the inverse operator of the auxiliary linear operator $\mathcal{L}$ defined by (12.43). According to the property (12.44), its common solution reads

$$f_m(\eta, \xi) = f_m^*(\eta, \xi) + C_1 + C_2 \exp(-\eta) + C_3 \exp(\eta),$$

where the coefficients C_1, C_2 , and C_3 are uniquely determined by the boundary conditions (12.37).

Note that the auxiliary linear operator (12.43) has nothing to do with the temporal variable ξ . Besides, the right-hand side term $\delta_{m-1}(\eta, \xi)$ of the m th-order deformation equation (12.36) is always known. Mathematically, the m th-order deformation equation (12.36) is a linear ODE, and thus can be solved in a similar way like the steady-state similarity boundary-layer flows described in Sect. 10.2, since the temporal variable ξ can be regarded as a constant. The only difference is that ξ must be regarded as a variable when calculating the term $\delta_{m-1}(\eta, \xi)$ by means of (12.39). In this way, it is easy to solve the *linear* high-order deformation equations (12.36) and (12.37), successively, especially by means of computer algebra system Mathematica.

In essence, the above approach transfers a nonlinear PDE into an infinite number of linear ODEs. More importantly, unlike perturbation techniques, the HAM provides us great freedom to choose the initial guess $f_0(\eta, \xi)$ and the auxiliary linear operator $\mathcal{L}$ so that the high-order approximations are easy to obtain by means of computer algebra system like Mathematica. In this way, we greatly simplify solving the original nonlinear PDE, and thus can obtain accurate analytic approximations valid in the whole spatial and temporal interval, as shown below.

12.3.2 Homotopy-approximation

As mentioned above, in the frame of the HAM, the original nonlinear PDE (12.7) describing the unsteady boundary-layer flow is transferred into an infinite number of linear ODEs in a similar way like the steady-state boundary-layer flow considered in Sect. 10.2. Note that, the high-order deformation equation (12.36) for the unsteady flow is quite similar to the high-order deformation equation (10.11) for the steady-state flow in Sect. 10.2: the auxiliary linear operator (12.43) for the *unsteady* flow is exactly the *same* as the auxiliary linear operator (10.16) for the *steady-state* flow! Therefore, like the non-similarity boundary-layer flow mentioned in Chapter 11, the unsteady boundary-layer flow governed by the nonlinear PDE (12.7) can be also solved by means of the BVPh 1.0, which is given in Chapter 7 and free available (Accessed 25 Nov 2011, will be updated in the future) at

<http://numericaltank.sjtu.edu.cn/BVPh.htm>

The corresponding input data file is given in the Appendix of this chapter and free available at the above website.

Fig. 12.1 Averaged squared residual of governing equation at the m th-order homotopy approximation versus c_0 by means of the auxiliary function $H(\eta) = 1$ and the initial guess (12.42). Dashed line: $m = 5$; Dash-dotted line: $m = 10$; Solid line: $m = 20$.

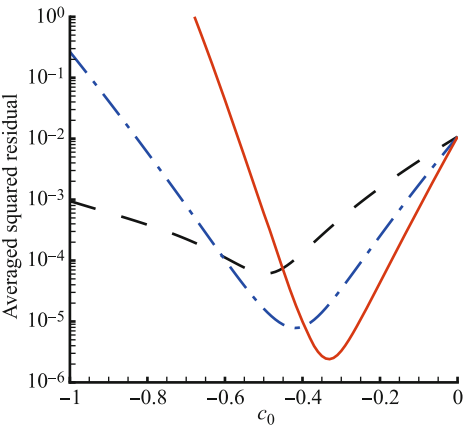


Table 12.1 Averaged squared residual of the governing equation (12.7) over $\eta \in [0, 10]$ and $\xi \in [0, 1]$ by means of $c_0 = -1/4$ and $H(\eta) = 1$.

Order of approximation	Averaged squared residual of (12.7)
1	6.5×10^{-3}
3	2.5×10^{-3}
5	9.5×10^{-4}
10	9.6×10^{-5}
15	1.1×10^{-5}
20	1.4×10^{-6}
25	3.5×10^{-7}

Note that we have great freedom to choose the auxiliary function $H(\eta)$. For simplicity, let us choose

$$H(\eta) = 1.$$

Then, $f_m(\eta, \xi)$ contains the so-called convergence-control parameter c_0 only, which has no physical meanings but provides us a convenient way to guarantee the convergence of the homotopy-series solution (12.35). The averaged squared residual E_m of the governing equation (12.7) at the m th-order homotopy-approximation is gained by first integrating the squared residual of (12.7) in the interval $\eta \in [0, 10]$ by means of the module `GetErr[m]` of the Mathematica package `BVPh` (version 1.0), which is then further integrated in the interval $\xi \in [0, 1]$. The curves of the averaged squared residual E_m versus c_0 are as shown in Fig. 12.1. It indicates that, at the 20th-order homotopy-approximation, the optimal convergence-control parameter c_0^* is about - 0.3. It is found that, by means of $c_0 = -1/4$, the corresponding 25th-order homotopy-approximation is accurate enough with the averaged squared residual 3.5×10^{-7} , as shown in Table 12.1.

It is found that, when $c_0 = -1/4$, the homotopy-approximations of $f''(0, 0)$ agree well with the exact result $f''(0, 0) = -1/\sqrt{\pi} \approx -0.56419$, as shown in Table 12.2, where the prime denotes the differentiation with respect to η . Besides, the accuracy

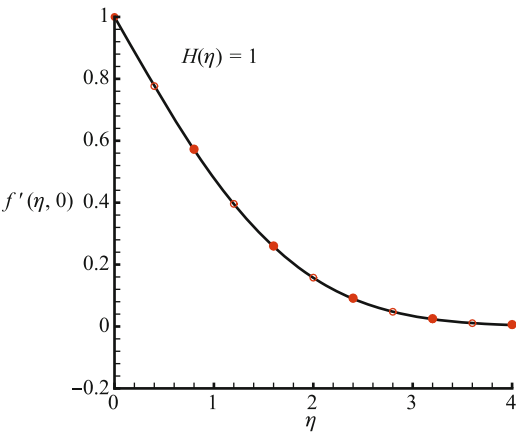
Table 12.2 The homotopy-approximations of $f''(0,0)$ by means of $c_0 = -1/4$ and $H(\eta) = 1$.

Order of approximation	$f''(0,0)$
5	-0.69303
10	-0.60114
15	-0.57440
20	-0.56693
25	-0.56491
30	-0.56438
35	-0.56424
40	-0.56420
45	-0.56419
50	-0.56419

Table 12.3 The $[m,m]$ homotopy-Pad  approximations of $f''(0,0)$ by means of $H(\eta) = 1$.

m	$f''(0,0)$
5	-0.56415
10	-0.56418
15	-0.56419
20	-0.56419
25	-0.56419
30	-0.56419

Fig. 12.2 The comparison of the exact solution (12.11) with the 20th-order homotopy-approximation at $\xi = 0$ by means of $c_0 = -1/4$, $H(\eta) = 1$ and its $[3,3]$ homotopy-Pad  approximation. Solid line: exact solution (12.11); Open circles: 20th-order homotopy-approximation at $\xi = 0$; Filled circles: the $[3,3]$ homotopy-Pad  approximation.



of the homotopy-approximation $f''(0,0)$ is greatly modified by means of the so-called homotopy-Pad  method (Liao, 2003b), as shown in Table 12.3. Furthermore, the 20th-order homotopy-approximation and its $[3,3]$ homotopy-Pad  approximation of the velocity profile $f'(\eta,0)$ agree well with the exact solution (12.11) in the whole region $0 \leq \eta < +\infty$, as shown in Fig. 12.2. This indicates that the initial solution (12.11), which contains the complementary error function $\text{erfc}(\eta/2)$ and the

Fig. 12.3 The homotopy-approximations of $f''(0, \xi)$ by means of $c_0 = -1/4$ and $H(\eta) = 1$. Solid line: 30th-order approximation; Symbols: 20th-order approximation.

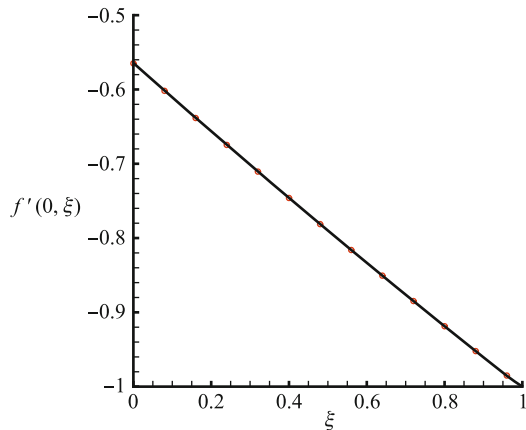
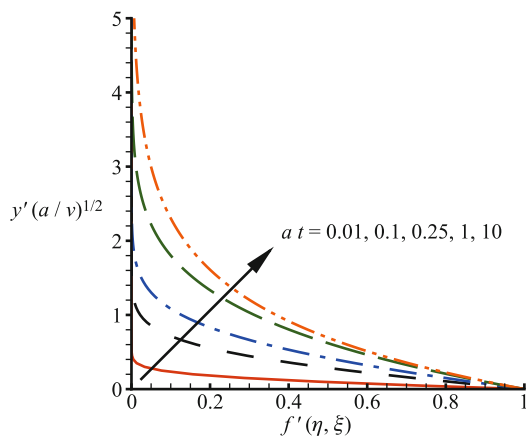


Fig. 12.4 The velocity profiles by means of $c_0 = -1/4$ and $H(\eta) = 1$ at different dimensionless time $\tau = at$. Solid line: $\tau = 0.01$; Dashed line: $\tau = 0.1$; Dash-dotted line: $\tau = 0.25$; Long-dashed line: $\tau = 1$; Dash-double-dotted line: $\tau = 10$.



Gaussian distribution function $\exp(-\eta^2/4)$, can be well approximated by the base functions (12.40).

Note that $f''(0, \xi)$ is related to the skin friction. It is found that, when $c_0 = -1/4$, the 20th-order homotopy-approximation of $f''(0, \xi)$ agrees well with the 30th-order one in the whole region $\xi \in [0, 1]$, as shown in Fig. 12.3. Besides, the corresponding velocity profiles are accurate in the whole interval $\xi \in [0, 1]$ and $0 \leq \eta < +\infty$, as shown in Fig. 12.4. Note that, the velocity profile varies smoothly as τ increase from 0 to $+\infty$.

The local skin friction is related to $f''(0, \xi)$. The 30th-order homotopy approximation of $f''(0, \xi)$ reads

$$\begin{aligned} f''(0, \xi) = & -0.5643747892 - 0.4653303619 \xi + 2.998049008 \times 10^{-2} \xi^2 \\ & - 2.518392990 \times 10^{-3} \xi^3 - 2.561860658 \times 10^{-5} \xi^4 \\ & - 2.531901893 \times 10^{-5} \xi^5 + 3.073353805 \times 10^{-5} \xi^6 \end{aligned}$$

$$\begin{aligned}
& +5.063224875 \times 10^{-5} \xi^7 + 5.780083670 \times 10^{-5} \xi^8 \\
& -3.019750875 \times 10^{-4} \xi^9 + 0.2746188078 \xi^{10} - 48.017634463 \xi^{11} \\
& +3.4358227736 \times 10^3 \xi^{12} - 1.2769915485 \times 10^5 \xi^{13} \\
& +2.8257114566 \times 10^6 \xi^{14} - 4.0636817506 \times 10^7 \xi^{15} \\
& +4.0325552732 \times 10^8 \xi^{16} - 2.8809318182 \times 10^9 \xi^{17} \\
& +1.5277371296 \times 10^{10} \xi^{18} - 6.1471279622 \times 10^{10} \xi^{19} \\
& +1.9058081506 \times 10^{11} \xi^{20} - 4.5980815420 \times 10^{11} \xi^{21} \\
& +8.6766946108 \times 10^{11} \xi^{22} - 1.2809186178 \times 10^{12} \xi^{23} \\
& +1.4720293398 \times 10^{12} \xi^{24} - 1.3019682527 \times 10^{12} \xi^{25} \\
& +8.6859786813 \times 10^{11} \xi^{26} - 4.2255442112 \times 10^{11} \xi^{27} \\
& +1.4139635601 \times 10^{11} \xi^{28} - 2.9087215325 \times 10^{10} \xi^{29} \\
& +2.7723411925 \times 10^9 \xi^{30}.
\end{aligned} \tag{12.45}$$

The corresponding local skin friction at the dimensionless time $\tau \in [0, +\infty)$ is shown in Fig. 12.5. Using the first four-terms of (12.45), we have the simplified local skin friction formula

$$\begin{aligned}
C_f^x(x, \xi) = & (\xi Re_x)^{-1/2} (-0.5643747892 - 0.4653303619 \xi \\
& + 2.998049008 \times 10^{-2} \xi^2 - 2.518392990 \times 10^{-3} \xi^3), \tag{12.46}
\end{aligned}$$

which agrees well with the exact 30th-order homotopy-approximation in the *whole* time interval $0 \leq \tau < +\infty$, as shown in Fig. 12.5. Thus, by means of a proper convergence-control parameter c_0 , we can obtain accurate analytic approximation of the unsteady boundary-layer flow governed by the nonlinear PDE (12.7), which is uniformly valid not only in the whole temporal interval $0 \leq \tau < +\infty$ but also in the whole spatial interval $0 \leq \eta < +\infty$. To the best of our knowledge, such a kind of simple and accurate analytic approximation of the skin friction for the unsteady boundary-layer flow has never been reported. This verifies once again the validity of the HAM for complicated nonlinear problems.

Note that the above homotopy-approximations are obtained by means of the simplest auxiliary function $H(\eta) = 1$. However, in the frame of the HAM, we have great freedom to choose the auxiliary function $H(\eta)$. For example, when we choose $H(\eta) = \exp(-\eta)$, the corresponding averaged squared residual of the governing equation (12.7) at the 20th-order of approximation has the minimum 1.0×10^{-4} , corresponding to the optimal convergence-control parameter $c_0^* = -0.44$, as shown in Fig. 12.6. Indeed, using the BVPh 1.0 as a tool, we can gain accurate analytic approximation by means of $H(\eta) = \exp(-\eta)$ and $c_0 = -2/5$, too. This verifies once again the validity and flexibility of the BVPh 1.0.

Fig. 12.5 The comparison of the four-term approximation (12.46) of $C_f^x \sqrt{Re_x}$ with the exact 30th-order homotopy-approximation (12.45) and the perturbation approximation (12.27). Solid line: the exact 30th-order homotopy-approximation (12.45); Dashed line: the perturbation approximation (12.27); Symbols: the simplified homotopy-approximation (12.46).

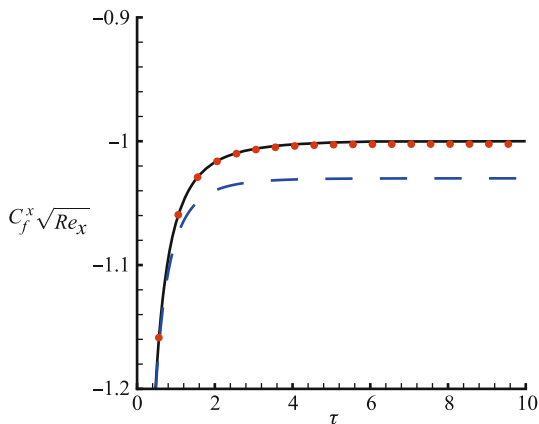
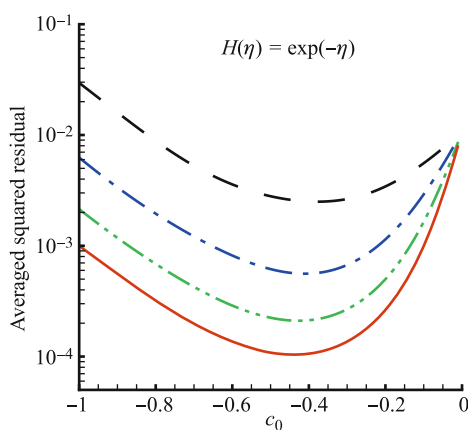


Fig. 12.6 The averaged squared residual of the governing equation (12.7) over $\eta \in [0, 10]$ and $\xi \in [0, 1]$ by means of $H(\eta) = \exp(-\eta)$ at the m th-order of approximation. Dashed line: $m = 5$; Dash-dotted line: $m = 10$; Dash-dot-dotted line: $m = 15$; Solid line: $m = 20$.



12.4 Concluding remarks

We illustrate in this chapter that, in the frame of the HAM, the nonlinear PDE (12.7) describing an unsteady boundary-layer flow can be transferred into an infinite number of linear ODEs, and thus can be solved by means of the BVPh 1.0 in a rather similar way to that for steady-state similarity ones. In other words, by means of the HAM, solving unsteady boundary-layer flows is as easy as steady-state ones. This verifies the validity of the BVPh 1.0 for some nonlinear PDEs, especially for those related to boundary-layer flows.

Note that, one can not gain such kind of accurate approximations by means of perturbation techniques, although the original nonlinear PDE (12.7) can be also transferred into an infinite number of linear ODEs by regarding ξ as a small perturbation quantity. This is mainly because, using perturbation methods, we have *no* freedom to choose the initial approximation (12.22) and the corresponding linear operator (12.26) in the high-order perturbation equations so that these high-order perturbation equations are rather difficult to solve. However, unlike perturbation tech-

niques, the HAM provides us extremely large freedom to choose the initial guess $f_0(\eta, \xi)$ and the auxiliary linear operator $\mathcal{L}$, therefore we can choose such a simple initial guess (12.42) and such a simple auxiliary linear operator (12.43) that the corresponding high-order deformation equation (12.36) can be easily solved. More importantly, in the frame of the HAM, the accuracy of the high-order homotopy-approximations is guaranteed by means of the convergence-control parameter c_0 . So, unlike perturbation methods, the HAM is valid for more complicated nonlinear problems.

As illustrated in Chapter 11, in the frame of the HAM, a nonlinear PDE describing a steady-state *non-similarity* boundary-layer flow can be transferred into an infinite number of linear ODEs governed by the auxiliary linear operator

$$\mathcal{L}(f) = \frac{\partial^3 f}{\partial \eta^3} - \frac{\partial f}{\partial \eta}. \quad (12.47)$$

Here, we further show that, in the frame of the HAM, the nonlinear PDE describing an *unsteady* similarity boundary-layer flow can be also transferred into an infinite number of linear ODEs, which are governed by the *same* auxiliary linear operator as above! It should be emphasized that the above auxiliary linear operator is also exactly the *same* as the auxiliary linear operator (10.16) used in Sect. 10.2 to solve a kind of *state-state similarity* boundary-layer flow with exponentially decaying solutions! Notice that the boundary-layer flows considered in these three chapters are quite different, not only physically but also mathematically. However, using the BVPh 1.0, *all* of them can be solved by the *same* auxiliary linear operator (12.47) in the frame of the HAM! These show the generality of the HAM and BVPh 1.0.

Therefore, the BVPh 1.0 can be used as a tool to solve many nonlinear PDEs in a similar way, especially those related to unsteady and/or non-similarity boundary-layer flows and heat transfer, although we should keep in mind that it does not work for *all* nonlinear PDEs.

Indeed, it might be impossible to develop an analytic approach valid for *all* nonlinear boundary-value problems. However, as pointed out by Rabindranth Tagore (1861 — 1941), “if you shut your door to all errors, truth will be shut out”. Therefore, our strategy is to develop an analytic approach valid for as many nonlinear boundary-value problems as possible. The BVPh 1.0 and its successful applications in some nonlinear ODEs and PDEs mentioned in Part II suggest that such a strategy should have a good prospect, although further modifications and more applications are needed for the development of the HAM-based Mathematica package BVPh.

Anyway, we start out on a promising march, no matter how far we can go!

Appendix 12.1 Input data of BVPh

```
(* Input Mathematica package BVPh version 1.0 *)
<<BVPh.txt;
```

```

(* Define the physical and control parameters *)
TypeEQ      = 1;
ApproxQ      = 0;
ErrReq       = 10^(-10);
NgetErr      = 100;
zRintegral   = 10;

(* Define the governing equation *)
f[z_,u_,lambda_] := Module[{temp},
temp[1] = D[u,{z,3}] + (1-t)*z/2*D[u,{z,2}]
        - t*(1-t)*D[u,z,t];
temp[2] = u*D[u,{z,2}] - D[u,z]^2;
temp[1] + t* temp[2] // Expand
];

(* Define boundary conditions *)
zR = infinity;
OrderEQ = 3;
BC[1,z_,u_,lambda_] := Limit[u, z -> 0];
BC[2,z_,u_,lambda_] := Limit[D[u,z] - 1, z -> 0];
BC[3,z_,u_,lambda_] := Limit[D[u,z] , z -> zR ];

(* Define initial guess *)
u[0] = 1 - Exp[-z];

(* Define the auxiliary linear operator *)
L[u_] := D[u,{z,3}] - D[u,z];

(* Define output term *)
output[z_,u_,k_] := Print["output" = ",
        D[u[k] , {z,2}]/.{z->0,t->0}/N];

(* Define Getdelta[k] *)
Getdelta[k_] := Module[{temp,i},
uz[k] = D[u[k],z]//Expand;
uzz[k] = D[uz[k],z]//Expand;
uzzz[k] = D[uzz[k],z]//Expand;
uzuz[k] = Sum[uz[i]*uz[k-i],{i,0,k}]//Expand;
uzzu[k] = Sum[uzz[i]*u[k-i],{i,0,k}]//Expand;
uzt[k] = D[uz[k],t]//Expand;
temp[1] = uzzz[k] + (1-t)*z/2*uzz[k]
        - t*(1-t)*uzt[k]//Expand;
temp[2] = t*(uzzu[k] - uzuz[k]);
delta[k] = temp[1] + temp[2]//Expand;
];

(* Print input and control parameters *)
PrintInput[u[z,t]];

(* Set convergence-control parameter c0 *)
c0 = -1/4;
Print["c0 = ",c0];

```

```

(* Gain 10th-order HAM approximation *)
Print[" c0 = ", c0];
BVPh[1, 10];

(* Calculate the squared residual *)
For[k=2, k<=10, k=k+2,
GetErr[k];
err[k] = Integrate[Err[k], {t, 0, 1}];
Print[" k = ", k, " Squared residual = ",
      err[k]/N];
];

```

The Mathematica package BVPh (version 1.0) and the above input data are free available at

<http://numericaltank.sjtu.edu.cn/BVPh.htm>

Note that, for nonlinear boundary-value problems defined in an infinite interval, we must set `zR = infinity` in the input data of BVPh (version 1.0). Besides, we must define the module `Getdelta` in the input data file, when a nonlinear PDE is solved by means of the BVPh (version 1.0).

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Part III

Applications in Nonlinear Partial Differential Equations

“The small truth has words which are clear; the great truth has great silence.”

by Rabindranth Tagore (1861 — 1941)

Chapter 13

Applications in Finance: American Put Options

Abstract The homotopy analysis method (HAM) is successfully combined with the Laplace transform to solve the famous American put option equation in finance. Unlike asymptotic and/or perturbation formulas that are often valid only a couple of days or weeks prior to expiry, our homotopy-approximation of the optimal exercise boundary $B(\tau)$ in polynomials of $\sqrt{\tau}$ to $o(\tau^M)$ may be valid a couple of dozen years, or even a half century, as long as M is large enough. It is found that the homotopy-approximation of $B(\tau)$ in polynomial of $\sqrt{\tau}$ to $o(\tau^{48})$ is often valid in so many years that the well-known theoretical perpetual optimal exercise price is accurate enough thereafter, so that the combination of them can be regarded as an analytic formula valid in the whole time interval $0 \leq \tau < +\infty$. A practical Mathematica code APOh is provided in the Appendix 13.2 for businessmen to gain accurate enough optimal exercise price of American put option at large expiration-time by a laptop only in a few seconds, which is free available (Accessed 25 Nov 2011, will be updated in the future) at

<http://numericaltank.sjtu.edu.cn/HAM.htm>

13.1 Mathematical modeling

There are many nonlinear partial differential equations (PDEs) in the field of finance. Traditionally, analytic approximations of these nonlinear PDEs are given by asymptotic methods. However, most of these asymptotic approximations are valid only for small time and/or some small quantities. In this chapter, we illustrate that, by means of the homotopy analysis method (HAM) (Liao, 1992, 1997, 1999a,b; Liao and Campo, 2002; Liao, 2003a,b, 2004, 2005, 2006, 2009, 2010a,b; Liao and Tan, 2007; Li et al., 2010; Xu et al., 2010), we can gain much better analytic approximations of some nonlinear PDEs in finance.

Let us consider a famous problem in finance: American put option with strike price X that expires at time T . Let $V(S, t)$ denote the value of an American put option, S the price of the underlying asset and t the time, respectively. Besides, let σ denote the volatility of the underlying, r the risk-free interest rate, respectively,

and both of them are constants. At any moment, there exists an optimal exercise boundary $B(t)$ such that it is optimal to exercise the put option when S is at or below $B(t)$. Hence, when $S \leq B(t)$, the put option is of value

$$V(S, t) = X - S, \quad (13.1)$$

where X is the strike price. When $S > B(t)$, $V(S, t)$ satisfies the famous Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} - r V = 0, \quad (13.2)$$

subject to the smooth pasting conditions at the exercise boundary $B(t)$:

$$\lim_{S \rightarrow B(t)} V(S, t) = X - B(t), \quad \lim_{S \rightarrow B(t)} \frac{\partial V}{\partial S} = -1, \quad (13.3)$$

the upper boundary condition:

$$\lim_{S \rightarrow +\infty} V(S, t) \rightarrow 0 \quad (13.4)$$

and the terminal condition:

$$\lim_{t \rightarrow T} V(S, t) = \max\{X - S, 0\}, \quad (13.5)$$

respectively.

Define the variable $\tau \equiv T - t$. When $r > 0$, the terminal condition (13.5) can be further simplified as

$$\lim_{\tau \rightarrow 0} V(S, \tau) = 0 \quad (13.6)$$

in the range

$$\Sigma_1 = \{(S, \tau) | B(\tau) \leq S < +\infty, 0 \leq \tau \leq T\}.$$

Kim (1990) and Carr et al. (1992) derived the formula

$$V(S, \tau) = V_E(S, \tau) + X \int_0^\tau r \exp(-r\xi) N(-d_{\xi,2}) d\xi \quad (13.7)$$

for the option price, where

$$V_E(S, \tau) = X \exp(-r\tau) N(-d_2) + S N(-d_1)$$

is the price of the European put option with the following definitions

$$d_1 = \frac{\ln(S/X) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}},$$

$$d_2 = d_1 - \sigma\sqrt{\tau},$$

$$d_{\xi,1} = \frac{\ln[S/B(\tau - \xi)] + (r + \sigma^2/2)\xi}{\sigma\sqrt{\xi}},$$

$$d_{\xi,2} = d_{\xi,1} - \sigma\sqrt{\xi},$$

and $N(x)$ is a cumulative distribution function for a standardized normal random variable defined by

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp\left(-\frac{w^2}{2}\right) dw. \quad (13.8)$$

Hence, by means of (13.7), it is easy to gain the option price $V(S, \tau)$, as long as the optimal exercise boundary $B(\tau)$ is known. So, the optimal exercise boundary $B(\tau)$ is the key point of this problem.

Due to the existence of the unknown moving boundary $B(\tau)$, this problem is nonlinear in essence, although the governing equation (13.2) is linear. The above PDE with an unknown moving boundary can be solved by means of numerical methods, such as the binomial/trinomial methods (Cox et al., 1979; Broadie and Detemple, 1996), the Monte Carlo simulation (Grant et al., 1996), the least squares method (Longstaff and Schwartz, 2001), the variational inequalities (Jaillet et al., 1990; Dempster, 1994), and the techniques based on solving PDEs (Brennan and Schwartz, 1977; Wu and Kwok, 1997; Allegretto et al., 2001; Hon and Mao, 1997; Chen et al., 2000; Broadie and Detemple, 1997). However, since most of market practitioners are not familiar with numerical methods, analytical approximations are extremely valuable in practice and theory.

Most of traditional analytic approaches are based on the perturbation or asymptotic methods, such as Barles et al. (1995), Kuske and Keller (1998), Alobaidi and Mallier (2001), Evans et al. (2002), Zhang and Li (2006) and Knessl (2001). As reported by Chen et al. (2000) and Chen and Chadam (2005), all of these approximations are valid for very short time prior to expiry, usually on the order of days and weeks.

Zhu (2006a) was the first who applied the HAM to the American put option in 2006. Using the Landau transform (Landau, 1950) and the HAM, Zhu (2006a) gave a solution in the form of infinite recursive series involving double integrals. With a 30th-order approximation through numerical integration, Zhu (2006a) numerically demonstrated the convergence of his results. This is a remarkable contribution to find an analytic formula without extra parameters involved. Besides, Zhu (2006b) also applied the HAM to gain an analytical solution for the valuation of convertible bonds with constant dividend yield. In 2010, Cheng et al. (2010) further applied the HAM to give an explicit analytic approximation of the optimal exercise boundary $B(\tau)$ in polynomial of $\sqrt{\tau}$ to $o(\tau^6)$, which is valid in a much longer time prior to expiry, usually on the order of years, and is as accurate as many numerical results. Their approach is based on the Laplace transform and has nothing to do with the Landau transform (Landau, 1950). By means of the HAM, Cheng (2008) gave an analytic approximation of $B(\tau)$ in the polynomial of $\sqrt{\tau}$ to $o(\tau^7)$. These successful

applications in American put option show the potential and validity of the HAM in finance.

In this chapter, we further modify the HAM-based approach of Cheng et al. (2010) and give more accurate explicit expressions of the optimal exercise boundary $B(\tau)$. Especially, we investigate the influence of the order $o(\tau^M)$ of the optimal exercise boundary $B(\tau)$ in polynomials of $\sqrt{\tau}$ up to $o(\tau^M)$, and illustrate that the maximum valid time of $B(\tau)$ given by the HAM is directly proportional to the order $o(\tau^M)$. Therefore, when M is large enough, $B(\tau)$ given by the HAM can be valid even up to a half century prior to expiry, which is about 1000 times longer than those given by perturbation or asymptotic methods! Based on this explicit approximations, a short Mathematica code APOh is given in the appendix of this chapter and available at

<http://numericaltank.sjtu.edu.cn/HAM.htm>

which can be used by businessmen to gain, only in a few seconds by a laptop, accurate optimal exercise price of American put option for a rather large expiry time.

13.2 Brief mathematical formulas

When $\sigma \neq 0$, we introduce the following dimensionless variables

$$V^* = \frac{V}{X}, \quad S^* = \frac{S}{X}, \quad \tau^* = \frac{\sigma^2}{2} \tau, \quad \gamma = \frac{2r}{\sigma^2}. \quad (13.9)$$

Dropping all stars from now on, the dimensionless governing equation becomes

$$-\frac{\partial V}{\partial \tau} + S^2 \frac{\partial^2 V}{\partial S^2} + \gamma S \frac{\partial V}{\partial S} - \gamma V = 0, \quad (13.10)$$

subject to the boundary conditions

$$V(B(\tau), \tau) = 1 - B(\tau), \quad (13.11)$$

$$\frac{\partial V}{\partial S}(B(\tau), \tau) = -1, \quad (13.12)$$

$$V(S, \tau) \rightarrow 0 \quad \text{as } S \rightarrow +\infty, \quad (13.13)$$

$$V(S, 0) = 0. \quad (13.14)$$

Let $V_0(S, \tau)$ and $B_0(\tau)$ denote the initial approximations of $V(S, \tau)$ and the optimal exercise boundary $B(\tau)$, respectively. Let $q \in [0, 1]$ denote the embedding parameter. In the frame of the HAM, we first construct such two continuous variations (or deformations) $\phi(S, \tau; q)$ and $\Lambda(\tau; q)$ that, as q increases from 0 to 1, $\phi(S, \tau; q)$ varies continuously from the initial guess $V_0(S, \tau)$ to the solution $V(S, \tau)$, so does $\Lambda(\tau; q)$ from the initial guess $B_0(\tau)$ to the optimal exercise boundary $B(\tau)$, respectively. Such kind of continuous variations are governed by the zeroth-order defor-

mation equation

$$-\frac{\partial \phi(S, \tau; q)}{\partial \tau} + S^2 \frac{\partial^2 \phi(S, \tau; q)}{\partial S^2} + \gamma S \frac{\partial \phi(S, \tau; q)}{\partial S} - \gamma \phi(S, \tau; q) = 0 \quad (13.15)$$

defined in the domain

$$\Lambda(\tau; q) \leq S < +\infty, \quad 0 \leq \tau \leq \tau_{\text{exp}},$$

subject to the initial/boundary conditions

$$\phi(S, 0; q) = 0, \quad (13.16)$$

$$\frac{\partial \phi(S, \tau; q)}{\partial S} = -1 \quad \text{at } S = \Lambda(\tau; q), \quad (13.17)$$

$$\phi(+\infty, \tau; q) \rightarrow 0, \quad (13.18)$$

and

$$(1 - q) [\Lambda(\tau; q) - B_0(\tau)] = c_0 q \{ \Lambda(\tau; q) + \phi[\Lambda(\tau; q), \tau; q] - 1 \}, \quad (13.19)$$

where $c_0 \neq 0$ is a convergence-control parameter and

$$\tau_{\text{exp}} = \frac{1}{2} \sigma^2 T$$

is the dimensionless expiring time.

When $q = 1$, the zeroth-order deformation equations (13.15) to (13.19) are equivalent to the original equations (13.10) to (13.14), thus

$$\phi(S, \tau; 1) = V(S, \tau), \quad \Lambda(\tau; 1) = B(\tau). \quad (13.20)$$

When $q = 0$, we have from (13.19) that

$$\Lambda(\tau; 0) = B_0(\tau), \quad (13.21)$$

and from (13.15) to (13.18), we further have the governing equation

$$-\frac{\partial V_0(S, \tau)}{\partial \tau} + S^2 \frac{\partial^2 V_0(S, \tau)}{\partial S^2} + \gamma S \frac{\partial V_0(S, \tau)}{\partial S} - \gamma V_0(S, \tau) = 0, \quad (13.22)$$

subject to the initial/boundary conditions

$$V_0(S, 0) = 0, \quad (13.23)$$

$$\frac{\partial V_0(S, \tau)}{\partial S} = -1 \quad \text{at } S = B_0(\tau), \quad (13.24)$$

$$V_0(+\infty, \tau) \rightarrow 0, \quad (13.25)$$

where

$$V_0(S, \tau) = \phi(S, \tau; 0). \quad (13.26)$$

For the sake of simplicity, we choose

$$B_0(\tau) = 1$$

as the initial guess of the optimal exercise boundary $B(\tau)$. Then, the initial guess $V_0(S, \tau)$ is governed by the linear PDE (13.22), subject to the linear initial condition (13.23), the linear boundary condition (13.24) at a *fixed* boundary $S = 1$, and the linear boundary condition (13.25) at infinity. Therefore, the zeroth-order deformation equations (13.15) to (13.19) indeed construct two continuous variations (or deformations) $\phi(S, \tau; q)$ and $\Lambda(\tau; q)$, which are two homotopies

$$\phi(S, \tau; q) : V_0(S, \tau) \sim V(S, \tau), \quad \Lambda(\tau; q) : B_0(\tau) \sim B(\tau)$$

in topology.

Expanding $\phi(S, \tau; q)$ and $\Lambda(\tau; q)$ in Maclaurin series with respect to $q \in [0, 1]$, we obtain the homotopy-Maclaurin series

$$\phi(S, \tau; q) = V_0(S, \tau) + \sum_{n=1}^{+\infty} V_n(S, \tau) q^n, \quad (13.27)$$

$$\Lambda(\tau; q) = B_0(\tau) + \sum_{n=1}^{+\infty} B_n(\tau) q^n, \quad (13.28)$$

where

$$\begin{aligned} V_n(S, \tau) &= \frac{1}{n!} \left. \frac{\partial^n \phi(S, \tau; q)}{\partial q^n} \right|_{q=0} = \mathcal{D}_n[\phi(S, \tau; q)], \\ B_n(\tau) &= \frac{1}{n!} \left. \frac{\partial^n \Lambda(\tau; q)}{\partial q^n} \right|_{q=0} = \mathcal{D}_n[\Lambda(\tau; q)], \end{aligned} \quad (13.29)$$

are the n th-order homotopy-derivatives of $\phi(S, \tau; q)$ and $\Lambda(q)$, respectively, and $\mathcal{D}_n$ is the n th-order homotopy-derivative operator. Note that the zeroth-order deformation equation contains the convergence-control parameter c_0 . Assuming that c_0 is chosen properly so that the above series are absolutely convergent at $q = 1$, we have the homotopy-series solution

$$V(S, \tau) = V_0(S, \tau) + \sum_{n=1}^{+\infty} V_n(S, \tau), \quad (13.30)$$

$$B(\tau) = B_0(\tau) + \sum_{n=1}^{+\infty} B_n(\tau). \quad (13.31)$$

The equations for $V_n(S, \tau)$ and $B_n(\tau)$ can be derived directly from the zeroth-order deformation equations (13.15) to (13.19). Substituting the series (13.27) and (13.28) into (13.15), (13.16) and (13.18), then equating the like-power of q , we have

the so-called n th-order deformation equation ($n \geq 1$)

$$-\frac{\partial V_n(S, \tau)}{\partial \tau} + S^2 \frac{\partial^2 V_n(S, \tau)}{\partial S^2} + \gamma S \frac{\partial V_n(S, \tau)}{\partial S} - \gamma V_n(S, \tau) = 0, \quad (13.32)$$

subject to the initial/boundary conditions

$$V_n(S, 0) = 0, \quad (13.33)$$

$$V_n(+\infty, \tau) \rightarrow 0. \quad (13.34)$$

For details about above formulas, please refer to Cheng et al. (2010).

Note that (13.17) and (13.19) are defined at the moving boundary $S = \Lambda(\tau; q)$, which itself is dependent upon q , so that (13.27) is invalid for them. Cheng et al. (2010) developed a Mathematica code to gain the Maclaurin series of $\phi(S, \tau; q)$ on the moving boundary $S = \Lambda(\tau; q)$ with respect to q . Unlike Cheng et al. (2010), we give in this chapter the *explicit* expression of it.

Let the prime denote the differentiation with respect to S . On the moving boundary $S = \Lambda(\tau; q)$, using (13.28) and expanding $\phi(S, \tau; q)$, $\phi'(S, \tau; q)$ into the Maclaurin series with respect to q , we have

$$\phi(S, \tau; q) = V_0(B_0, \tau) + \sum_{n=1}^{+\infty} [V_n(B_0, \tau) + f_n(\tau)] q^n \quad (13.35)$$

and

$$\frac{\partial \phi(S, \tau; q)}{\partial S} = V'_0(B_0, \tau) + \sum_{n=1}^{+\infty} [V'_n(B_0, \tau) + g_n(\tau)] q^n \quad (13.36)$$

where

$$f_n(\tau) = \sum_{j=0}^{n-1} \alpha_{j, n-j}(\tau), \quad g_n(\tau) = \sum_{j=0}^{n-1} \beta_{j, n-j}(\tau), \quad (13.37)$$

with the following explicit definitions

$$\alpha_{n,i}(\tau) = \sum_{m=1}^i \psi_{n,m}(\tau) \mu_{m,i}(\tau), \quad i \geq 1, \quad (13.38)$$

$$\beta_{n,i}(\tau) = \sum_{m=1}^i (m+1) \psi_{n,m+1}(\tau) \mu_{m,i}(\tau), \quad i \geq 1, \quad (13.39)$$

$$\psi_{n,0}(\tau) = V_n(B_0, \tau), \quad \psi_{n,m}(\tau) = \frac{1}{m!} \frac{\partial^m V_n(S, \tau)}{\partial S^m} \Big|_{S=1}, \quad (13.40)$$

and the recursion formula

$$\mu_{1,n}(\tau) = B_n(\tau), \quad \mu_{m+1,n}(\tau) = \sum_{i=m}^{n-1} \mu_{m,i}(\tau) B_{n-i}(\tau). \quad (13.41)$$

The detailed derivation of the above explicit expressions is given in Appendix 13.1. These explicit expressions greatly modify the computational efficiency. More importantly, by means of these explicit formulas, it becomes easier to investigate the influence of the order $o(\tau^M)$ of the optimal exercise boundary $B(\tau)$ in polynomials of $\sqrt{\tau}$, as shown later.

Then, substituting (13.36) into (13.17) and equating the like-power of q , we have the boundary condition

$$\frac{\partial V_n(S, \tau)}{\partial S} = -g_n(\tau) \quad \text{at } S = B_0(\tau) = 1 \text{ for } n \geq 1. \quad (13.42)$$

Similarly, substituting (13.35) into (13.19) and equating the like-power of q , we have

$$B_n(\tau) = \begin{cases} c_0 V_0(B_0, \tau), & n = 1, \\ B_{n-1}(\tau) + c_0 [B_{n-1}(\tau) + V_{n-1}(B_0, \tau) + f_{n-1}(\tau)], & n > 1, \end{cases} \quad (13.43)$$

where $B_0 = 1$.

For the sake of simplicity, define $g_0(\tau) = 1$. Then, (13.22) to (13.25) for the unknown initial guess $V_0(S, \tau)$ have the same forms as the n th-order deformation equations (13.32) to (13.34) and (13.42) for $V_n(S, \tau)$, respectively. Using the explicit recursion formula mentioned above, we have

$$\begin{aligned} g_0(\tau) &= 1, \\ g_1(\tau) &= 2\psi_{0,2}(\tau) B_1(\tau), \\ g_2(\tau) &= 2\psi_{0,2}(\tau) B_2(\tau) + 3\psi_{0,3}(\tau) B_1^2(\tau) + 2\psi_{1,2}(\tau) B_1(\tau), \\ g_3(\tau) &= 2\psi_{0,2}(\tau) B_3(\tau) + 6\psi_{0,3}(\tau) B_1(\tau) B_2(\tau) + 4\psi_{0,4}(\tau) B_1^3(\tau) \\ &\quad + 2\psi_{1,2}(\tau) B_2(\tau) + 3\psi_{1,3}(\tau) B_1^2(\tau) + 2\psi_{2,2}(\tau) B_1(\tau), \\ &\vdots \end{aligned}$$

and

$$\begin{aligned} f_0(\tau) &= 0, \\ f_1(\tau) &= \psi_{0,1}(\tau) B_1(\tau), \\ f_2(\tau) &= \psi_{0,1}(\tau) B_2(\tau) + \psi_{0,2}(\tau) B_1^2(\tau) + \psi_{1,1}(\tau) B_1(\tau), \\ f_3(\tau) &= \psi_{0,1}(\tau) B_3(\tau) + 2\psi_{0,2}(\tau) B_1(\tau) B_2(\tau) + \psi_{0,3}(\tau) B_1^3(\tau) \\ &\quad + \psi_{1,1}(\tau) B_2(\tau) + \psi_{1,2}(\tau) B_1^2(\tau) + \psi_{2,1}(\tau) B_1(\tau), \\ &\vdots \end{aligned}$$

Note that $g_n(\tau)$ depends only upon $V_m(S, \tau)$ and $B_m(\tau)$, where $m = 0, 1, 2, \dots, n-1$. So, for the n th-order deformation equation, $g_n(\tau)$ is always known. Therefore, $V_n(S, \tau)$ for $n \geq 0$ is governed by the linear PDE (13.32) subject to the linear initial condition (13.33), the linear boundary condition (13.34) at infinity, and the linear

condition (13.42) at the fixed boundary $S = B_0(\tau) = 1$. As long as $V_n(S, \tau)$ at $S = B_0(\tau) = 1$ is known, it is convenient to gain $f_n(\tau)$ by means of the explicit formula (13.37) with the recursion formulas mentioned above, and then $B_{n+1}(\tau)$ by means of (13.43).

However, the linear PDE (13.32) contains variable coefficients and thus is not easy to solve. Cheng et al. (2010) solved such kind of system of linear PDEs with variable coefficients by means of the Laplace transform. Let

$$\hat{V}_n(S, \zeta) = \mathcal{L}_T[V_n(S, \tau)]$$

and

$$\hat{g}_n(\zeta) = \mathcal{L}_T[g_n(\tau)]$$

denote the Laplace transforms of $V_n(S, \tau)$ and $g_n(\tau)$ about τ , respectively, where $\mathcal{L}_T$ is the operator of Laplace transform, ζ is a complex number corresponding to τ . By means of the Laplace transform and using the initial condition (13.23), one has

$$\mathcal{L}_T \left[\frac{\partial V_n(S, \tau)}{\partial \tau} \right] = \zeta \mathcal{L}_T[V_n(S, \tau)] - V_n(S, 0) = \zeta \hat{V}_n(S, \zeta)$$

and

$$\mathcal{L}_T \left[\frac{\partial^m V_n(S, \tau)}{\partial S^m} \right] = \frac{\partial^m}{\partial S^m} \{ \mathcal{L}_T[V_n(S, \tau)] \} = \frac{\partial^m \hat{V}_n(S, \zeta)}{\partial S^m}$$

for $m \geq 0$. Therefore, by means of the Laplace transform, the high-order deformation equation (13.32) with the initial/boundary conditions (13.33), (13.34) and (13.42) become a linear ordinary differential equation (ODE)

$$S^2 \frac{\partial^2 \hat{V}_n}{\partial S^2} + \gamma S \frac{\partial \hat{V}_n}{\partial S} - (\gamma + \zeta) \hat{V}_n = 0, \quad (13.44)$$

subject to the boundary conditions

$$\frac{\partial \hat{V}_n}{\partial S} = -\hat{g}_n(\zeta), \quad \text{at } S = 1, \quad (13.45)$$

$$\hat{V}_n(S, \zeta) \rightarrow 0, \quad \text{as } S \rightarrow +\infty. \quad (13.46)$$

The above linear ordinary ODE has the closed-form solution

$$\hat{V}_n(S, \zeta) = \hat{K}(S, \zeta) \hat{g}_n(\zeta), \quad (13.47)$$

where

$$\hat{K}(S, \zeta) = -\frac{S^\lambda}{\lambda}, \quad \lambda = \frac{1 - \gamma - \sqrt{4\zeta + (1 + \gamma)^2}}{2}. \quad (13.48)$$

Note that $\hat{g}_n(\zeta) = \mathcal{L}_T[g_n(\tau)]$ is the Laplace transform of $g_n(\tau)$. Then, using the inverse Laplace transform, we have

$$V_n(S, \tau) = \mathcal{L}_T^{-1} [\hat{K}(S, \zeta) \hat{g}_n(\zeta)] \quad (13.49)$$

and its derivative

$$V_n^{(m)}(S, \tau) = \frac{\partial^m V_n(S, \tau)}{\partial S^m} = \mathcal{L}_T^{-1} [\hat{K}^{(m)}(S, \zeta) \hat{g}_n(\zeta)], \quad (13.50)$$

where

$$\hat{K}^{(m)}(S, \zeta) = \frac{\partial^m \hat{K}(S, \zeta)}{\partial S^m},$$

and $\mathcal{L}_T^{-1}$ denotes the inverse operator of the Laplace transform.

According to (13.40), (13.42), (13.43) and the related recursion formulas, $g_n(\tau)$, $f_n(\tau)$ and $B_n(\tau)$ depend only on $V_n^{(m)}(1, \tau)$ for $0 \leq m \leq n-1$, where the superscript (m) denotes the m th-order differentiation with respect to S . Therefore, it is unnecessary to gain the general expression $V_n^{(m)}(S, \tau)$ for all S but only on $S = B_0(\tau) = 1$. So, the key is to gain $V_n^{(m)}(1, \tau)$ efficiently. From (13.50), it holds

$$V_n^{(m)}(1, \tau) = \mathcal{L}_T^{-1} [\hat{K}^{(m)}(1, \zeta) \hat{g}_n(\zeta)]. \quad (13.51)$$

According to (13.48), it is convenient to gain

$$\begin{aligned} \hat{K}(1, \zeta) &= -\frac{2}{(1-\gamma) - \sqrt{4\zeta + (1+\gamma)^2}}, \\ \hat{K}^{(1)}(1, \zeta) &= \left. \frac{\partial \hat{K}(S, \zeta)}{\partial S} \right|_{S=1} = -1, \\ \hat{K}^{(2)}(1, \zeta) &= \frac{1+\gamma}{2} + \frac{1}{2} \sqrt{4\zeta + (1+\gamma)^2}, \\ &\vdots \end{aligned}$$

and so on. Their inverse Laplace transforms read

$$\begin{aligned} K(1, \tau) &= \frac{1}{\sqrt{\pi\tau}} \exp\left[-\frac{1}{4}(1+\gamma)^2\tau\right] + \frac{(1-\gamma)}{2} \exp(-\gamma\tau) \operatorname{Erfc}\left[-\frac{(1-\gamma)}{2}\sqrt{\tau}\right], \\ K^{(1)}(1, \tau) &= -\delta(\tau), \\ K^{(2)}(1, \tau) &= -\frac{1}{2\sqrt{\pi\tau^3}} \exp\left[-\frac{1}{4}(1+\gamma)^2\tau\right] + \frac{1}{2}(1+\gamma)\delta(\tau), \\ &\vdots \end{aligned}$$

where $\operatorname{Erfc}(x)$ is the complementary error function and $\delta(\tau)$ is the Dirac delta function, respectively.

As mentioned above, we only need obtain $V_n^{(m)}(1, \tau)$ so as to gain the optimal exercise boundary $B(\tau)$. The procedure starts from the initial guess $B_0(\tau) = 1$. Since $g_0(\tau) = 1$, we have its Laplace transform

$$\hat{g}_0(\zeta) = \mathcal{L}_T[g_0(\tau)] = \frac{1}{\zeta}.$$

Then, $V_0(1, \tau)$ and its derivatives are given by the inverse Laplace transform (13.51), $B_1(\tau)$ is gained by (13.43), $g_1(\tau)$ and $f_1(\tau)$ are obtained by (13.37) with the explicit formulas (13.38) to (13.41), successively. Similarly, $\hat{g}_1(\zeta)$ is given by means of the Laplace transform, then $V_1(1, \tau)$ and its derivatives are given by the inverse Laplace transform (13.51), $B_2(\tau)$ is gained by (13.43), $g_2(\tau)$ and $f_2(\tau)$ are obtained by (13.37) with the explicit formulas (13.38) to (13.41), successively. In theory, we can gain $B_m(\tau)$ successively in this way, where $m = 1, 2, 3, \dots$.

Note that the inverse Laplace transform (13.51) can be expressed by a convolution integral of $K^{(m)}(1, \tau)$ and $g_n(\tau)$, say,

$$V_n^{(m)}(1, \tau) = \int_0^\tau K^{(m)}(1, \tau - t) g_n(t) dt. \quad (13.52)$$

Therefore, since $K(1, \tau)$ contains the complementary error function, the expressions of $V_n^{(m)}(1, \tau)$, and then $B_n(\tau)$, $g_n(\tau)$ and $f_n(\tau)$, become more and more complicated as n increases. Thus, it becomes more and more difficult to exactly calculate the Laplace transform $\hat{g}_n(\zeta) = \mathcal{L}_T[g_n(\tau)]$ and especially the inverse Laplace transform (13.51). Therefore, it becomes more and more difficult to exactly solve the high-order deformation equations.

In order to obtain the high-order approximations, Zhu (2006a) employed a numerical method to calculate the related integrals similar to (13.52). Unlike Zhu (2006a), Cheng et al. (2010) used analytic approximations of $B(\tau)$ in powers of $\sqrt{\tau}$ about the expiry date $\tau = 0$. Note that $K(1, \tau)$ is expressed by the exponential function and the complementary error function, whose Taylor series at $\tau = 0$ have the infinite radius of convergence. Besides, the dimensionless τ to expiry is often small in practice. Therefore, $B(\tau)$ may be well approximated by a polynomial of $\sqrt{\tau}$ as long as many enough terms are used, say, the order $o(\tau^M)$ is high enough.

Cheng et al. (2010) used the following strategy. First, both of $K^{(m)}(1, \tau)$ and $g_n(\tau)$ are expanded in power series of τ about $\tau = 0$ up-to the order $o(\tau^M)$, denoted by $\bar{K}^{(m)}(1, \tau)$ and $\bar{g}_n(\tau)$, respectively. It is found that they can be expressed by

$$\bar{K}^{(m)}(1, \tau) = \sum_{n=0}^{2M} a_{m,n} (\sqrt{\tau})^n \quad (13.53)$$

and

$$\bar{g}_n(\tau) = \sum_{n=0}^{2M} d_n (\sqrt{\tau})^n, \quad (13.54)$$

which are Hölder continuous with exponent $1/2$ in time and therefore agree well with Blanchet's (Blanchet, 2006) proof. Then, it is easy to gain their Laplace transforms

$$\check{K}^{(m)}(1, \zeta) = \mathcal{L}_T[\bar{K}^{(m)}(1, \tau)], \quad \check{g}_m(\zeta) = \mathcal{L}_T[\bar{g}_m(\tau)].$$

More importantly, it is rather convenient to gain their inverse Laplace transform

$$V_n^{(m)}(1, \tau) = \mathcal{L}_T^{-1} \left[\check{K}^{(m)}(1, \zeta) \check{g}_m(\zeta) \right],$$

since both of $\check{K}^{(m)}(1, \zeta)$ and $\check{g}_m(\zeta)$ are expressed in polynomials of $\zeta^{-1/2}$.

The N th-order homotopy-approximation of the dimensionless $B(\tau)$ in polynomial of $\sqrt{\tau}$ to the order $o(\tau^M)$ is given *explicitly* by

$$B(\tau) \approx \sum_{m=0}^N B_m(\tau) = \sum_{k=0}^{2M} b_k (\sqrt{\tau})^k, \quad (13.55)$$

where the coefficient b_k is dependent upon $\gamma = 2r/\sigma^2$ and the convergence-control parameter c_0 . To further modify the above approximation of $B(\tau)$, we first write $z = \sqrt{\tau}$, and then calculate the Padé approximation to $B = \sum_{n=0}^{2M} b_n z^n$, centered at $z = 0$ of degree $[M, M]$. Then, replacing z by $\sqrt{\tau}$, we gain the $[M, M]$ Padé approximant of $B(\tau)$ to the order $o(\tau^M)$.

13.3 Validity of the explicit homotopy-approximations

To show the accuracy and validity of the explicit expression (13.55) of the optimal exercise boundary $B(\tau)$ given by the above HAM approach, we compare it with some published analytic results. As pointed out by Cheng et al. (2010), all published explicit approximations of $B(\tau)$ are valid for $\tau \ll \tau_{\text{exp}}$. For example,

$$B(\tau) = \exp(-2\sqrt{\alpha\tau}), \quad \tau \ll \tau_{\text{exp}}, \quad (13.56)$$

where

$$\alpha = -\frac{1}{2} \ln(9\pi\gamma^2\tau), \quad \text{by Kuske and Keller (1998),} \quad (13.57)$$

$$\alpha = -\frac{1}{2} \ln \frac{4e\gamma^2\tau}{2 - B_p^2}, \quad \text{by Bunch and Johnson (2000),} \quad (13.58)$$

with the perpetual optimal exercise price

$$B_p = \frac{\gamma}{1 + \gamma}. \quad (13.59)$$

Besides, Knessl (2001) gave the following asymptotic formula

$$\ln[B(\tau)] = -\sqrt{2\tau|\ln(4\pi\gamma^2\tau)|} \left[1 + |\ln(4\pi\gamma^2\tau)|^{-2} \right], \quad \tau \ll \tau_{\text{exp}}. \quad (13.60)$$

By means of two examples, these asymptotic or perturbation formulas are compared with the explicit formula (13.55) given by the HAM approach mentioned above. All results reported below are gained by a Mathematica code given in the Appendix 13.2, which is free available at

<http://numericaltank.sjtu.edu.cn/HAM.htm>

Besides, all results given below have units, say, they are not dimensionless.

Example 13.1

Let us first consider the following sample case discussed by Wu and Kwok (1997), Carr and Faguet (1994), Zhu (2006a), Cheng et al. (2010) and also Cheng (2008):

- strike price $X = \$100$.
- risk-free interest rate $r = 0.1$.
- volatility $\sigma = 0.3$.
- time to expiration $T = 1$ (year).

Note that the zeroth-order deformation equation contains the convergence-control parameter c_0 , which is used in the frame of the HAM to guarantee the convergence of the homotopy-series. In this example, we choose $c_0 = -1$ for the sake of simplicity.

It is found that, neither of (13.57) given by Kuske and Keller (1998), nor of (13.60) by Knessl (2001), nor of (13.58) by Bunch and Johnson (2000), is valid for more than a couple of weeks prior to expiry, as shown in Fig. 13.1.

By means of the HAM-based approach mentioned above, we obtain the corresponding optimal exercise boundary $B(\tau)$ in the polynomial of $\sqrt{\tau}$ to the order $o(\tau^8)$, which agrees well with the results of Zhu (2006a) given by the numerical integral in the *whole* time, as shown in Fig. 13.1. At the expiration time $T = 1$ (year), the optimal exercise boundary $B(T)$ is \$76.25 in Wu and Kwok's (Wu and Kwok, 1997) numerical solution, \$76.11 by Zhu (2006a) numerical integral, and \$76.17 by the 20th-order homotopy-approximation in polynomial to the order $o(\tau^8)$ given by the HAM, as shown in Table 13.1. This indicates the validity of the analytic approach based on the HAM described above. Therefore, unlike the asymptotic and perturbation approximations mentioned above, which are often valid for only a couple of days or weeks prior to expiry, the explicit polynomial approximation of the optimal exercise boundary given by the HAM-based approach is accurate and valid even up-to several years. Note that even the 12th-order homotopy-approximation of $B(\tau)$ in polynomial of $\sqrt{\tau}$ to the order $o(\tau^8)$ is accurate enough, as shown in Fig. 13.1. Mathematically speaking, all asymptotic and perturbation formulas are valid only for $\tau \ll T$, but the homotopy-approximation in polynomials of $\sqrt{\tau}$ to the order $o(\tau^8)$ is valid even at $\tau = T$, i.e. the time to expiry!

It is found that the 10th-order homotopy-approximation of $B(\tau)$ in polynomial of $\sqrt{\tau}$ to the order $o(\tau^8)$ by means of $c_0 = -1$ is accurate enough up to 3 years

Fig. 13.1 Approximations of the optimal exercise boundary in the case of Example 13.1: $X = \$100$, $r = 0.1$, $\sigma = 0.3$ and $T = 1$ (year). Solid line: the 12th-order homotopy-approximation in polynomial of $\sqrt{\tau}$ to $o(\tau^8)$ by means of $c_0 = -1$; Symbols: result given by Zhu (2006a) with numerical integral; Dashed line A: (13.57) given by Kuske and Keller (1998); Dashed line B: (13.60) by Knessl (2001); Dashed line C: (13.58) by Bunch and Johnson (2000).

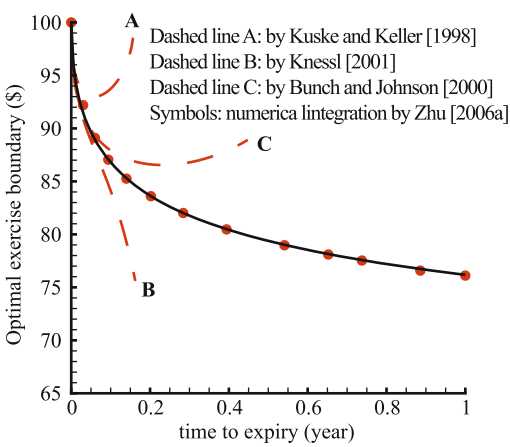


Table 13.1 The m th-order homotopy-approximation of the optimal exercise boundary $B(\tau)$ in polynomial of $\sqrt{\tau}$ up to $o(\tau^8)$ by means of $c_0 = -1$ at different time in case of $X = \$100$, $r = 0.1$, $\sigma = 0.3$ and $T = 1$ (year).

Order of approx. m	3 month	6 month	9 month	12 month
4	84.02	79.86	77.44	75.84
8	82.86	79.22	77.24	75.96
12	82.63	79.27	77.41	76.15
16	82.60	79.35	77.49	76.18
18	82.62	79.38	77.50	76.18
20	82.63	79.40	77.50	76.17

prior to expiry, as shown in Fig. 13.2, which is about 60 times longer than those of other asymptotic and/or perturbation formulas mentioned above! Besides, the Padé technique can be used to further enlarge the valid time of $B(\tau)$: writing $z = \sqrt{\tau}$ and then using Padé technique, we gain the [8,8] Padé approximant of $B(\tau)$ centered at $z = 0$, say,

$$B(\tau) \approx 100 \left[\frac{1 + \check{b}_1(\tau)}{1 + \check{b}_2(\tau)} \right] \tag{13.61}$$

where

$$\begin{aligned} \check{b}_1(\tau) &= -0.758595\tau^{1/2} + 0.8748811\tau - 0.474758\tau^{3/2} + 0.209634\tau^2 \\ &\quad - 0.0551746\tau^{5/2} + 0.01333\tau^3 - 0.00210274\tau^{7/2} + 2.15592 \times 10^{-4}\tau^4, \\ \check{b}_2(\tau) &= -0.228157\tau^{1/2} + 0.297781\tau - 0.0406677\tau^{3/2} + 0.0324789\tau^2 \\ &\quad - 0.0018480\tau^{5/2} + 0.0016651\tau^3 + 3.545 \times 10^{-6}\tau^{7/2} + 4.245 \times 10^{-5}\tau^4. \end{aligned}$$

Fig. 13.2 The $[8,8]$ Padé approximant of the optimal exercise boundary $B(\tau)$ in the case of Example 13.1: $X = \$100$, $r = 0.1$, $\sigma = 0.3$. Dashed-line: the 10th-order homotopy-approximation of $B(\tau)$ in the polynomial of $\sqrt{\tau}$ to $o(\tau^8)$ by $c_0 = -1$; Solid line: the corresponding $[8,8]$ Padé approximant of $B(\tau)$; Dash-dotted line: perpetual optimal exercise price.

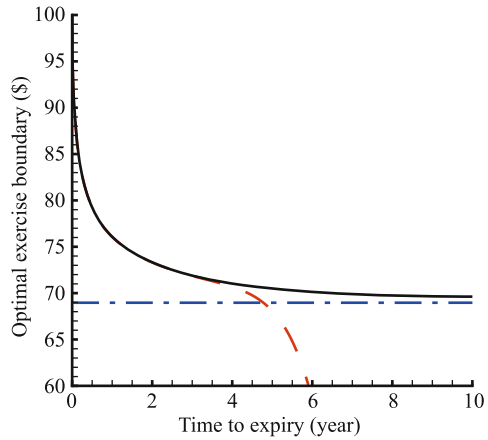
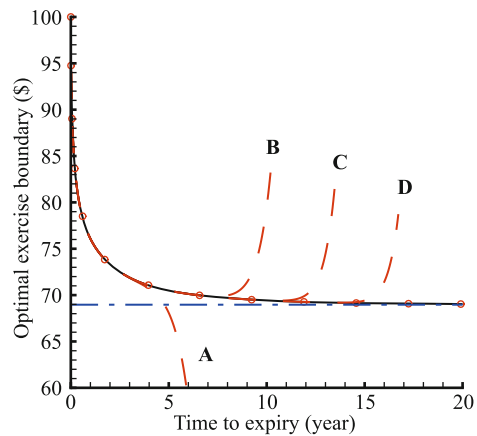


Fig. 13.3 The influence of the order $o(\tau^M)$ to the 10th-order homotopy-approximations of the optimal exercise boundary by means of $c_0 = -1$ in the case of Example 13.1: $X = \$100$, $r = 0.1$ and $\sigma = 0.3$. Dashed line A: $M = 8$; Dashed line B: $M = 16$; Dashed line C: $M = 24$; Dashed line D: $M = 32$; Solid line: $M = 48$; Symbols: $[48,48]$ Padé approximant; Dash-dotted line: perpetual optimal exercise price.



As shown in Fig. 13.2, this $[8,8]$ Padé approximant of $B(\tau)$ is accurate enough even up to 10 years prior to expiry, about 200 times longer than those of other asymptotic and/or perturbation formulas mentioned above!

For the same problem, Cheng et al. (2010) and Cheng (2008) gave homotopy-approximations of $B(\tau)$ up to $o(\tau^6)$ and $o(\tau^7)$ by means of the HAM, respectively. Both of them are valid even at $\tau = T = 1$ year, i.e. the time to expiry. In this chapter, we gained the 10th-order homotopy-approximations of $B(\tau)$ up to $o(\tau^8)$, which is accurate up to 3 years prior to expiry. In theory, $B(\tau)$ can be expanded into the polynomial of $\sqrt{\tau}$ to the order $o(\tau^M)$ for arbitrary positive integer $M \geq 1$. Does the maximum valid time of the homotopy-approximation of $B(\tau)$ up to $o(\tau^M)$ strongly depend upon M ?

Using the explicit formulas given in the Appendix 13.1, it is convenient now for us to investigate the influence of the order $o(\tau^M)$ of $B(\tau)$. It is found that, the higher the order of $o(\tau^M)$, the longer the maximum valid time of the 10th-order homotopy-approximation of $B(\tau)$ prior to expiry, as shown in Fig. 13.3. Note that,

Fig. 13.4 The influence of the order $o(\tau^M)$ to the maximum valid time (year) prior to expiry for the 10th-order homotopy-approximations of the optimal exercise boundary by $c_0 = -1$ in the case of Example 13.1: $X = \$100$, $r = 0.1$ and $\sigma = 0.3$. Dash-dotted line: formula $1.6008 + 0.3853M$ given by a least-squares fit; Symbols: the maximum valid time prior to expiry for $B(\tau)$ in polynomial of $\sqrt{\tau}$ to $o(\tau^M)$.

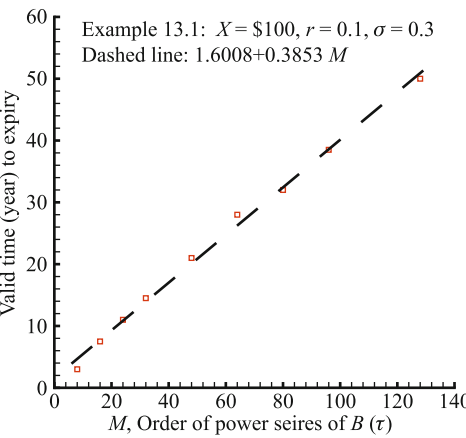


Table 13.2 The maximum valid time (years) prior to expiry for the 10th-order homotopy-approximation of the optimal exercise boundary $B(\tau)$ in polynomial of $\sqrt{\tau}$ to different orders $o(\tau^M)$ in case of Example 13.1 with $X = \$100$, $r = 0.1$ and $\sigma = 0.3$ by means of $c_0 = -1$.

The order $o(\tau^M)$ of the polynomial for $B(\tau)$	Maximum valid time (years) prior to expiry
$M = 8$	3
$M = 16$	7.5
$M = 24$	11
$M = 32$	14.5
$M = 48$	21
$M = 64$	28
$M = 80$	32
$M = 96$	38.5
$M = 128$	50

the 10th-order homotopy-approximation of $B(\tau)$ in polynomial of $\sqrt{\tau}$ to $o(\tau^{48})$ is accurate enough even up to 20 years prior to expiry, which is about 400 times longer than those of asymptotic and/or perturbation formulas mentioned above! As shown in Fig.13.3, the optimal exercise price given by $M = 48$ tends to the perpetual optimal exercise price B_p so closely that the combination of the 10th-order homotopy-approximation of $B(\tau)$ in polynomial to $o(\tau^{48})$ with the known perpetual optimal exercise price $B_p = \gamma/(1 + \gamma)$ gives accurate enough optimal exercise price even in the *whole* time ($0 \leq \tau < +\infty$) prior to expiry! Using the Mathematica code given in the Appendix 13.2, such kind of accurate analytic approximation is gained by means of a laptop (MacBook Pro with 2.8 GHz and 4GB MHz DDR3) in 102 seconds only.

As shown in Fig. 13.4 and Table 13.2, the 10th-order homotopy approximation of $B(\tau)$ in polynomial of $\sqrt{\tau}$ to $o(\tau^{128})$ is valid even up to a *half century* prior to expiry, which is about *one thousand* times longer than those of asymptotic and/or perturbation formulas mentioned above!

Especially, it is found that the maximum valid time of the 10th-order homotopy-approximation of $B(\tau)$ in polynomial of $\sqrt{\tau}$ to $o(\tau^M)$ is directly proportional to M ,

although approximately, as shown in Fig. 13.4. It suggests that, given an arbitrary time T prior to expiry, we can always gain the corresponding accurate enough optimal exercise price, as long as the N th-order homotopy-approximation of $B(\tau)$ in polynomial of $\sqrt{\tau}$ to $o(\tau^M)$ has large enough M with a reasonably high order N . This is an excellent example to illustrate that the HAM can give much better explicit, analytic approximations of some nonlinear problems than asymptotic and/or perturbation methods!

Indeed, *a truly new method always gives something new and/or different*. This example shows the originality and great potential of the HAM once again.

Example 13.2

The second example is a long term option considered by Chen et al. (2000):

- strike price $X = \$1$.
- risk-free interest rate $r = 0.08$.
- volatility $\sigma = 0.4$.
- time to expiration $T = 3$ (year).

As shown in Fig. 13.5, the 10th-order homotopy-approximation of $B(\tau)$ in the polynomial of $\sqrt{\tau}$ to $o(\tau^{48})$ by means of $c_0 = -1$ is valid even up to 20 years prior to expiry, while neither of (13.57) given by Kuske and Keller (1998), nor of (13.60) by Knessl (2001), nor of (13.58) by Bunch and Johnson (2000), is valid for more than one month prior to expiry! Note that our homotopy-approximation fits well with its [48,48] Padé approximant in the whole time interval $0 \leq \tau \leq 20$ year. Such kind of accurate analytic approximation is gained by means of a laptop (MacBook Pro with 2.8 GHz and 4GB MHz DDR3) in only 57 seconds. At the expiry time $T = 20$ year, the 10th-order homotopy-approximation and the corresponding [48,48] Padé approximant give the same optimal exercise price \$0.5029, which is so close to the perpetual optimal exercise price \$0.5 that $B(\tau) = 0.5$ is an accurate enough approximation for $\tau > 20$. So, combining the 10th-order homotopy-approximation of $B(\tau)$ in the polynomial of $\sqrt{\tau}$ to $o(\tau^{48})$ for $0 \leq \tau \leq 20$ (year) and the perpetual optimal exercise price $B(\tau) = 0.5$ for $\tau > 20$ (year), we have an analytic approximation accurate enough in the *whole* time domain $0 \leq \tau < +\infty$, i.e. for *arbitrary* time to expiry!

Like Cheng et al. (2010) and Cheng (2008), we obtained all of above homotopy-approximations by means of the convergence-control parameter $c_0 = -1$. However, as proved in Chapter 4 and shown in this book, the convergence-control parameter c_0 provides us a convenient way to guarantee the convergence of homotopy-series. As shown in Fig. 13.6, the 10th-order homotopy-approximation of $B(\tau)$ in polynomial of $\sqrt{\tau}$ to $o(\tau^8)$ by $c_0 = -1$ is valid about 6 years prior to expiry. However, when $c_0 = -1/2$, it is valid about 10 years prior to expiry: the maximum valid time of $B(\tau)$ increases about 66%. So, the convergence-control parameter c_0 also provides us an

Fig. 13.5 The optimal exercise boundary $B(\tau)$ in case of Example 13.2: $X = \$1$, $r = 0.08$ and $\sigma = 0.4$. Dashed line A: (13.57) by Kuske and Keller (1998); Dashed line B: (13.60) by Knessl (2001); Dashed line C: (13.58) by Bunch and Johnson (2000); Solid line: 10th-order homotopy-approximation of $B(\tau)$ by $c_0 = -1$ in polynomial of $\sqrt{\tau}$ to $o(\tau^{48})$; Symbols: 10th-order homotopy-approximation of $B(\tau)$ by Padé method; Dash-dotted line: perpetual optimal exercise price \$0.5.

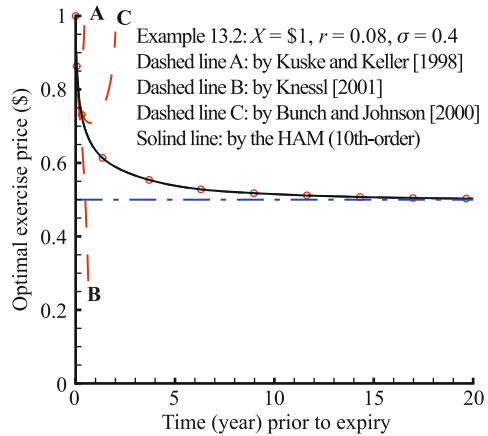
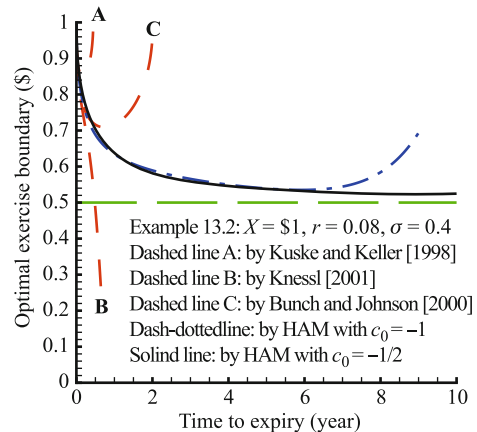


Fig. 13.6 The optimal exercise boundary in case of Example 13.2: $X = \$1$, $r = 0.08$ and $\sigma = 0.4$. Dashed line A: (13.57) by Kuske and Keller (1998); Dashed line B: (13.60) by Knessl (2001); Dashed line C: (13.58) by Bunch and Johnson (2000); Dash-dotted line: 10th-order homotopy-approximation of $B(\tau)$ in polynomial of $\sqrt{\tau}$ to $o(\tau^8)$ by $c_0 = -1$; Solid line: 10th-order homotopy-approximation of $B(\tau)$ in polynomial of $\sqrt{\tau}$ to $o(\tau^8)$ by $c_0 = -1/2$; Long-dashed line: perpetual optimal exercise price \$0.5.



alternative way to enlarge the maximum valid time of the homotopy-approximation of $B(\tau)$ prior to expiry.

These two examples illustrate that, unlike other asymptotic and /or perturbation formulas which are often valid only a couple of days or weeks prior to expiry, the HAM provides us an accurate approximation of the optimal exercise boundary $B(\tau)$ in polynomial of $\sqrt{\tau}$ to $o(\tau^M)$, which is often valid a couple of dozen years, or even a half century, prior to expiry, as long as M and the order of approximation are reasonably large with a properly chosen convergence-control parameter c_0 .

As pointed out by Kim (1990) and Carr et al. (1992), when the optimal exercise boundary $B(\tau)$ is known, it is easy to gain the price $V(S, \tau)$ of American put option by means of (13.7). So, by means of the explicit analytic approximation (13.55) of $B(\tau)$ in polynomial of $\sqrt{\tau}$ given by the HAM-based approach mentioned in this

chapter, it is convenient to gain accurate approximation of the price $V(S, \tau)$, which maybe valid even up to a half century prior to expiry!

13.4 A practical code for businessmen

For all examples considered by Cheng et al. (2010) and Cheng (2008), we gain accurate optimal exercise price in much longer time intervals (prior to expiry) by means of the 10th-order homotopy-approximation of $B(\tau)$ in polynomial of $\sqrt{\tau}$ to $o(\tau^{48})$. Besides, it is found that the 10th-order homotopy-approximation of $B(\tau)$ in polynomial of $\sqrt{\tau}$ to $o(\tau^{48})$ by means of $c_0 = -1$ is often valid to such a long time that the known perpetual optimal exercise price becomes a good enough approximation thereafter. Therefore, practically speaking, we have the accurate optimal exercise price $B(\tau)$ and the option price $V(S, \tau)$ for *arbitrarily* large expiration-time $0 < T < +\infty$. This suggests that the homotopy-approximation of $B(\tau)$ in the polynomial of $\sqrt{\tau}$ to $o(\tau^{48})$ is valid in general and thus can be widely used in business related to American put option.

As mentioned above, by means of a laptop (MacBook Pro with 2.8 GHz and 4GB MHz DDR3), only 102 and 57 seconds CPU time is needed to gain the 10th-order homotopy-approximations of $B(\tau)$ in polynomial of $\sqrt{\tau}$ to $o(\tau^{48})$ by means of $c_0 = -1$ for Examples 13.1 and 13.2, respectively. This is fast enough for a scholar, but not for a businessman. For convenience sake, a practical Mathematica code is provided in Appendix 13.3, which can give accurate enough optimal exercise price of American put option for rather large expiration-time T in a few seconds!

Note that the dimensionless equations (13.10) to (13.14) contain only one dimensionless parameter $\gamma = 2r/\sigma^2$. So, the dimensionless $B(\tau)$ is only dependent upon γ . Using the Mathematica code given in Appendix 13.2 and a high-performance computer, we obtained the 10th-order homotopy-approximation of the dimensionless $B(\tau)$ in polynomial of $\sqrt{\tau}$ to $o(\tau^M)$ by means of $c_0 = -1$ for the unknown parameter γ , where M is a reasonably large integer such as $M = 24, 36, 48$ and so on. Then, these lengthy homotopy-approximations of the dimensionless $B(\tau)$ are saved in data files with different names such as APO-48-10.txt, corresponding to the 10th-order homotopy-approximation to the order $o(\tau^{48})$. A short Mathematica code, namely APOh, is given in Appendix 13.3, which first reads all dimensionless homotopy-approximations of $B(\tau)$ saved in such a data file, and then calculates the dimensional optimal exercise price according to a given strike price X , risk-free interest rate r , volatility σ and time to expiration T (year). Using a laptop, one can often gain an accurate enough exercise price of American put option for rather large expiration-time T in only a few seconds! This practical Mathematica code with a simple users guide is available at

<http://numericaltank.sjtu.edu.cn/HAM.htm>

Note that such kind of *explicit* homotopy-approximations of $B(\tau)$ in polynomial of $\sqrt{\tau}$ are *analytic*, because they are continuous for any $0 < \tau < +\infty$ and their

derivatives $B'(\tau)$, $B''(\tau)$ and so on exist for all $\tau > 0$. Thus, any interpolations are unnecessary at all. It is true that such kind of analytic approximations given by the HAM are very lengthy, which might be hundreds pages long if printing out. From the traditional points of view, such kind of explicit, lengthy analytic formula might be useless. Fortunately, we are now in the times of computer: a laptop can read and calculate such kind of explicit, lengthy analytic formulas in a few seconds, which is even faster than calculating a simple, half-page length analytic formula by hands! So, if we regard the keyboard of a laptop as a pen, its hard disk as papers, and its central processing unit (CPU) as a brain of human-being, then an explicit, analytic formula in the times of computer can be very lengthy, as shown in this section. Note that the traditional concept “analytic” appeared several hundreds of years ago, when our computational tools were rather inefficient. The practical Mathematica code APOh is an excellent example to verify that our traditional concept “analytic” is *out of date*, and thus should be greatly *modified* in the times of computer and internet. This also shows, on the other hand, the originality of the HAM.

Indeed, *a truly new method always gives something new and/or different.*

13.5 Concluding remarks

The American put option is governed by a linear PDE with variable coefficients, subject to some linear initial and boundary conditions. However, it is in principle a nonlinear problem, since the two boundary conditions are satisfied on an unknown moving boundary $B(\tau)$. The asymptotic and/or perturbation formulas (13.57) given by Kuske and Keller (1998), (13.58) by Bunch and Johnson (2000) and (13.60) by Knessl (2001), respectively, are often valid only a couple of days or weeks prior to expiry, which is too short for the practical use in business.

It was Cheng et al. (2010) who first combined the HAM with the Laplace transform in such a way that the homotopy-approximations of the optimal exercise boundary $B(\tau)$ in polynomial of $\sqrt{\tau}$ to $o(\tau^6)$ were obtained. Cheng (2008) further gave a homotopy-approximation of $B(\tau)$ in polynomial to $o(\tau^7)$. Unlike the asymptotic and/or perturbation formulas, these homotopy approximations of $B(\tau)$ are valid for several years, and thus are much better than all asymptotic and/or perturbation formulas mentioned above.

In this chapter, we further modified the HAM-based approach of Cheng et al. (2010) and Cheng (2008) by means of deriving explicit formulas (13.35) to (13.41) related to the unknown moving boundary. Especially, we investigate, for the first time, the influence of the order $o(\tau^M)$ of polynomials of $B(\tau)$ to its maximum valid time prior to expiry. It is found that, the maximum valid time of $B(\tau)$ in polynomial of $\sqrt{\tau}$ to $o(\tau^M)$ is directly proportional to M , so that, as long as M is large enough, the explicit homotopy-approximation of $B(\tau)$ can be valid even up to a half-century prior to expiry, about 1000 times longer than those of asymptotic and/or perturbation formulas mentioned above, as shown in Example 13.1. Besides, it is found that the 10th-order homotopy-approximation of $B(\tau)$ in polynomial of $\sqrt{\tau}$ to $o(\tau^{48})$ by $c_0 =$

-1 is often valid up to so many years prior to expiry that the theoretical perpetual optimal exercise price

$$B_p = \left(\frac{\gamma}{1 + \gamma} \right) X$$

is accurate enough thereafter. Therefore, the combination of such kind of homotopy-approximation of $B(\tau)$ in polynomial of $\sqrt{\tau}$ to $o(\tau^{48})$ and the theoretical perpetual optimal exercise price B_p mentioned above can be regarded as an accurate analytic approximation of $B(\tau)$ valid in the *whole* time domain $0 \leq \tau < +\infty$. Based on the HAM approach, a Mathematica code is given in Appendix 13.2 and free available at

<http://numericaltank.sjtu.edu.cn/HAM.htm>

Using this code with a high-performance computer, we gained the 10th-order (dimensionless) homotopy-approximation of $B(\tau)$ in polynomial of $\sqrt{\tau}$ to $o(\tau^{48})$ for the unknown dimensionless parameter $\gamma = 2r/\sigma^2$, and saved it in a data-file by the name of `APO-48-10.txt`, which can be directly used to gain accurate enough optimal exercise price of American put option in a few seconds by means of the short, practical Mathematica code `APOh`. Both of the practical code `APOh` and the data-file `APO-48-10.txt` are available at the same website mentioned above. Obviously, the `APOh` may provide businessmen a convenient tool in business.

In addition, we also investigate, for the first time, the influence of the convergence-control parameter c_0 to the maximum valid time (prior to expiry) of the homotopy-approximation of $B(\tau)$ in polynomial of $\sqrt{\tau}$. As shown in Fig. 13.6, the maximum valid time of the 10th-order homotopy-approximation of $B(\tau)$ in polynomial of $\sqrt{\tau}$ to $o(\tau^8)$ given by $c_0 = -1/2$ is about 66% longer than that by $c_0 = -1$. This suggests that the convergence-control parameter c_0 also provides us a convenient way to enlarge the maximum valid time of $B(\tau)$ prior to expiry.

Note that Landau transform (Landau, 1950) is not used here. So, this analytic approach based on the HAM and Laplace transform has quite general meanings, and thus can be widely applied to solve similar problems in finance, such as the optimal exercise boundary of American options on an underlying asset with dividend yield (Zhu, 2006b) and so on.

Finally, we emphasize that, unlike all asymptotic and/or perturbation formulas mentioned above, which are often valid only a couple of days or weeks prior to expiry, the analytic homotopy-approximations of $B(\tau)$ in polynomials of $\sqrt{\tau}$ may be valid a couple of dozen years, or even a half century! Besides, although these accurate homotopy-approximations of $B(\tau)$ saved in the data-file `APO-48-10.txt` might be several hundreds pages long when printed out, we can gain accurate optimal exercise price of American put option in a few seconds by means of the short, practical Mathematica code `APOh` in a laptop! This is an excellent example to show that the traditional concept of “analytic” solutions is *out of date* and thus should be *modified* in the times of computer and internet.

Acknowledgements The formulas about the inverse Laplace transform used in Appendix 13.2 are provided by Dr. Jun Cheng.

Appendix 13.1 Detailed derivation of $f_n(\tau)$ and $g_n(\tau)$

According to (13.28), we define

$$[\Lambda(\tau; q) - B_0(\tau)]^m = \left[\sum_{i=1}^{+\infty} B_i(\tau) q^i \right]^m = \sum_{n=m}^{+\infty} \mu_{m,n}(\tau) q^n, \quad (13.62)$$

with the definition

$$\mu_{1,n}(\tau) = B_n(\tau), \quad n \geq 1. \quad (13.63)$$

Then, it holds

$$\begin{aligned} [\Lambda(\tau; q) - B_0(\tau)]^{m+1} &= \sum_{n=m+1}^{+\infty} \mu_{m+1,n}(\tau) q^n \\ &= \left[\sum_{n=m}^{+\infty} \mu_{m,n}(\tau) q^n \right] \left[\sum_{i=1}^{+\infty} B_i(\tau) q^i \right] = \sum_{n=m+1}^{+\infty} q^n \left[\sum_{i=m}^{n-1} \mu_{m,i}(\tau) B_{n-i}(\tau) \right] \end{aligned} \quad (13.64)$$

which gives the recursion formula

$$\mu_{m+1,n}(\tau) = \sum_{i=m}^{n-1} \mu_{m,i}(\tau) B_{n-i}(\tau). \quad (13.65)$$

For the sake of simplicity, define

$$\psi_{n,0}(\tau) = V_n(B_0, \tau), \quad \psi_{n,m}(\tau) = \frac{1}{m!} \frac{\partial^m V_n(S, \tau)}{\partial S^m} \Big|_{S=B_0(\tau)}. \quad (13.66)$$

Then, on the moving boundary $S = \Lambda(\tau; q)$, We have by means of Taylor expansion at $S = B_0(\tau)$ that

$$\begin{aligned} V_n(S, \tau) &= \psi_{n,0}(\tau) + \sum_{m=1}^{+\infty} \psi_{n,m}(\tau) [\Lambda(\tau; q) - B_0(\tau)]^m \\ &= \psi_{n,0}(\tau) + \sum_{m=1}^{+\infty} \psi_{n,m}(\tau) \left[\sum_{i=m}^{+\infty} \mu_{m,i}(\tau) q^i \right] \\ &= \psi_{n,0}(\tau) + \sum_{i=1}^{+\infty} q^i \left[\sum_{m=1}^i \psi_{n,m}(\tau) \mu_{m,i}(\tau) \right] \\ &= V_n(B_0, \tau) + \sum_{i=1}^{+\infty} \alpha_{n,i}(\tau) q^i, \end{aligned} \quad (13.67)$$

where

$$\alpha_{n,i}(\tau) = \sum_{m=1}^i \psi_{n,m}(\tau) \mu_{m,i}(\tau), \quad i \geq 1. \quad (13.68)$$

Therefore, on the moving boundary $S = \Lambda(\tau; q)$, we have

$$\begin{aligned}\phi(S, \tau; q) &= \sum_{n=0}^{+\infty} V_n(S, \tau) q^n = \sum_{n=0}^{+\infty} q^n \left[V_n(B_0, \tau) + \sum_{i=1}^{+\infty} \alpha_{n,i}(\tau) q^i \right] \\ &= \sum_{n=0}^{+\infty} q^n \left[V_n(B_0, \tau) + \sum_{j=0}^{n-1} \alpha_{j,n-j}(\tau) \right] \\ &= \sum_{n=0}^{+\infty} [V_n(B_0, \tau) + f_n(\tau)] q^n,\end{aligned}\tag{13.69}$$

where

$$f_n(\tau) = \sum_{j=0}^{n-1} \alpha_{j,n-j}(\tau).\tag{13.70}$$

When $q = 0$, it holds on $S = \Lambda(\tau; 0) = B_0(\tau)$ that

$$\phi(\Lambda, \tau; q) = \phi(B_0, \tau; 0) = V_0(B_0, \tau).$$

So, we have $f_0(\tau) = 0$ and thus it holds

$$\phi(S, \tau; q) = V_0(B_0, \tau) + \sum_{n=1}^{+\infty} [V_n(B_0, \tau) + f_n(\tau)] q^n\tag{13.71}$$

on the moving boundary $S = \Lambda(\tau; q)$.

Similarly, we have on the moving boundary $S = \Lambda(\tau; q)$ that

$$\begin{aligned}\frac{\partial V_n(S, \tau)}{\partial S} &= \psi_{n,1}(\tau) + \sum_{m=1}^{+\infty} (m+1) \psi_{n,m+1}(\tau) [\Lambda(\tau; q) - B_0(\tau)]^m \\ &= V'_n(B_0, \tau) + \sum_{i=1}^{+\infty} \beta_{n,i}(\tau) q^i,\end{aligned}\tag{13.72}$$

where the prime denotes the differentiation with respect to S , and

$$\beta_{n,i}(\tau) = \sum_{m=1}^i (m+1) \psi_{n,m+1}(\tau) \mu_{m,i}(\tau), \quad i \geq 1.\tag{13.73}$$

Furthermore, it holds on the moving boundary $S = \Lambda(\tau; q)$ that

$$\frac{\partial \phi(S, \tau; q)}{\partial S} = V'_0(B_0, \tau) + \sum_{n=1}^{+\infty} [V'_n(B_0, \tau) + g_n(\tau)] q^n,\tag{13.74}$$

where

$$g_n(\tau) = \sum_{j=0}^{n-1} \beta_{j,n-j}(\tau), \quad n \geq 1.\tag{13.75}$$

Appendix 13.2 Mathematica code for American put option

The American put option equation is solved by means of the HAM with the Laplace transform. This code is available at

<http://numericaltank.sjtu.edu.cn/HAM.htm>

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Mathematica code for American put option

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December 2010

```
<<Calculus`Pade`;
<<Graphics`Graphics`;

(* Define approx[f] for Taylor expansion of f *)
approx[f_] := Module[{temp},
temp[0] = Series[f, {t, 0, OrderTaylor}]/Normal;
temp[1] = temp[0]/.t^(n_)*Derivative[j_][DiracDelta][0]->0;
temp[2] = temp[1]/.t^(n_)*DiracDelta[0]->0;
temp[3] = temp[2]/.DiracDelta[0]->0;
temp[4] = temp[3]/.Derivative[j_][DiracDelta][0]->0;
temp[5] = N[temp[4], 60]/Expand;
```

```

If[KeyCutOff == 1, temp[5] = temp[5]//Chop];
temp[5]
];

(* Define approx2[f] for Taylor expansion of f *)
approx2[f_] := Module[{temp},
temp[0] = Expand[f];
temp[1] = temp[0] /. Derivative[n_] [DiracDelta][t] -> dd[n];
temp[2] = temp[1] /. DiracDelta[t] -> dd[0];
temp[3] = Series[temp[2],{t, 0, OrderTaylor}]/Normal;
temp[4] = temp[3] /. dd[0] -> DiracDelta[t];
temp[5]=temp[4]/.dd[n_]->Derivative[n] [DiracDelta][t];
temp[6] = N[temp[5],60]//Expand;
If[KeyCutOff == 1, temp[6] = temp[6]//Chop];
temp[6]
];

(* Define GetLK[n] *)
lamda      := (1 - gamma)/2 - 1/2*Sqrt[(4 p + (1 + gamma)^2)];
kernel[s_] := -s^lamda/lamda;
LK0[0]      = -1/lamda;
LK0[i_]     := D[kernel[s], {s, i}] /. s -> 1 // Expand;

GetLK[m0_,m1_,Nappr_] := Module[{temp,K1,K2,lK1,lK2},
For[i = Max[m0,0], i <= m1, i++,
  K[i] = invl[LK0[i]];
  K1[i] = K[i]/.{Derivative[_] [DiracDelta][t_]->0,
    DiracDelta[t_]->0};
  K2[i] = Collect[K[i]-K1[i],{DiracDelta[t],
    Derivative[Blank[]] [DiracDelta][t]}];
  temp = Series[K1[i],{t,0,Nappr}]/Normal;
  lK1[i] = LaplaceTransform[temp, t, p];
  lK2[i] = LaplaceTransform[K2[i],t, p];
  LK[i] = Collect[lK1[i] + lK2[i], p];
];
];

(* Define Getf[n] and Getg[n] *)
mu[m_,n_] := If[m == 1,b[n],Sum[mu[m-1,i]*b[n-i],{i,m-1,n-1}]];
psi[n_,m_] := dV[n,m]/m!;
alpha[n_,i_] := Sum[psi[n,m]*mu[m,i],{m,1,i}];
beta[n_,i_] := Sum[(m+1)*psi[n,m+1]*mu[m,i],{m,1,i}];
f[0] := 0;
g[0] := 1;
Getf[n_] := Sum[alpha[j,n-j],{j,0,n-1}];
Getg[n_] := Sum[beta[j,n-j],{j,0,n-1}];

(* Define Getb[n] *)
b[0] := 1;
BB[0] := 1;
B[0] := X;
Getb[n_] := Module[{temp},
If[n == 1,
  b[1] = c0*dV[0, 0] // Expand,

```



```

temp = b[n - 1] + c0*(b[n - 1]+dV[n-1,0]+f[n-1])//Expand;
b[n] = approx[temp];
];

(* Define GetDV[m,n] *)
GetDV[m_, n_] := Module[{temp},
If[n == 1, DV[m, 1] = -g[m],
temp[1] = Expand[LK[n]*Lg[m]];
DV[m, n] = invl[temp[1]];
];
DV[m, n] = approx[DV[m, n]];
];

(* Define dV[m,n] *)
dV[m_,n_] := Module[{temp},
If[NumberQ[flag[m,n]],
Goto[100],
GetDV[m,n];
flag[m,n] = 1
];
Label[100];
DV[m,n]
];

(* Define hp[f_,m_,n_] *)
hp[f_,m_,n_] := Module[{k,i,df,res,q},
df[0] = f[0];
For[k = 1, k <= m+n, k++, df[k] = f[k] - f[k-1]//Expand];
res = df[0] + Sum[df[i]*q^i,{i,1,m+n}];
Pade[res,{q,0,m,n}]/.q->1
];

(* Get [m,n] Pade approximant of B *)
pade[order_] := Module[{temp,s,i,j},
temp[0] = BB[order] /. t^i_ -> s^(2*i);
temp[1] = Pade[temp[0],{s,0,OrderTaylor,OrderTaylor}];
If[KeyCutoff == 1, temp[1] = temp[1]//Chop];
BBpade[order] = temp[1] /. s^j_ -> t^(j/2);
Bpade[order] = X*BBpade[order]/. t -> (sigma^2*t/2);
];

(* Define inverse Laplace transformation *)
invl[Sqrt[p]] := -1/(2*Sqrt[Pi]*t^(3/2));

invl[p^n_] := Module[{temp, nInt},
nInt = IntegerPart[n];
If[n > 1/2 && n > nInt,
Goto[100],
temp[2] = InverseLaplaceTransform[p^n, p, t];
Goto[200];
];
Label[100];

```

```

temp[1] = -1/2/Sqrt[Pi]/t^(3/2);
temp[2] = D[temp[1], {t, nInt}];
Label[200];
temp[2]//Expand
];

invl[d_./(c_ + a_.*Sqrt[4p + b_.])] := Module[{temp},
temp[1] = d/(4a)*Exp[-b*t/4];
temp[2] = 2/Sqrt[Pi*t];
temp[3] = c/a*Exp[c^2*t/(4a^2)]*Erfc[c*Sqrt[t]/(2a)];
temp[1]*(temp[2]-temp[3])//Expand
];

invl[d_./(p*(c_ + a_.*Sqrt[4p + b_.]))] := Module[{temp},
temp[1] = Sqrt[b]*Erf[Sqrt[b*t]/2];
temp[2] = c/a*Exp[-(b-(c/a)^2)*t/4]*Erfc[c*Sqrt[t]/(2a)];
temp[3] = -1/(b - (c/a)^2)*d/a*(c/a-temp[1]-temp[2]);
temp[3]//Expand
];

invl[p^i_.*Sqrt[c_.*p + a_.]] := Module[{temp},
temp = D[-Exp[-a*t/c]/(2*c*Sqrt[Pi]*(t/c)^(3/2)), {t, i}];
temp//Expand
];

invl[Sqrt[c_.*p+a_.]]:=-Exp[-a*t/c]/(2*c*Sqrt[Pi]*(t/c)^(3/2));
invl[f_] := InverseLaplaceTransform[f, p, t] // Expand;
invl[p_Plus] := Map[invl, p];
invl[c_*f_] := c*invl[f] /; FreeQ[c, p];

(* Main code *)
ham[m0_, m1_] := Module[{temp, k, n},
If[m0 == 1,
Print[" Strike price = ?"];
temp[0] = Input[];
If[!NumberQ[temp[0]],Goto[100]];
X = IntegerPart[temp[0]*10^10]/10^10;
Print[" Risk-free interest rate = ?"];
temp[0] = Input[];
If[!NumberQ[temp[0]],Goto[100]];
r = IntegerPart[temp[0]*10^10]/10^10;
Print[" Volatility = ?"];
temp[0] = Input[];
If[!NumberQ[temp[0]],Goto[100]];
sigma = IntegerPart[temp[0]*10^10]/10^10;
Print[" Time to expiry = ?"];
temp[0] = Input[];
If[!NumberQ[temp[0]],Goto[100]];
T = IntegerPart[temp[0]*10^10]/10^10;
gamma = 2*r/sigma^2;
texp = sigma^2*T/2;
Bp = X*gamma/(1 + gamma);
Label[100];

```

```

If[!NumberQ[gamma],
  X      = .;
  r      = .;
  sigma  = .;
  gamma  = .;
  T      = .;
];
Print["-----"];
Print[" INPUT PARAMETERS: "];
Print["   Strike price           (X) = ",X," ($) "];
Print["   Risk-free interest rate (r) = ",r];
Print["   Volatility                (sigma) = ",sigma];
Print["   Time to expiry           (T) = ",T," (year)"];
Print["-----"];
Print[" CORRESPONDING PARAMETERS: "];
Print["   gamma = ",gamma];
Print["   dimensionless time to expiry (texp)=" ,texp//N];
Print["   perpetual optimal exercise price (Bp)=" ,Bp//N," ($) "];
Print["-----"];
Print[" CONTROL PARAMETERS: "];
Print["   OrderTaylor = ",OrderTaylor];
Print["   c0          = ",c0];
Print["-----"];
KeyCutOff = If[OrderTaylor < 80 && NumberQ[gamma], 1, 0];
If[KeyCutOff == 1,
  Print["Command Chop is used to simplify the result"],
  Print["Command Chop is NOT used "];
];
If[NumberQ[gamma],
  Print["Pade technique is used"],
  Print["Pade technique is NOT used"]
];
Clear[f,flag,DV];
];
For[k = Max[1, m0], k <= m1, k++,
  Print[" k = ", k];
  If[k == 1, GetKK[]; GetBJ[]; GetKn[]];
  If[k == 1, Lg[0] = LaplaceTransform[g[0], t, p]];
  If[k == 1, GetLK[0,2,OrderTaylor],
    GetLK[k+1,k+1,OrderTaylor]];

  Getb[k];
  BB[k] = Collect[BB[k - 1] + b[k], t];
  temp[0] = X*BB[k] /. t->(sigma^2*t/2)//Expand;
  B[k] = Collect[temp[0],t];
  If[NumberQ[gamma],pade[k]];
  temp[1] = Getg[k];
  temp[2] = Getf[k];
  g[k] = approx2[temp[1]];
  f[k] = approx2[temp[2]];
  Lg[k] = LaplaceTransform[g[k], t, p];
  If[NumberQ[gamma] && NumberQ[sigma] && NumberQ[X],
    Print[" Optimal exercise price at the time
           to expiration = ", B[k]/.t->T//N];
    Print[" Modified result given

```

```

        by Pade technique = ",Bpade[k]/.t->T//N];
    ];
];
Print[" Well done !"];
If[NumberQ[gamma] && NumberQ[sigma] && NumberQ[X],
  Plot[{Bp,B[m1],Bpade[m1]},{t,0,1.25*T},
    PlotRange -> {0.8*Bp,X}, PlotStyle ->
      {RGBColor[1,0,0],RGBColor[0,1,0],RGBColor[0,0,1]}}];
Print[" Order of homotopy-approximation : ",m1];
Print[" Green line : optimal exercise boundary B
      in polynomial "];
Print[" Blue line : optimal exercise boundary B
      by Pade method "];
Print[" Red line : perpetual optimal exercise price "];
];
];

(* Dimensionless analytic formula given by Kuske and Keller *)
GetKK[] := Module[{alpha},
  alpha = -Log[9*Pi*gamma^2*t]/2;
  KK0 = Exp[-2*Sqrt[alpha*t]];
  KK = X*KK0 /. t->sigma^2/2*t;
];

(* Dimensionless formula given by Bunch and Johnson *)
GetBJ[] := Module[{alpha},
  Bp0 = gamma/(1+gamma);
  alpha = -Log[4*E*gamma^2*t/(2 - Bp0^2)]/2;
  BJ0 = Exp[-2*Sqrt[alpha*t]];
  BJ = X*BJ0 /. t->sigma^2/2*t;
];

(* Dimensionless formula given by Knessl *)
GetKn[] := Module[{z},
  z = Abs[Log[4*Pi*gamma^2*t]];
  Kn0 = Exp[-Sqrt[2*t*z]*(1+1/z^2)];
  Kn = X*Kn0 /. t->sigma^2/2*t;
];

(* Define the order of Taylor's series expansion *)
OrderTaylor = 8;

(* Assign the convergence-control parameter *)
c0 = -1;

(* Get 8th-order homotopy-approximation of B *)
ham[1,8];

(* Get 10th-order homotopy-approximation of B *)
ham[8,10]

```

Appendix 13.3 Mathematica code APOh for businessmen

The Mathematica code APOh gives the accurate optimal exercise price at a given time prior to expiry in a few second.

This practical code first reads the dimensionless results from the data-file named APO-48-10.txt, which were gained by Shijun Liao using the Mathematica code in Appendix 13.2 and a high-performance computer for the unknown dimensionless parameter $\gamma = 2r/\sigma^2$ in general, and then calculate the dimensional optimal exercise price up to the time to expiry. Unlike other asymptotic and/or perturbation formulas that are often valid a couple of days or weeks prior to expiry, this code often gives an accurate approximation that is often valid a couple of dozen years. So, it is especially useful for businessmen.

The code APOh is available at

<http://numericaltank.sjtu.edu.cn/HAM.htm>

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A Simple Users Guide

APOh [order_] This module gives the homotopy-approximation of dimensional optimal exercise price $B(\tau)$, with order denoting the order of homotopy-approximation. The code first reads the data-file APO-48-10.txt, then asks the user to input strike price X , risk-free interest rate r , volatility σ and time to expiration T (year), and finally lists the results of $B(\tau)$ at different order of approximations, their modified approximations by the Padé method, and plots a

curve of $B(\tau)$ in the interval $0 \leq \tau \leq 1.25T$ with the theoretical perpetual optimal exercise price $B_p = X\gamma/(1 + \gamma)$. To begin a new case, simply run the code APOh once again and input new parameters.

B [n] The n th-order homotopy-approximation of the optimal exercise price with dimension (\$), in polynomial of $\sqrt{\tau}$ to $o(\tau^M)$, where $M = \text{OrderTaylor}$.

Bpade [n] The $[M, M]$ Padé approximant of the n th-order homotopy approximation of $B(\tau)$ in polynomial of $\sqrt{\tau}$ to $o(\tau^M)$, where $M = \text{OrderTaylor}$.

OrderHAM The highest order of homotopy-approximations of the dimensionless results in the data file APO-48-10.txt. Its defaults is 10 in the data file APO-48-10.txt.

OrderTaylor The order of $B(\tau)$ in polynomial of $\sqrt{\tau}$. Its defaults is 48 in the data file APO-48-10.txt.

Bp The perpetual optimal exercise price $B_p = X\gamma/(1 + \gamma)$.

Mathematica code APOh for businessmen

by Shijun LIAO
Shanghai Jiao Tong University
December 2010

```
<<Calculus`Pade`;
<<Graphics`Graphics`;

(* Input dimensionless results given by means of the HAM *)
<<APO-48-10.txt;
Print["-----"];
Print["OrderTaylor = ", OrderTaylor];
Print["OrderHAM    = ", OrderHAM];
Print["-----"];

(* Define APOh[Order_] *)
APOh[Order_] := Module[{temp, n, i, j, s},
Print[" OrderTaylor = ", OrderTaylor];
Print[" strike price = ?"];
temp[0] = Input[];
X = IntegerPart[temp[0]*10^10]/10^10;
Print[" risk-free interest rate = ?"];
temp[0] = Input[];
r = IntegerPart[temp[0]*10^10]/10^10;
Print[" volatility = ?"];
temp[0] = Input[];
sigma = IntegerPart[temp[0]*10^10]/10^10;
Print[" time to expiry (year) = ?"];
```

```

temp[0] = Input[];
T = IntegerPart[temp[0]*10^10]/10^10;
gamma   = 2*r/sigma^2;
texp    = sigma^2*T/2;
Bp      = X*gamma/(1 + gamma);
Print["-----"];
Print[" INPUT PARAMETERS: "];
Print["   Strike price           (X) = ",X," ($) "];
Print["   Risk-free interest rate (r) = ",r];
Print["   Volatility             (sigma) = ",sigma];
Print["   Time to expiry          (T) = ",T," (year)"];
Print["-----"];
Print[" CORRESPONDING PARAMETERS: "];
Print["   gamma = ",gamma];
Print["   dimensionless time to expiry (texp)=",texp//N];
Print["   perpetual optimal exercise price (Bp)=",Bp//N," ($)"];
Print["-----"];
Print[" CONTROL PARAMETERS: "];
Print["   OrderTaylor = ",OrderTaylor];
Print["   c0          = ",c0];
Print["-----"];
For[n = 1,n <= Min[Order,OrderHAM], n++,
  Print[" n = ",n];
  temp[0] = X*BB[n] /. t-> (sigma^2*t/2)//Expand;
  B[n] = Collect[temp[0],t];
  If[NumberQ[gamma],
    temp[0] = BB[n] /. t^i_ -> s^(2*i);
    temp[1] = Pade[temp[0],{s,0,OrderTaylor,OrderTaylor}];
    BBpade[n] = temp[1] /. s^j_ -> t^(j/2);
    Bpade[n] = X*BBpade[n]/. t -> (sigma^2*t/2);
  ];
  If[NumberQ[gamma] && NumberQ[sigma] && NumberQ[X],
    Print["Optimal exercise price at the time to expiration
          = ", B[n]/.t->T//N];
    Print["Modified result given by Pade technique = ",
          Bpade[n]/.t->T//N];
  ];
];
Print["Well done"];
If[NumberQ[gamma] && NumberQ[sigma] && NumberQ[X],
  n = Min[Order,OrderHAM];
  Plot[{Bp, B[n], Bpade[n]}, {t, 0, 1.25*T},
    PlotRange -> {0.8*Bp, X}, PlotStyle ->
    {RGBColor[1,0,0],RGBColor[0,1,0],RGBColor[0,0,1]};
  Print[" Order of homotopy-approximation : ",n];
  Print[" Green line : optimal exercise boundary B
        in polynomial "];
  Print[" Blue line : optimal exercise boundary B
        by Pade method "];
  Print[" Red line : perpetual optimal exercise price "];
];
];

```

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Chapter 14

Two and Three Dimensional Gelfand Equation

Abstract Using the two-dimensional (2D) and 3D Gelfand equation as an example, we illustrate that the homotopy analysis method (HAM) can be used to solve a 2nd-order nonlinear partial differential equation (PDE) in a rather easy way by transforming it into an infinite number of the 4th or 6th-order linear PDEs. This is mainly because the HAM provides us extremely large freedom to choose auxiliary linear operator and besides a convenient way to guarantee the convergence of solution series. To the best of our knowledge, such kind of transformation has never been used by other analytic/numerical methods. This illustrates the originality and great flexibility of the HAM for strongly nonlinear problems. It also suggests that we must keep an open mind, since we might have much larger freedom to solve nonlinear problems than we thought traditionally.

14.1 Introduction

In Chapter 2, we illustrate that the homotopy analysis method (HAM) (Liao, 1992, 1997, 1999a,b; Liao and Campo, 2002; Liao, 2003a,b, 2004, 2005, 2006, 2009, 2010a,b; Liao and Tan, 2007; Xu et al., 2010) provides us extremely large freedom to choose the auxiliary linear operator: the 2nd-order nonlinear ordinary differential equation (ODE) describing a periodic oscillation can be transferred into an infinite number of linear 2κ -order linear ODEs, where $\kappa \geq 1$ is any a positive integer, and besides the convergence of the series solution is guaranteed by means of the so-called convergence-control parameter c_0 . In addition, it is due to this extremely large freedom that the nonlinear PDEs describing the non-similarity boundary-layer flows (in Chapter 11) and the unsteady boundary-layer flows (in Chapter 12) can be transferred into an infinite number of linear ODEs, and thus can be solved by means of the HAM-based Mathematica package BVPh 1.0. In this chapter, we further illustrate that, such kind of freedom on the choice of the auxiliary linear operator $\mathcal{L}$ can greatly simplify solving some high dimensional nonlinear PDEs.

For example, let us consider the high dimensional Gelfand equation (Liouville, 1853; Boyd, 1986)

$$\begin{cases} \Delta u + \lambda e^u = 0, & \mathbf{x} \in \Omega \subset \mathbf{R}^N, \\ u = 0, & \mathbf{x} \in \partial\Omega, \end{cases} \quad (14.1)$$

where Δ denotes the Laplace operator, λ is an eigenvalue, u is the eigenfunction, $\mathbf{x}$ is a vector for the spatial variables, $N = 1, 2, 3$ corresponds to the number of the dimension, Ω is the domain, $\partial\Omega$ denotes the boundary, respectively. Physically, Gelfand equation arises in several contexts, such as chemical reactor theory, the steady-state equation for a nonlinear heat conduction problem, questions on geometry and relativity about the expansion of universe, and so on. Mathematically, the governing equation has strong nonlinearity, since it contains the exponential term $\exp(u)$.

Generally speaking, it is difficult to gain accurate analytic approximations of eigenvalues and eigenfunctions of a high-dimensional PDE with strong nonlinearity. The investigation on Gelfand problem (Liouville, 1853; Boyd, 1986) has a long history. Liouville (1853) gave a closed-form expression of eigenvalue for one-dimensional (1D) Gelfand equation. Based on Chebyshev functions, Boyd (1986) proposed an analytic approach and a numerical method for two-dimensional (2D) Gelfand equation on the square $[-1, 1] \times [-1, 1]$, and suggested a one-point analytic approximation

$$\lambda = 3.2 A e^{-0.64 A}, \quad (14.2)$$

and a three-point analytic approximation

$$\lambda = (2.667 A + 4.830 B + 0.127 C) e^{-0.381 A - 0.254 B - 0.018 C}, \quad (14.3)$$

where $A = u(0, 0)$ and

$$\begin{aligned} B &= A \left(0.829 - 0.566 e^{0.463 A} - 0.0787 e^{-0.209 A} \right) / G, \\ C &= A \left(-1.934 + 0.514 e^{0.463 A} + 1.975 e^{-0.209 A} \right) / G, \\ G &= 0.2763 + e^{0.463 A} + 0.0483 e^{-0.209 A}. \end{aligned}$$

This problem still attracts the attention of current researchers (McGough, 1998; Jacobsen and Schmitt, 2002).

In this chapter, the HAM is employed to solve the 2nd-order 2D (or 3D) Gelfand equation with the high nonlinearity in a rather easy way by transferring it into an infinite number of the 4th (or 6th) order linear 2D (or 3D) PDEs.

14.2 Homotopy-approximations of 2D Gelfand equation

14.2.1 Brief mathematical formulas

Following Boyd (1986), let us consider the 2D Gelfand equation

$$\Delta u + \lambda e^u = 0, \quad -1 < x < 1, \quad -1 < y < 1, \quad (14.4)$$

subject to the boundary condition on the four walls

$$u(x, \pm 1) = 0, \quad u(\pm 1, y) = 0. \quad (14.5)$$

The above nonlinear eigenvalue equation has an infinite number of eigenvalues and eigenfunctions. Obviously, different eigenfunctions have different values at the origin $(0, 0)$. So, we can use the different values of

$$A = u(0, 0) \quad (14.6)$$

to distinguish different eigenfunctions and the corresponding eigenvalues. Writing

$$u(x, y) = A + w(x, y), \quad (14.7)$$

where A is a given constant, the original 2D Gelfand equation becomes

$$\Delta w + \lambda e^A e^w = 0, \quad -1 < x < 1, \quad -1 < y < 1, \quad (14.8)$$

subject to the boundary conditions on the four walls

$$w(x, \pm 1) = -A, \quad w(\pm 1, y) = -A, \quad (14.9)$$

with the restriction

$$w(0, 0) = 0. \quad (14.10)$$

Let $w_0(x, y)$ and λ_0 denote the initial guesses of the eigenfunction $w(x, y)$ and the eigenvalue λ , respectively. Here, the initial guess $w_0(x, y)$ is unnecessary to satisfy the boundary conditions (14.9) and the restriction (14.10). Let $q \in [0, 1]$ denote an embedding parameter. In the frame of the HAM, we should first of all construct such two continuous variations (or deformations) $\phi(x, y; q)$ and $\Lambda(q)$ that, as q increases from 0 to 1, $\phi(x, y; q)$ varies continuously from the initial guess $w_0(x, y)$ to the eigenfunction $w(x, y)$, and at the same time, $\Lambda(q)$ varies continuously from the initial guess λ_0 to the eigenvalue λ , respectively. Such two continuous variations are governed by the zeroth-order deformation equation

$$(1 - q) \mathcal{L} [\phi(x, y; q) - w_0(x, y)] = c_0 q \mathcal{N} [\phi(x, y; q), \Lambda(q)] \quad (14.11)$$

on the square $(x, y) \in [-1, 1] \times [-1, 1]$, subject to the boundary conditions on the four walls

$$(1 - q) [\phi(\pm 1, y; q) - w_0(\pm 1, y)] = c_0 q [\phi(\pm 1, y; q) + A], \quad (14.12)$$

$$(1 - q) [\phi(x, \pm 1; q) - w_0(x, \pm 1)] = c_0 q [\phi(x, \pm 1; q) + A], \quad (14.13)$$

with the additional restriction at the origin

$$(1 - q) [\phi(0, 0; q) - w_0(0, 0)] = c_0 q \phi(0, 0; q), \quad (14.14)$$

where

$$\begin{aligned} & \mathcal{N}[\phi(x, y; q), \Lambda(q)] \\ &= \frac{\partial^2 \phi(x, y; q)}{\partial x^2} + \frac{\partial^2 \phi(x, y; q)}{\partial y^2} + e^A \Lambda(q) \exp[\phi(x, y; q)] \end{aligned} \quad (14.15)$$

is a nonlinear operator corresponding to the 2D Gelfand equation (14.8), $\mathcal{L}$ is an auxiliary linear operator with the property $\mathcal{L}(0) = 0$, and $c_0 \neq 0$ is the convergence-control parameter, respectively. It should be emphasized here that we have extremely large freedom to choose the auxiliary linear operator $\mathcal{L}$ and the convergence-control parameter c_0 , as shown later.

When $q = 0$, since $\mathcal{L}(0) = 0$, the zeroth-order deformation equations (14.11) to (14.14) have the solution

$$\phi(x, y; 0) = w_0(x, y). \quad (14.16)$$

When $q = 1$, since $c_0 \neq 0$, the zeroth-order deformation equations (14.11) to (14.14) are equivalent to the original PDEs (14.8) to (14.10), provided

$$\phi(x, y; 1) = w(x, y), \quad \Lambda(1) = \lambda. \quad (14.17)$$

Thus, as the embedding parameter $q \in [0, 1]$ increases from 0 to 1, $\phi(x, y; q)$ indeed varies continuously from the initial guess $w_0(x, y)$ to the eigenfunction $w(x, y)$, so does $\Lambda(q)$ from the initial guess λ_0 to the eigenvalue λ . So, mathematically, the zeroth-order deformation equations (14.11) to (14.14) construct two continuous homotopies

$$\phi(x, y; q) : w_0(x, y) \sim w(x, y), \quad \Lambda(q) : \lambda_0 \sim \lambda.$$

Expanding $\phi(x, y; q)$ and $\Lambda(q)$ into Maclaurin series with respect to the embedding parameter q and using (14.16), we have the homotopy-Maclaurin series

$$\phi(x, y; q) = w_0(x, y) + \sum_{n=1}^{+\infty} w_n(x, y) q^n, \quad (14.18)$$

$$\Lambda(q) = \lambda_0 + \sum_{n=1}^{+\infty} \lambda_n q^n, \quad (14.19)$$

where

$$w_n(x, y) = \frac{1}{n!} \left. \frac{\partial^n \phi(x, y; q)}{\partial q^n} \right|_{q=0} = \mathcal{D}_n [\phi(x, y; q)], \quad (14.20)$$

$$\lambda_n = \frac{1}{n!} \left. \frac{\partial^n \Lambda(q)}{\partial q^n} \right|_{q=0} = \mathcal{D}_n [\Lambda(q)], \quad (14.21)$$

are the so-called n th-order homotopy-derivatives of $\phi(x, y; q)$ and $\Lambda(q)$, and $\mathcal{D}_n$ is the n th-order homotopy-derivative operator, respectively. It should be emphasized that, in the frame of the HAM, we have great freedom to choose the auxiliary linear operator $\mathcal{L}$, the initial guess $w_0(x, y)$, and especially the convergence-

control parameter c_0 : all of them influence the convergence of the series (14.18) and (14.19). Assume that they are chosen so properly that the homotopy-Maclaurin series (14.18) and (14.19) absolutely converge at $q = 1$. Then, due to (14.17), we have the homotopy-series solution

$$w(x, y) = w_0(x, y) + \sum_{n=1}^{+\infty} w_n(x, y), \quad (14.22)$$

$$\lambda = \lambda_0 + \sum_{n=1}^{+\infty} \lambda_n. \quad (14.23)$$

The differential equations for the unknown $w_n(x, y)$ and λ_{n-1} ($n \geq 1$) can be deduced directly from the zeroth-order deformation equations (14.11) to (14.14): taking the n th-order homotopy-derivative on both sides of the zeroth-order deformation equations (14.11) to (14.14), we have the so-called n th-order deformation equation

$$\mathcal{L}[w_n(x, y) - \chi_n w_{n-1}(x, y)] = c_0 \delta_{n-1}(x, y), \quad (14.24)$$

subject to the boundary conditions on the four walls

$$w_n(x, \pm 1) = \mu_n(x, \pm 1), \quad w_n(\pm 1, y) = \mu_n(\pm 1, y), \quad (14.25)$$

and the additional restriction at the origin

$$w_n(0, 0) = (\chi_n + c_0) w_{n-1}(0, 0), \quad (14.26)$$

where

$$\chi_n = \begin{cases} 1, & n > 1, \\ 0, & n = 1, \end{cases} \quad (14.27)$$

and

$$\begin{aligned} \delta_k(x, y) &= \mathcal{D}_k \{ \mathcal{N}[\phi(x, y; q)] \} \\ &= \Delta w_k(x, y) + e^A \sum_{j=0}^k \lambda_{k-j} \mathcal{D}_j [e^{\phi(x, y; q)}], \end{aligned} \quad (14.28)$$

$$\mu_n(x, y) = (\chi_n + c_0) w_{n-1}(x, y) + c_0 (1 - \chi_n) A, \quad (14.29)$$

are gained by Theorem 4.1. According to Theorem 4.7, we have the recursion formula

$$\mathcal{D}_0 [e^{\phi(x, y; q)}] = e^{w_0(x, y)}, \quad \mathcal{D}_k [e^{\phi(x, y; q)}] = \sum_{j=0}^{k-1} \left(1 - \frac{j}{k} \right) w_{k-j} \mathcal{D}_j [e^{\phi(x, y; q)}]. \quad (14.30)$$

Thus, it is easy to calculate the term $\delta_{n-1}(x, y)$ of (14.24) by means of computer algebra system like Mathematica. Note that (14.24) can be gained by Theorem 4.15 directly. For details, please refer to Chapter 4.

As mentioned before, in the frame of the HAM, we have extremely large freedom to choose the auxiliary linear operator $\mathcal{L}$ and the initial guess $w_0(x, y)$. Note that the Gelfand equation contains a linear operator, i.e. the Laplace operator Δ . However, even the 2D linear PDE

$$\Delta u(x, y) = 0, \quad -1 \leq x \leq 1, \quad -1 \leq y \leq 1,$$

has a complicated common solution

$$\begin{aligned} u(x, y) = & \sum_{k=0}^{+\infty} \left(B_{k,1} e^{-\alpha k x} + B_{k,2} e^{\alpha k x} \right) [B_{k,3} \sin(\alpha k y) + B_{k,4} \cos(\alpha k y)] \\ & + \sum_{k=1}^{+\infty} \left(B_{k,5} e^{-\beta k y} + B_{k,6} e^{\beta k y} \right) [B_{k,7} \sin(\beta k x) + B_{k,8} \cos(\beta k x)], \end{aligned}$$

where the coefficients α, β and $B_{k,i}$ are determined by the boundary conditions. For example, on the boundary $x = 1$, the above expression reads

$$\begin{aligned} u(1, y) = & \sum_{k=0}^{+\infty} \left(B_{k,1} e^{-\alpha k} + B_{k,2} e^{\alpha k} \right) [B_{k,3} \sin(\alpha k y) + B_{k,4} \cos(\alpha k y)] \\ & + \sum_{k=1}^{+\infty} \left(B_{k,5} e^{-\beta k y} + B_{k,6} e^{\beta k y} \right) [B_{k,7} \sin(\beta k) + B_{k,8} \cos(\beta k)]. \end{aligned}$$

So, it is not easy to satisfy the boundary condition on $x = 1$ by means of the above expression. Therefore, if we choose the Laplace operator Δ as the auxiliary linear operator $\mathcal{L}$, it is not easy to solve the high-order deformation equations (14.24) to (14.26). Thus, we should choose an auxiliary linear operator $\mathcal{L}$ better than the Laplace operator Δ . Fortunately, in the frame of the HAM, we have extremely large freedom to choose the auxiliary linear operator $\mathcal{L}$. Using this kind of freedom, we indeed could choose such an auxiliary linear operator $\mathcal{L}$ that it is rather easy to solve the high-order deformation equation, as shown below.

Note that the boundary conditions (14.5) and the boundary

$$\partial\Omega : (x, y) \in [-1, 1] \times [-1, 1]$$

itself are symmetric about x and y axes. Besides, it is easy to prove that, if $w(x, y)$ is a solution of the 2D Gelfand equation, then $w(\pm x, \pm y)$ is also its solution. So, $w(x, y)$ is symmetric about x and y axis, and therefore can be expressed by the base functions

$$\{x^{2m} y^{2n} \mid m = 1, 2, 3, \dots, n = 1, 2, 3, \dots\} \quad (14.31)$$

in the form

$$w(x, y) = \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} b_{m,n} x^{2m} y^{2n}, \quad (14.32)$$

where $b_{m,n}$ is a coefficient to be determined. It provides us the so-called *solution expression* of $w(x, y)$. Our aim is to find a convergent series solution of the eigen-

function $w(x, y)$ in the form (14.32) and the corresponding convergent series of the eigenvalue λ , for a given value of A .

To satisfy the additional restriction (14.10) and the solution-expression (14.32), we choose the simplest initial guess

$$w_0(x, y) = 0. \quad (14.33)$$

Note that this initial guess satisfies the restriction (14.10), but not the boundary conditions (14.9) on the four walls.

Next, we should choose the auxiliary linear operator $\mathcal{L}$. To obey the solution expression (14.32) of $w(x, y)$, it should hold

$$\mathcal{L}(C_1) = 0 \quad (14.34)$$

for any a non-zero constant C_1 . Besides, since $w_0(x, y) = 0$, it holds $\delta_0(x, y) = \lambda_0 e^A$, therefore, $\delta_{n-1}(x, y)$ may contain a non-zero constant. So, to obey the solution expression (14.32), the inverse operator $\mathcal{L}^{-1}$ of the auxiliary linear operator $\mathcal{L}$ should have the property

$$\mathcal{L}^{-1}(1) = C_2 x^2 y^2, \quad (14.35)$$

where C_2 is a non-zero constant. Especially, the linear auxiliary operator $\mathcal{L}$ should be chosen properly so that it is rather easy to solve the high-order deformation equation (14.24) with the boundary conditions (14.25) at the four walls. Let $w_m^*(x, y)$ denote a special solution of (14.24). Obviously,

$$w_n^*(x, y) - w_n^*(x, \pm 1) - w_n^*(\pm 1, y) + w_n^*(\pm 1, \pm 1)$$

vanishes on the four walls, and besides

$$\mu_n(x, \pm 1) + \mu_n(\pm 1, y) - \mu_n(\pm 1, \pm 1)$$

satisfies the boundary conditions (14.25) on the four walls, where $\mu_n(x, y)$ is defined by (14.29). Therefore,

$$\begin{aligned} w_n(x, y) = & w_n^*(x, y) - w_n^*(x, \pm 1) - w_n^*(\pm 1, y) + w_n^*(\pm 1, \pm 1) \\ & + \mu_n(x, \pm 1) + \mu_n(\pm 1, y) - \mu_n(\pm 1, \pm 1) \end{aligned} \quad (14.36)$$

satisfies the n th-order deformation equation (14.24) and the boundary conditions (14.25), as long as the auxiliary linear operator $\mathcal{L}$ has the property

$$\mathcal{L}[f(x)] = \mathcal{L}[g(y)] = 0 \quad (14.37)$$

for arbitrary smooth functions $f(x)$ and $g(y)$.

There exist an infinite number of linear differential operators satisfying the above-mentioned properties (14.34), (14.35) and (14.37), such as the 2nd-order linear operator

$$\mathcal{L}u = c_2 \frac{1}{xy} \frac{\partial^2 u}{\partial x \partial y}, \quad (14.38)$$

and the 4th-order linear operator

$$\mathcal{L}(u) = c_4 \frac{\partial^4 u}{\partial x^2 \partial y^2}, \quad (14.39)$$

where c_2 and c_4 are constants. These two linear operators are special cases of the more general linear operator

$$\mathcal{L}(u) = \frac{c_2}{xy} \frac{\partial^2 u}{\partial x \partial y} + c_4 \frac{\partial^4 u}{\partial x^2 \partial y^2}, \quad (14.40)$$

whose inverse operator is

$$\mathcal{L}^{-1} \left(x^k y^n \right) = \frac{x^{k+2} y^{n+2}}{(k+2)(n+2)[c_2 + c_4(k+1)(n+1)]}. \quad (14.41)$$

Using the above inverse operator $\mathcal{L}^{-1}$, it is very easy to gain a special solution

$$w_n^*(x, y) = c_0 \mathcal{L}^{-1} [\delta_{n-1}(x, y)] + \chi_n w_{n-1}(x, y) \quad (14.42)$$

of the n th-order deformation equation (14.24). Thereafter, the solution $w_n(x, y)$ of the high-order deformation equations (14.24) to (14.25) is obtained by means of (14.36). Then, λ_{n-1} is determined by the linear algebraic equation (14.26). For more details, please refer to Liao and Tan (2007).

Note that the above approach needs only algebraic calculations. Thus, it is rather easy to obtain high-order approximations of the eigenfunction $w(x, y)$ and the eigenvalue λ , especially by means of algebra computer system such as Mathematica, Maple, and so on. In this way, we greatly simplifies solving the 2D Gelfand equation, as shown later.

The corresponding Mathematica code for the 2D Gelfand equation is given in the Appendix 14.1 and free available (Accessed 25 Nov 2011, will be updated in the future) at

<http://numericaltank.sjtu.edu.cn/HAM.htm>

14.2.2 Homotopy-approximations

In the frame of the HAM, the convergence-control parameter c_0 provides us a convenient way to guarantee the convergence of solution series. It should be emphasized that the auxiliary linear operator (14.40) contains two parameters c_2 and c_4 , which can be also regarded as the convergence-control parameters like c_0 . So, we have now three convergence-control parameters c_0, c_2 and c_4 , which are used to guarantee the convergence of (14.22) for the eigenfunction and (14.23) for the eigenvalue.

It is found that, for the arbitrary values of c_2 and c_4 , the n th-order approximation of the eigenfunction $u(x, y)$ on the four walls reads

$$A(1 + c_0)^n,$$

which vanishes as $n \rightarrow +\infty$ only if

$$|1 + c_0| < 1. \quad (14.43)$$

The above expression restricts the choice of the convergence-control parameter c_0 . Especially, when $c_0 = -1$, the boundary conditions on the four walls are exactly satisfied at every order of approximation. Due to this reason, we choose $c_0 = -1$ for the sake of simplicity. Then, there are two unknown convergence-control parameters c_2 and c_4 left.

Define the averaged squared residual of the 2D Gelfand equation

$$E_m = \frac{1}{100} \sum_{i=0}^9 \sum_{j=0}^9 \left[\Delta u(i\Delta x, j\Delta y) + \lambda e^{u(i\Delta x, j\Delta y)} \right]^2, \quad \Delta x = \Delta y = \frac{1}{10}, \quad (14.44)$$

where u and λ are the m th-order homotopy-approximation of the eigenfunction and the eigenvalue, respectively. Obviously, E_m is dependent upon the two convergence-control parameters c_2 and c_4 .

Since the original Gelfand equation (14.8) is *2nd-order*, we first use the *2nd-order* linear operator (14.38) as the auxiliary linear operator, corresponding to $c_4 = 0$ of the linear operator (14.40). Due to $c_0 = -1$ and $c_4 = 0$, there exists only one non-zero convergence-control parameter c_2 now. Without loss of generality, let us consider the case of $A = 1$. It is found that the minimum of the averaged squared residual E_m of the governing equation (14.8) does *not* decrease as m increases, as shown in Table 14.1. Besides, there does not exist an interval of c_2 , where the averaged squared residual E_m of the governing equation (14.8) decreases as m increases, as shown in Fig. 14.1. For example, in case of $A = 1$ and $c_2 = -20$, the corresponding homotopy-series is divergent. Therefore, we can not gain convergent series solution of the 2D Gelfand equation (14.8) by using the 2nd-order linear operator (14.38) as the auxiliary linear operator.

Then, we use the *4th-order* linear operator (14.39) as the auxiliary linear operator, corresponding to $c_2 = 0$ of the linear operator (14.40). The minimum of

Table 14.1 The minimum of the averaged squared residual of the 2D Gelfand equation (14.8) when $A = 1$ by means of $c_0 = -1$ and the 2nd-order auxiliary linear operator (14.38).

Order of approximation m	Minimum of squared residual	Optimal value of c_2
10	1.23×10^{-2}	-5.3545
15	1.57×10^{-2}	-7.9349
20	1.89×10^{-2}	-10.5021

Table 14.2 The minimum of the averaged squared residual of the 2D Gelfand equation when $A = 1$ by means of $c_0 = -1$ and the 4th-order auxiliary linear operator (14.39).

Order of approximation m	Minimum of squared residual	Optimal value of c_4
3	2.65×10^{-2}	-0.4443
5	9.07×10^{-3}	-0.3144
8	3.17×10^{-3}	-0.2866
10	1.83×10^{-3}	-0.2763
15	6.31×10^{-4}	-0.2555
20	2.85×10^{-4}	-0.2494

Fig. 14.1 Averaged squared residual of (14.8) versus c_2 when $A = 1$ by means of $c_0 = -1$ and using the 2nd-order linear operator (14.38) as the auxiliary linear operator. Dashed line: 10th-order homotopy-approximation; Dash-dotted line: 15th-order homotopy-approximation; Solid line: 20th-order homotopy-approximation.

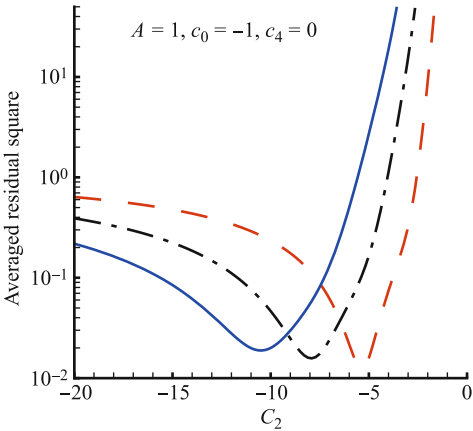
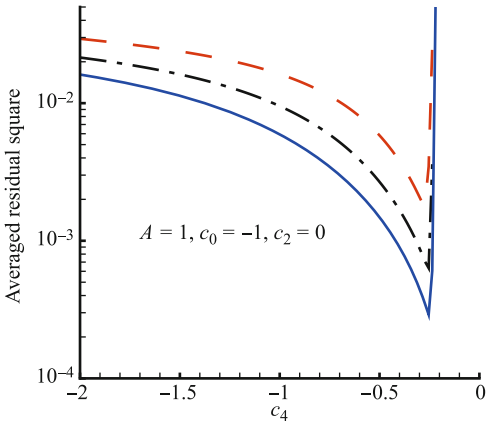


Fig. 14.2 Averaged squared residual of (14.8) versus c_4 when $A = 1$ by means of $c_0 = -1$ and using the 4th-order linear operator (14.39) as the auxiliary linear operator. Dashed line: 10th-order homotopy-approximation; Dash-dotted line: 15th-order homotopy-approximation; Solid line: 20th-order homotopy-approximation.

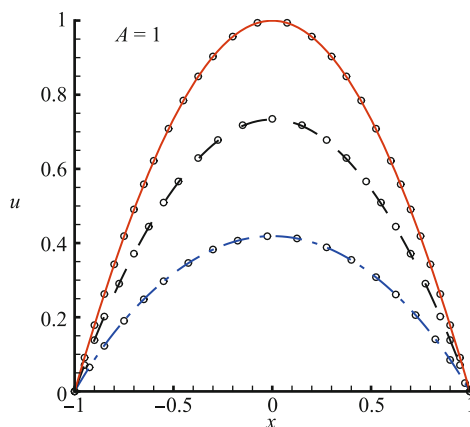


the averaged squared residual of (14.8) decreases to 2.85×10^{-4} at the 20th-order homotopy-approximation by means of the optimal convergence-control parameter $c_4^* = -0.2494$, as listed in Table 14.2. Besides, for arbitrary convergence-control parameter $c_4 \leq -1/4$, the averaged squared residual E_m of (14.8) decreases as m , the order of approximation, increases, as shown in Fig. 14.2. This suggests that

Table 14.3 The eigenvalue λ and averaged squared residual E_m of (14.8) when $A = 1$, $c_0 = -1$ by means of the 4th-order linear operator (14.39) with $c_4 = -1/4$ as the auxiliary linear operator.

Order of approximation m	Minimum of squared residual	Eigenvalue λ
3	0.574	1.5765
5	9.4×10^{-2}	1.6059
10	4.9×10^{-3}	1.6212
15	6.7×10^{-4}	1.6237
20	2.8×10^{-4}	1.6233
25	1.4×10^{-4}	1.6231
30	7.6×10^{-5}	1.6231
40	2.5×10^{-5}	1.62311
50	8.9×10^{-6}	1.6231158
60	3.3×10^{-6}	1.6231158
70	1.3×10^{-6}	1.6231158
80	5.5×10^{-7}	1.62311584
90	2.6×10^{-7}	1.62311584
100	1.5×10^{-7}	1.62311584

Fig. 14.3 Profiles of $u(x, y)$ of 2D Gelfand equation (14.8) when $A = 1$, $c_0 = -1$ by means of the 4th-order linear operator (14.39) with $c_4 = -1/4$ as the auxiliary linear operator. Lines: 20th-order homotopy-approximation; Symbols: 10th-order homotopy-approximation; Solid line: $y = 0$; Dashed line: $y = 1/2$; Dash-dotted line: $y = 1/4$.



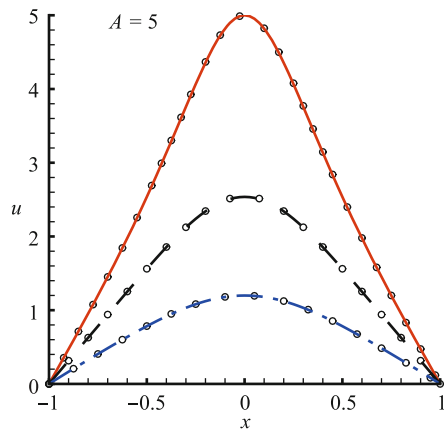
we can gain convergent series solution of the eigenfunction and the eigenvalue by using the 4th-order linear operator (14.39) with the convergence-control parameter $c_4 \leq -1/4$ as the auxiliary linear operator. This is indeed true: when $A = 1$, the averaged squared residual of the 2D Gelfand equation (14.8) monotonously decreases to 1.5×10^{-7} at the 100th-order homotopy-approximation by means of the 4th-order auxiliary linear operator (14.39) with $c_4 = -1/4$, and besides the corresponding eigenvalue λ converges to 1.62311584, as shown in Table 14.3. Note that even the 10th-order homotopy-approximation of the eigenfunction $u(x, y)$ is accurate enough, as shown in Fig. 14.3.

It is found that, for other values of A , we can gain convergent eigenfunction and eigenvalue by means of the 4th-order auxiliary linear operator (14.39) with a proper convergence-control parameter c_4 in a similar way. For example, when

Table 14.4 The eigenvalue λ and the averaged squared residual E_m of (14.8) when $A = 5$, $c_0 = -1$ by means of the 4th-order linear operator (14.39) with $c_4 = -2/5$ as the auxiliary linear operator.

Order of approximation m	Averaged squared residual	Eigenvalue λ
5	21.1	0.471
10	3.31	0.490
15	1.11	0.501
20	0.56	0.507
30	0.18	0.512
40	7.3×10^{-2}	0.514
50	3.3×10^{-2}	0.515
60	1.6×10^{-2}	0.516
70	8.3×10^{-3}	0.516
80	4.6×10^{-3}	0.516

Fig. 14.4 Profiles of $u(x, y)$ of 2D Gelfand equation (14.8) when $A = 5$, $c_0 = -1$ by means of the 4th-order linear operator (14.39) with $c_4 = -2/5$ as the auxiliary linear operator. Lines: 50th-order homotopy-approximation; Symbols: 30th-order homotopy-approximation; Solid line: $y = 0$; Dashed line: $y = 1/2$; Dash-dotted line: $y = 1/4$.



$A = 5$, we gain the convergent eigenfunction and eigenvalue by means of the optimal convergence-control parameter $c_4 = -2/5$: as shown in Table 14.4, the averaged squared residua of (14.8) decreases monotonously and the corresponding eigenvalue converges to $\lambda = 0.516$. Besides, the 30th-order homotopy-approximation of the eigenfunction agrees well with the 50th-order ones, as shown in Fig. 14.4.

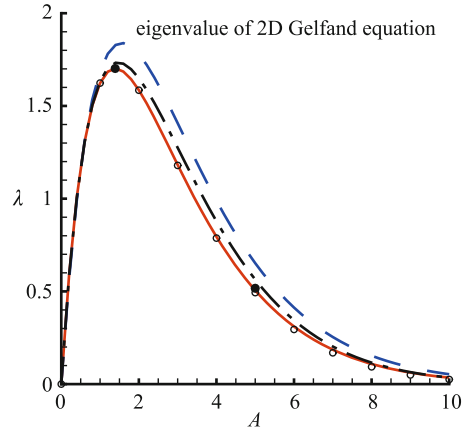
Regarding A as an unknown parameter, we obtain the 30th-order homotopy-approximation of the eigenvalue by means of $c_0 = -1$ and the 4th-order auxiliary linear operator (14.39) with $c_4 = -1$. It is found that the eigenvalue of the 2D Gelfand equation has the maximum value 1.702 at $A = 1.391$, which agrees well with Boyd's numerical result $\lambda_{\max} = 1.702$ at $A = 1.39$, as shown in Table 14.5. Besides, the simplified formula

$$\begin{aligned} \lambda \approx e^{-A} & \left(3.39403927 A + 0.85308129 A^2 + 0.14514688 A^3 \right. \\ & + 1.83020402 \times 10^{-2} A^4 + 1.65401606 \times 10^{-3} A^5 \\ & \left. + 1.03116169 \times 10^{-4} A^6 + 5.35313091 \times 10^{-6} A^7 \right) \quad (14.45) \end{aligned}$$

Table 14.5 Comparison of the maximum eigenvalue $\lambda_{\max}$ of the 2D Gelfand equation with Boyd’s analytic and numerical results. The homotopy-approximations are obtained by means of $q_0 = -1$ and the 4th-order auxiliary linear operator (14.39) with $c_4 = -1$.

	$\lambda_{\max}$	The corresponding value of A
5th-order HAM approx.	1.701	1.383
10th-order HAM approx.	1.702	1.389
15th-order HAM approx.	1.702	1.391
20th-order HAM approx.	1.702	1.391
25th-order HAM approx.	1.702	1.391
30th-order HAM approx.	1.702	1.391
Boyd’s 1-point formula (14.2)	1.84	1.56
Boyd’s 3-point formula (14.3)	1.735	1.465
Boyd’s numerical result	1.702	1.39

Fig. 14.5 Comparison of the eigenvalue of 2D Gelfand equation given by different methods. Solid line: the simplified formula (14.45) given by the first 7 terms of the 30th-order homotopy-approximation; Open circles: 20th-order homotopy-approximation; Filled circles: Numerical results given by Boyd (1986); Dashed line: Boyd’s 1-point approximation (14.2); Dash-dotted line: Boyd’s 3-points approximation (14.3).



given by the first seven terms of the 30th-order homotopy-approximation of the eigenvalue agrees well with the 20th-order homotopy-approximation and Boyd’s numerical results (Boyd, 1986) in $0 \leq A \leq 10$, as shown in Fig. 14.5.

All of these verify that, in the frame of the HAM, the 2nd-order nonlinear PDE (14.4) can be transferred into an infinite number of the 4th-order linear PDEs (14.24) governed by the 4th-order auxiliary linear operator

$$\mathcal{L}(u) = c_4 \frac{\partial^4 u}{\partial x^2 \partial y^2}.$$

Note that, using above 4th-order auxiliary linear operator, there exist an infinite number of smooth functions, such as

$$\sin x, \exp(y), x f(y), y g(x)$$

and so on, which satisfy

$$\mathcal{L}(\sin x) = \mathcal{L}[\exp(y)] = \mathcal{L}[x f(y)] = \mathcal{L}[y g(x)] = 0,$$

where $f(y)$ and $g(x)$ are *arbitrary* smooth functions except the polynomials of y and x , respectively. However, all of them are *not* allowed in the solution of the high-order deformation equation (14.24), because they disobey *the solution-expression* (14.32). In other words, if $w_n^*(x, y)$ is a special solution of (14.24), then

$$w_n^*(x, y) + B_1 \sin x + B_2 \exp(y) + B_3 x f(y) + B_4 y g(x) + \dots$$

also satisfies (14.24), where B_1, B_2, B_3 and B_4 are coefficients, and $f(y)$ and $g(x)$ are not polynomials of y and x , respectively. However, to obey the solution-expression (14.32), we *had to* enforce

$$B_1 = B_2 = B_3 = B_4 = \dots = 0$$

in the frame of the HAM. This illustrates that the so-called *solution-expression* indeed plays an important role in the frame of the HAM, which however can greatly simplify solving some nonlinear problems, if properly used.

It should be emphasized that the convergence of (14.22) for the eigenfunction and (14.23) for the eigenvalue is guaranteed by two convergence-control parameters c_0 and c_4 . In fact, it is these two convergence-control parameters that provide a strong support for the extremely large freedom of the HAM on the choice of the auxiliary linear operator $\mathcal{L}$, because a divergent series has no meanings at all.

Note that, by means of perturbation methods (Cole, 1992; Nayfeh, 1973; Von Dyke, 1975; Lagerstrom, 1988; Hinch, 1991; Murdock, 1991; Awrejcewicz et al., 1998; Nayfeh, 2000), it is possible to transfer a nonlinear differential equation into an infinite number of *lower-order* linear differential equations, when the highest derivative is multiplied by a perturbation quantity. However, to the best of the author's knowledge, a *2nd-order* nonlinear PDE has *never* been transferred into an infinite number of the *4th-order* linear PDEs in such a way by any other analytic and numerical methods! It suggests that we might have much larger freedom to solve nonlinear problems than we thought and believed traditionally! This shows the originality and great flexibility of the HAM for nonlinear problems.

Indeed, *a truly new method always gives something new and/or different.*

14.3 Homotopy-approximations of 3D Gelfand equation

Let us further consider the 3D Gelfand equation

$$\Delta u + \lambda e^u = 0, \quad -1 \leq x, y, z \leq 1 \quad (14.46)$$

subject to the boundary conditions

$$u(\pm 1, y, z) = 0, \quad u(x, \pm 1, z) = 0, \quad u(x, y, \pm 1) = 0. \quad (14.47)$$

The above 2nd-order nonlinear PDE can be easily solved by means of the HAM in a rather similar way as mentioned above. A corresponding Mathematica code is given in the Appendix 14.2 and available at

<http://numericaltank.sjtu.edu.cn/HAM.htm>

Write

$$A = u(0, 0, 0) \quad (14.48)$$

and

$$u(x, y, z) = A + w(x, y, z). \quad (14.49)$$

The original 3D Gelfand equation become

$$\Delta w + \lambda e^A e^w = 0, \quad -1 < x, y, z < 1, \quad (14.50)$$

subject to the boundary conditions on the six walls

$$w(\pm 1, y, z) = w(x, \pm 1, z) = w(x, y, \pm 1) = -A, \quad (14.51)$$

with the restriction

$$w(0, 0, 0) = 0. \quad (14.52)$$

Let $w_0(x, y, z)$ and λ_0 denote the initial guesses of the eigenfunction and eigenvalue, $q \in [0, 1]$ the embedding parameter, respectively. In the frame of the HAM, we first construct such two continuous variations $\phi(x, y, z; q)$ and $\Lambda(q)$ that, as $q \in [0, 1]$ increases from 0 to 1, $\phi(x, y, z; q)$ varies from the initial guess $w_0(x, y, z)$ to the eigenfunction $w(x, y, z)$, and at the same time, $\Lambda(q)$ varies from the initial guess λ_0 to the eigenvalue λ , respectively. Such kind of two continuous variations $\phi(x, y, z; q)$ and $\Lambda(q)$ are governed by the zeroth-order deformation equation

$$(1 - q)\mathcal{L}[\phi(x, y, z; q) - w_0(x, y, z)] = q c_0 \mathcal{N}[\phi(x, y, z; q), \Lambda(q)] \quad (14.53)$$

on the cubic $-1 < x, y, z < +1$, subject to the boundary conditions on the six walls

$$(1 - q)[\phi(\pm 1, y, z; q) - w_0(\pm 1, y, z)] = c_0 q [\phi(\pm 1, y, z; q) + A], \quad (14.54)$$

$$(1 - q)[\phi(x, \pm 1, z; q) - w_0(x, \pm 1, z)] = c_0 q [\phi(x, \pm 1, z; q) + A], \quad (14.55)$$

$$(1 - q)[\phi(x, y, \pm 1; q) - w_0(x, y, \pm 1)] = c_0 q [\phi(x, y, \pm 1; q) + A], \quad (14.56)$$

with the additional restriction at the origin

$$(1 - q)[\phi(0, 0, 0; q) - w_0(0, 0, 0)] = c_0 q \phi(0, 0, 0; q), \quad (14.57)$$

where

$$\begin{aligned} & \mathcal{N}[\phi(x, y, z; q), \Lambda(q)] \\ &= \frac{\partial^2 \phi(x, y, z; q)}{\partial x^2} + \frac{\partial^2 \phi(x, y, z; q)}{\partial y^2} + \frac{\partial^2 \phi(x, y, z; q)}{\partial z^2} \end{aligned}$$

$$+e^A \Lambda(q) \exp[\phi(x, y, z; q)], \quad (14.58)$$

is a nonlinear operator corresponding to (14.50), $\mathcal{L}$ is an auxiliary linear operator with the property $\mathcal{L}(0) = 0$, and $c_0 \neq 0$ is the convergence-control parameter, respectively. Note that we have great freedom to choose the auxiliary linear operator $\mathcal{L}$ and the convergence-control parameter c_0 in the frame of the HAM.

Similarly, we have the homotopy-series solution

$$w(x, y, z) = w_0(x, y, z) + \sum_{n=1}^{+\infty} w_n(x, y, z), \quad (14.59)$$

$$\lambda = \lambda_0 + \sum_{n=1}^{+\infty} \lambda_n. \quad (14.60)$$

According to Theorem 4.15, $w_n(x, y, z)$ is governed by the n th-order deformation equation

$$\mathcal{L}[w_n(x, y, z) - \chi_n w_{n-1}(x, y, z)] = c_0 \delta_{n-1}(x, y, z), \quad -1 < x, y, z < 1, \quad (14.61)$$

subject to the boundary conditions on the six walls

$$w_n(\pm 1, y, z) = \mu_n(\pm 1, y, z), \quad w_n(x, \pm 1, z) = \mu_n(x, \pm 1, z), \quad w_n(x, y, \pm 1) = \mu_n(x, y, \pm 1),$$

where

$$\mu_n(x, y, z) = (\chi_n + c_0) w_{n-1}(x, y, z) + c_0 (1 - \chi_n) A, \quad (14.62)$$

$$\delta_k(x, y, z) = \Delta w_k(x, y, z) + e^A \sum_{j=0}^k \lambda_{k-j} \mathcal{D}_j(e^\phi), \quad (14.63)$$

are gained by Theorem 4.1, and the term $\mathcal{D}_j(e^\phi)$ is given by the recursion formula (14.30).

Similarly, the solution of the above linear PDE reads

$$\begin{aligned} w_n(x, y, z) = & w_n^*(x, y, z) - w_n^*(\pm 1, y, z) - w_n^*(x, \pm 1, z) - w_n^*(x, y, \pm 1) \\ & + w_n^*(x, \pm 1, \pm 1) + w_n^*(\pm 1, y, \pm 1) + w_n^*(\pm 1, \pm 1, z) \\ & - w_n^*(\pm 1, \pm 1, \pm 1) \\ & + \mu_n(\pm 1, y, z) + \mu_n(x, \pm 1, z) + \mu_n(x, y, \pm 1) \\ & - \mu_n(x, \pm 1, \pm 1) - \mu_n(\pm 1, y, \pm 1) - \mu_n(\pm 1, \pm 1, z) \\ & + \mu_n(\pm 1, \pm 1, \pm 1), \end{aligned} \quad (14.64)$$

where

$$w_n^*(x, y, z) = c_0 \mathcal{L}^{-1}[\delta_{n-1}(x, y, z)] + \chi_n w_{n-1}(x, y, z) \quad (14.65)$$

is a special solution of (14.61). Besides, the eigenvalue λ_{n-1} is determined by the linear algebraic equation

$$w_n(0, 0, 0) = (\chi_n + c_0) w_{n-1}(0, 0, 0). \quad (14.66)$$

We also choose the same initial guess

$$w_0(x, y, z) = 0$$

for the 3D Gelfand equation. Quite similarly, we choose an auxiliary linear operator $\mathcal{L}$ in the form

$$\mathcal{L}(w) = \frac{c_3}{xyz} \frac{\partial^3 w}{\partial x \partial y \partial z} + c_6 \frac{\partial^6 w}{\partial x^2 \partial y^2 \partial z^2}, \quad (14.67)$$

where c_3 and c_6 are constants. Its inverse operator reads

$$\mathcal{L}^{-1}(x^m y^n z^k) = \frac{x^{m+2} y^{n+2} z^{k+2}}{(m+2)(n+2)(k+2)[c_3 + c_6(m+1)(n+1)(k+1)]} \quad (14.68)$$

for arbitrary positive integers m, n, k . Especially, when $c_3 = 0$, we have the *6th-order* auxiliary linear operator

$$\mathcal{L}(w) = c_6 \frac{\partial^6 w}{\partial x^2 \partial y^2 \partial z^2}. \quad (14.69)$$

For more details, please refer to Liao and Tan (2007).

Note that there exist three convergence-control parameters c_0, c_3 and c_6 . All of them have no physical meanings, but provide a convenient way to guarantee the convergence of the homotopy-series (14.59) and (14.60). Without loss of generality, let us first consider a special case $A = 1$. Similarly, it is found that the n th-order approximation of $u(x, y, z)$ on the six walls reads

$$A(1 + c_0)^n,$$

which vanishes as $n \rightarrow +\infty$ when $|1 + c_0| < 1$, i.e. $-2 < c_0 < 0$. Similarly, it is also found that we can not obtain convergent series of eigenvalue and eigenfunction when $c_3 \neq 0$ and $-2 < c_0 < 0$. Therefore, we choose $c_0 = -1$ and $c_3 = 0$ so that the boundary conditions on the six walls are exactly satisfied. Then, we focus our attention on the influence of the convergence-control parameter c_6 in (14.69) to the convergence of the homotopy-series (14.59) and (14.60).

Similarly, we define the averaged squared residual E_m of the 3D Gelfand equation in a similar way to (14.44), which is dependent only upon c_6 when $A = 1$, $c_0 = -1$ by means of the 6th-order auxiliary linear operator (14.69), as shown in Fig. 14.6. The minimum of the averaged squared residual E_m of the 3D Gelfand equation and the corresponding optimal values of the convergence-control parameter c_6 are listed in Table 14.6. It is found that, as the order of approximation increases, the averaged squared residual E_m decreases for an arbitrary convergence-control parameter c_6 in the interval $0.1 \leq c_6 < +\infty$, and besides the optimal value of c_6 is close to 0.1. This

Fig. 14.6 Averaged squared residual of the 3D Gelfand equation versus c_6 when $A = 1$, $c_0 = -1$ by means of the 6th-order auxiliary linear operator (14.69). Dashed line: 5th-order homotopy-approximation; Dash-dotted line: 10th-order homotopy-approximation; Solid line: 15th-order homotopy-approximation.

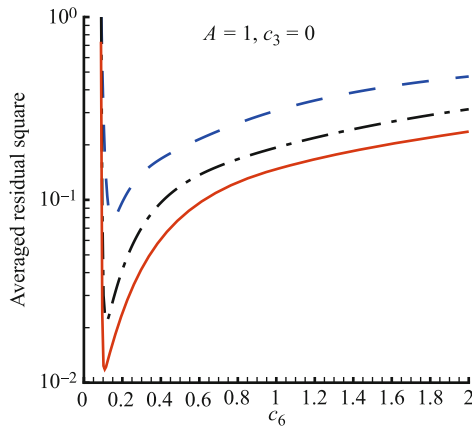


Table 14.6 The minimum of the averaged squared residual of the 3D Gelfand equation when $A = 1$, $c_0 = -1$ by means of the 6th-order auxiliary linear operator (14.69).

Order of approximation m	Minimum of squared residual	Optimal value of c_6
3	0.127	0.1614
5	7.7×10^{-2}	0.1478
10	2.2×10^{-2}	0.1185
15	1.2×10^{-2}	0.1088
20	7.9×10^{-3}	0.0994
25	5.8×10^{-3}	0.0982

Table 14.7 The eigenvalue λ and the averaged squared residual E_m of the 3D Gelfand equation when $A = 1$, $c_0 = -1$ by means of the 6th-order auxiliary linear operator (14.69) with $c_6 = 1/8$.

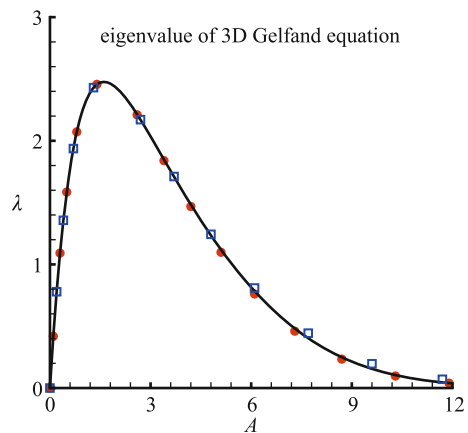
Order of approximation m	Averaged squared residual	Eigenvalue λ
1	0.77	1.6878
3	0.33	2.1757
5	9.6×10^{-2}	2.2935
10	2.2×10^{-2}	2.2668
15	1.3×10^{-2}	2.2635
20	9.5×10^{-3}	2.2636
25	7.3×10^{-3}	2.2636
30	5.9×10^{-3}	2.2636

is indeed true: when $A = 1$, we gain the convergent eigenfunction and its corresponding eigenvalue $\lambda = 2.2636$ by means of $c_0 = -1$ and the 6th-order auxiliary linear operator (14.69) with $c_6 = 1/8$, as shown in Table 14.7.

Regarding A as an unknown parameter, by means of $c_0 = -1$ and the 6th-order auxiliary linear operator (14.69) with $c_6 = 1$, we gain the 25th-order approximation of the eigenvalue using the Mathematica code in the Appendix 14.2. It is found that the eigenvalue of the 3D Gelfand equation has the maximum value $A = 2.476$ at

Table 14.8 The maximum eigenvalue $\lambda_{\max}$ of the 3D Gelfand equation by means of $c_0 = -1$ and the 6th-order auxiliary linear operator (14.69) with $c_6 = 1$.

	$\lambda_{\max}$	The corresponding value of A
4th-order HAM approx.	2.477	1.603
8th-order HAM approx.	2.476	1.600
12th-order HAM approx.	2.476	1.602
16th-order HAM approx.	2.476	1.605
20th-order HAM approx.	2.476	1.607
25th-order HAM approx.	2.476	1.610

Fig. 14.7 Eigenvalue of the 3D Gelfand equation by means of $c_0 = -1$ and the 6th-order auxiliary linear operator (14.69) with $c_6 = 1$. Solid line: 25th-order HAM approximation; Circles: 20th-order HAM approximation; Squares: simplified formula (14.70) by means of the first 8 terms of the 25th-order approximation.

$A \approx 1.61$, as shown in Table 14.8. The 25th-order homotopy-approximation of the eigenvalue is valid in the interval $0 \leq A \leq 12$, and the simplified formula given by its first eight terms, i.e.

$$\begin{aligned} \lambda \approx e^{-A} & \left(4.48514605A + 1.30348867A^2 \right. \\ & + 0.31378876A^3 + 0.056269253A^4 \\ & + 7.77343016 \times 10^{-3}A^5 + 8.58885688 \times 10^{-4}A^6 \\ & \left. + 6.90477890 \times 10^{-5}A^7 + 2.20710623 \times 10^{-6}A^8 \right), \end{aligned} \quad (14.70)$$

is a good approximation of λ , as shown in Fig. 14.7.

All of these verify that, in the frame of the HAM, the 2nd-order 3D nonlinear PDE (14.46) can be transferred into an infinite number of the 6th-order linear PDEs whose solutions can be obtained very easily by means of algebra calculations only. In this way, the original 3D nonlinear Gelfand equation is solved in a rather easy way. This is mainly because the HAM provides us extremely large freedom to choose the auxiliary linear operator, and at the same time, it also provides us a convenient way to guarantee the convergence of series solution by means of the convergence-control parameters. To the best of our knowledge, such kind of trans-

form has *never* been reported by means of other analytic and numerical methods. This suggests that we might have much larger freedom to solve nonlinear problems than we thought and believed traditionally!

14.4 Concluding remarks

In this chapter, a simple but rather efficient analytic approach is proposed to solve the high-dimensional Gelfand equation with strong nonlinearity. In the frame of the HAM, the *2nd-order* nonlinear PDE is transferred into an infinite number of *4th-order* 2D (or *6th-order* 3D) linear PDEs, which are easy to solve under the restriction of the so-called solution-expression.

By means of the HAM, the 3rd-order nonlinear PDE describing a *non-similarity* boundary-layer flow can be transferred into an infinite number of the 3rd-order linear ODEs so that the non-similarity boundary-layer flow can be solved in a rather similar way like *similarity* boundary-layer ones, as shown in Chapter 11. Besides, the 3rd-order nonlinear PDE describing the *unsteady* boundary-layer flow can be transferred into an infinite number of the 3rd-order linear ODEs so that the *unsteady* boundary-layer flow can be solved in a very similar way like *steady-state* boundary-layer ones, as shown in Chapter 12. Here, we further illustrate that, in the frame of the HAM, the *2nd-order* nonlinear PDE (Gelfand equation) can be transferred into an infinite number of the *4th-order* 2D (or *6th-order* 3D) linear PDEs. All of these verify that the HAM indeed provides us extremely large freedom to choose the auxiliary linear operator $\mathcal{L}$. Using this kind of extremely large freedom on the choice of the auxiliary linear operator $\mathcal{L}$, some nonlinear differential equations may be solved in a much easier way, as mentioned above.

It should be emphasized that the convergence-control parameters c_0, c_4 and c_6 play a very important role in guaranteeing the convergence of the series of the eigenfunction and eigenvalue of the Gelfand equation. For the 2D Gelfand equation, the convergent results are gained by means of $c_0 = -1$ and the 4th-order auxiliary linear operator (14.39) with *negative* convergence-control parameter c_4 . However, for the 3D Gelfand equation, the convergent results are obtained by means of $c_0 = -1$ and the 6th-order auxiliary linear operator (14.69) with *positive* convergence-control parameter c_6 . Therefore, without these properly choosing convergence-control parameters, it is very difficult to gain convergent results. Note that a divergent series has no meanings at all. Thus, in the frame of the HAM, the freedom on the choice of the auxiliary linear operator $\mathcal{L}$ is in essence based on this kind of guarantee of the convergence of homotopy-series solution, otherwise such kind of freedom has no meanings at all. This indicates once again the importance of the convergence-control parameters (Liang and Jeffrey, 2009): it is the convergence-control parameter that essentially distinguishes the HAM from *all* other analytic methods.

It should be emphasized that, the transform used in this chapter has *never* been reported by any other analytic and numerical methods, to the best of our knowledge. This reveals the originality and great flexibility of the HAM. Besides, it also sug-

gests that we might have much larger freedom to solve nonlinear problems than we thought and believed traditionally. Indeed, it is a good example to keep us an open mind for nonlinear problems: it is some of our “traditional” thoughts that might be the largest restriction to our mind.

It is a pity that many things about the HAM are still unclear now for nonlinear differential equations in general. For example, how to find the *best* or the *optimal* auxiliary linear operator among an infinite number of possible ones? Can we give some rigorous mathematical proofs in general? The freedom on the choice of the auxiliary linear operator might bring forward some new and interesting problems in applied and pure mathematics, and might, I wish, finally give us the “true” freedom on solving highly nonlinear differential equations, if such kind of freedom really exists.

Appendix 14.1 Mathematica code of 2D Gelfand equation

The 2D Gelfand equation

$$\Delta u + \lambda e^u = 0, \quad -1 < x < 1, \quad -1 < y < 1,$$

subject to the boundary condition on the four walls

$$u(x, \pm 1) = 0, \quad u(\pm 1, y) = 0,$$

is solved by means of the HAM. This Mathematica code is free available at

<http://numericaltank.sjtu.edu.cn/HAM.htm>

Mathematica code of 2D Gelfand equation

by Shijun LIAO
Shanghai Jiao Tong University
August 2010

```
(*****)
```

```
(*
(* For given A, we find such an eigenvalue lambda and
(*   a normalized eigenfunction w(x,y) satisfying:
(*       w_{xx} + w_{yy} + lambda * Exp[w(x)] = 0
(*   subject to the boundary conditions:
(*       w(1,y)=w(-1,y)=w(x,1)=w(x,-1)=-A, w(0,0)=0,
(*
(*
(*****)
```

```
<<Calculus`Pade`;
<<Graphics`Graphics`;

(*****)
```

```
(* Define initial guess of w(x)
(*)
```

```

(*****
w[0] = 0;
U[0] = A + w[0];

(*****
(* Define the function chi[k] *)
(*****
chi[k_] := If[ k <= 1, 0, 1];

(*****
(* Define the the auxiliary linear operator L *)
(*****
L[f_] := Module[{temp},
Expand[c2*D[f,{x,1},{y,1}]/x/y + c4*D[f,{x,2},{y,2}]
];

(*****
(* Define the inverse operator of L *)
(*****
Linvs[x^m_. * y^n_.] := x^(m+2)*y^(n+2)/(m+2)/(n+2)
/ (c2+c4*(n+1)*(m+1));
Linvs[x^m_.] := x^(m+2) * y^2/(m+2)/2/(c2+c4*(m+1));
Linvs[y^n_.] := x^2 * y^(n+2)/(n+2)/2/(c2+c4*(n+1));
Linvs[c_] := c*x^2*y^2/4/(c2+c4)/;FreeQ[c,x] && FreeQ[c,y];

(*****
(* The linear property of the inverse operator of L *)
(* Linvs[f_+g_] := Linvs[f]+Linvs[g]; *)
(*****
Linvs[p_Plus] := Map[Linvs,p];
Linvs[c_*f_] := c*Linvs[f] /; FreeQ[c,x] && FreeQ[c,y];

(*****
(* Define Getdelta[k] *)
(*****
Getdelta[k_] := Module[{temp,n},
temp[1] = D[w[k],{x,2}] + D[w[k],{y,2}];
temp[2] = Sum[lambda[n]*Dexp[k-n],{n,0,k}];
delta[k] = Expand[ temp[1] + temp[2] ];
];

(*****
(* Define GetDexp[n] *)
(*****
GetDexp[n_] := Module[{j},
If[n==0,Dexp[0]=Exp[w[0]]];
If[n > 0,Dexp[n]=Sum[(1-j/n)*Dexp[j]*w[n-j],{j,0,n-1}]/Expand];
];

(*****
(* Define Getlambda *)
(* This module gets lambda[k-1] *)
(*****

```



```

Getlambda[k_]:=Module[{eq,temp},
eq  = w[k]-(c0+chi[k])*w[k-1] /. {x -> 0,y->0};
temp = Solve[eq == 0, lambda[k-1]];
lambda[k-1] = temp[[1,1,2]]//Expand;
];

(*****
(* Define GetwSpecial *)
(* This module gets a special solution of (14.24) and (14.25)*)
(*****
GetwSpecial[k_]:=Module[{temp},
temp[0] = Expand[RHS[k]];
temp[1] = Linv[temp[0]];
temp[2] = temp[1] + chi[k]*w[k-1]//Simplify;
wSpecial = temp[2]//Expand;
];

(*****
(* Define Getw *)
(* This module gets a solution of (14.24) and (14.25) *)
(*****
Getw[k_]:=Module[{temp,alpha},
alpha = (c0+chi[k])*w[k-1]+c0*(1-chi[k])*A;
temp[1] = wSpecial /. y->1;
temp[2] = wSpecial /. x->1;
temp[3] = wSpecial /. {x->1, y->1};
temp[4] = alpha /. x->1;
temp[5] = alpha /. y->1;
temp[6] = alpha /. {x->1,y->1};
temp[7] = wSpecial - temp[1] - temp[2] + temp[3]
+ temp[4] + temp[5] - temp[6];
w[k] = Simplify[temp[7]]//Expand;
];

(*****
(* Define GetErr[k] *)
(*****
GetErr[k_]:=Module[{temp,sum,dx,dy,Num,i,j,X,Y},
err[k] = D[U[k],{x,2}] + D[U[k],{y,2}] + LAMBDA[k]*Exp[U[k]];
Nx = 10;
Ny = 10;
dx = N[1/Nx,100];
dy = N[1/Ny,100];
sum = 0;
Num = 0;
For[i = 0, i <= Nx-1, i++,
X = i*dx;
For[j = 0, j <= Ny - 1, j++,
Y = j*dy;
temp = err[k]^2 /. {x->X, y->Y};
sum = sum + temp;
Num = Num + 1;
];
];

```

```

Err[k] = sum/Num;
If[NumberQ[Err[k]], Print["Squared Residual of G.E. = ",
                          Err[k]/N]]
];

(*****
(* Define Body1[A] *)
(* Boyd's one-point formula *)
(*****
Boyd1[A_] := 3.2*A*Exp[-0.64*A];

(*****
(* Define Body3[A] *)
(* Boyd's three-point formula *)
(*****
Boyd3[A_] := Module[{temp,B,C,G},
G = 0.2763 + Exp[0.463*A] + 0.0483*Exp[-0.209*A];
B = A*( 0.829-0.566*Exp[0.463*A]-0.0787*Exp[-0.209*A])/G;
C = A*(-1.934+0.514*Exp[0.463*A]+1.9750*Exp[-0.209*A])/G;
temp[1] = 2.667*A + 4.830*B + 0.127*C;
temp[2] = 0.381*A + 0.254*B + 0.018*C;
temp[1]*Exp[-temp[2]]
];

(*****
(* Main Code *)
(*****
ham[m0_,m1_] := Module[{temp,k,j},
For[k=Max[1,m0], k<=m1, k=k+1,
Print[" k = ",k];
GetDexp[k-1];
Getdelta[k-1];
RHS[k] = c0*delta[k-1]/Expand;
GetwSpecial[k];
Getw[k];
U[k] = U[k-1] + w[k]/Simplify;
Getlambda[k];
Lambda[k-1] = Sum[lambda[j],{j,0,k-1}]/Expand;
LAMBDA[k-1] = Lambda[k-1]*Exp[-A];
Print[k-1,"th approximation of LAMBDA = ",N[LAMBDA[k-1],10] ];
];
Print["Successful !"];
];

(*****
(* Define the parameters *)
(*****
c0 = -1;
A = 1;
c2 = 0;
c4 = -1/4;

(*****
(* Print the parameters *)

```

```
(*****)
Print["    A    = ",A];
Print["    c0    = ",c0];
Print["    c2    = ",c2];
Print["    c4    = ",c4];

(* Get the 20th-order approximation *)
ham[1,21];

(* Get squared residual of up to 20th-order approximation *)
For[k=5, k<=20, k=k+5, Print["k = ",k]; GetErr[k]]
```

Appendix 14.2 Mathematica code of 3D Gelfand equation

The 3D Gelfand equation

$$\Delta u + \lambda e^u = 0, \quad -1 \leq x, y, z \leq 1,$$

subject to the boundary conditions

$$u(\pm 1, y, z) = 0, \quad u(x, \pm 1, z) = 0, \quad u(x, y, \pm 1) = 0,$$

is solved by means of the HAM. This Mathematica code is free available at

<http://numericaltank.sjtu.edu.cn/HAM.htm>

Mathematica code of 3D Gelfand equation

by Shijun LIAO
Shanghai Jiao Tong University
August 2010

```
(*****)
(*
(*   For given A, we find such an eigenvalue lambda and
(*   a normalized eigenfunction w(x,y) that:
(*       w_{xx}+w_{yy}+w_{zz}+lambda*Exp[w(x)] = 0
(*   subject to the boundary conditions:
(*       w(1,y)=w(-1,y)=w(x,1)=w(x,-1)=-A, w(0,0)=0,
(*
(*
<<Calculus`Pade`;
<<Graphics`Graphics`;
<<tool2010.nb;

(*****)
(* Define initial guess
(*****)
w[0] = 0;
```

```

U[0] = A + w[0];

(*****)
(* Define chi[k] *)
(*****)
chi[k_] := If[k <= 1, 0, 1];

(*****)
(* Define inverse operator of auxiliary linear operator L *)
(*****)
Liniv[x^m_ * y^n_ * z^k_] := x^(m+2)*y^(n+2)*z^(k+2)/(m+2)
/ (n+2)/(k+2)/(c3+c6*(n+1)*(m+1)*(k+1));
Liniv[y^n_ * z^k_] := x^2*y^(n+2)*z^(k+2)/2/(n+2)/(k+2)
/ (c3+c6*(n+1)*(k+1));
Liniv[x^m_ * z^k_] := x^(m+2)*y^2*z^(k+2)/(m+2)/2/(k+2)
/ (c3+c6*(m+1)*(k+1));
Liniv[x^m_ * y^n_] := x^(m+2)*y^(n+2)*z^2/(m+2)/(n+2)/2
/ (c3+c6*(n+1)*(m+1));
Liniv[x^m_] := x^(m+2)*y^2*z^2/(m+2)/2/(c3+c6*(m+1));
Liniv[y^n_] := x^2*y^(n+2)*z^2/2/(n+2)/2/(c3+c6*(n+1));
Liniv[z^k_] := x^2*y^2*z^(k+2)/2/2/(k+2)/(c3+c6*(k+1));
Liniv[c_] := c*x^2*y^2*z^2/8/(c3+c6) /; FreeQ[c,x]
&& FreeQ[c,y] && FreeQ[c,z];

(*****)
(* The linear property of the inverse operator of L *)
(* Liniv[f_+g_] := Liniv[f]+Liniv[g]; *)
(*****)
Liniv[p_Plus] := Map[Liniv,p];
Liniv[c_*f_] := c*Liniv[f]/; FreeQ[c,x] && FreeQ[c,y] && FreeQ[c,z];

(*****)
(* Define GetR[k] *)
(*****)
Getdelta[k_] := Module[{temp,n},
temp[1] = D[w[k],{x,2}] + D[w[k],{y,2}] + D[w[k],{z,2}];
temp[2] = Sum[lambda[n]*Dexp[k-n],{n,0,k}];
delta[k] = Expand[ temp[1] + temp[2] ];
];

(*****)
(* Define GetDexp[n] *)
(*****)
GetDexp[n_] := Module[{},
If[n == 0, Dexp[0] = 1];
If[n > 0, Dexp[n] = Sum[(1-j/n)*Dexp[j]*w[n-j],{j,0,n-1}]/Expand];
];

(*****)
(* Define Getlambda *)
(* This module gets lambda[k-1] *)
(*****)
Getlambda[k_] := Module[{eq,temp},
eq = w[k] - (c0+chi[k])*w[k-1] /. {x->0, y->0, z->0};

```

```

temp = Solve[eq == 0, lambda[k-1]];
lambda[k-1] = temp[[1,1,2]]//Expand;
];

(*****
(* Define GetwSpecial                                     *)
(*      This module gets a special solution of (14.61)    *)
(*****
GetwSpecial[k_] := Module[{temp},
temp[0] = Expand[RHS[k]];
temp[1] = Linv[temp[0]];
temp[2] = temp[1] + chi[k]*w[k-1];
wSpecial = temp[2]//Expand;
];

(*****
(* Define Getw                                           *)
(*      This module gets a solution of the high-order EQ  *)
(*****
Getw[k_] := Module[{temp, mu},
mu = (c0+chi[k])*w[k-1] + c0*(1-chi[k])*A;
temp[1] = wSpecial /. x->1;
temp[2] = wSpecial /. y->1;
temp[3] = wSpecial /. z->1;
temp[4] = wSpecial /. {x->1, y->1};
temp[5] = wSpecial /. {x->1, z->1};
temp[6] = wSpecial /. {y->1, z->1};
temp[7] = wSpecial /. {x->1, y->1, z->1};
temp[8] = wSpecial-temp[1]-temp[2]-temp[3]+temp[4]
        +temp[5]+temp[6]-temp[7];
temp[1] = mu /. x->1;
temp[2] = mu /. y->1;
temp[3] = mu /. z->1;
temp[4] = mu /. {x->1, y->1};
temp[5] = mu /. {x->1, z->1};
temp[6] = mu /. {y->1, z->1};
temp[7] = mu /. {x->1, y->1, z->1};
temp[9] = temp[8]+temp[1]+temp[2]+temp[3]-temp[4]
        -temp[5]-temp[6]+temp[7];
w[k] = Expand[temp[9]];
];

(*****
(* Define GetErr[k]                                       *)
(*****
GetErr[k_] := Module[
{temp, sum, dx, dy, dz, Num, i, j, m, X, Y, Z, Nx, Ny, Nz},
err[k] = D[U[k], {x, 2}] + D[U[k], {y, 2}] + D[U[k], {z, 2}]
        + LAMBDA[k]*Exp[U[k]];
Nx = 10;
Ny = 10;
Nz = 10;
dx = N[1/Nx, 100];
dy = N[1/Ny, 100];

```

```

dz = N[1/Nz,100];
sum = 0;
Num = 0;
For[i = 0, i <= Nx-1, i++,
  X = i*dx;
  For[j = 0, j <= Ny - 1, j++,
    Y = j*dy;
    For[m = 1, m <= Nz-1, m++,
      Z = m*dz;
      temp = err[k]^2 /. {x->X, y->Y, z->Z};
      sum = sum + temp;
      Num = Num + 1;
    ];
  ];
];
Err[k] = sum/Num;
If[NumberQ[Err[k]], Print["Squared Residual of G.E. = ",
  Err[k]/N]]
];

(*****
(* Define HP[F,m,n] *)
(* This module gives [m,n] homotopy-Pade approximation *)
(* of the series : F[k] = sum[f[i],{i,0,k}] *)
(*****
hp[F_,m_,n_] := Block[{i,k,dF,temp,q},
dF[0] = F[0];
For[k = 1, k <= m+n, k = k+1, dF[k] = Expand[F[k]-F[k-1]]];
temp = dF[0]+Sum[dF[i]*q^i,{i,1,m+n}];
Pade[temp,{q,0,m,n}]/.q->1
];

(*****
(* Main Code *)
(*****
ham[m0_,m1_] := Module[{temp,k,j},
For[k=Max[1,m0], k<=m1, k=k+1,
  Print[" k = ",k];
  GetDexp[k-1];
  Getdelta[k-1];
  RHS[k] = c0 * delta[k-1]//Expand;
  GetwSpecial[k];
  Getw[k];
  U[k] = U[k-1] + w[k]//Expand;
  Getlambda[k];
  Lambda[k-1] = Expand[Sum[lambda[j],{j,0,k-1}]];
  LAMBDA[k-1] = Lambda[k-1]*Exp[-A];
  Print[k-1,"th approximation of lambda = ",
    N[LAMBDA[k-1],10] ];
];
Print["Successful !"];
];

(*****

```

```

(* Physical and control parameters *)
(*****)
A      =      1;
c0      =      -1;
c3      =      0;
c6      =      1/8;

(*****)
(* Print physical and control parameters *)
(*****)
Print[" A = ",A];
Print[" c0 = ",c0];
Print[" c3 = ",c3];
Print[" c6 = ",c6];

(* Get the 10th-order approximation of LAMBDA *)
ham[1,11];

(* Get squared residual of up to 10th-order approximation *)
For[k=2, k<=10, k=k+2, Print["k = ",k]; GetErr[k]]

```

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Chapter 15

Interaction of Nonlinear Water Wave and Nonuniform Currents

Abstract In this chapter, we illustrate the validity of the homotopy analysis method (HAM) for a complicated nonlinear PDE describing the nonlinear interaction of a periodic traveling wave on a non-uniform current with exponential distribution of vorticity. In the frame of the HAM, the original highly nonlinear PDE with variable coefficient is transferred into an infinite number of much simpler linear PDEs, which are rather easy to solve. Physically, it is found that Stokes' criterion of wave breaking is still correct for traveling waves on non-uniform currents. It verifies that the HAM can be used to solve some complicated nonlinear PDEs so as to deepen and enrich our physical understanding about some interesting nonlinear phenomena.

15.1 Introduction

In science and engineering, there are many complicated nonlinear partial differential equations (PDEs). In this chapter, the homotopy analysis method (HAM) (Liao, 1992, 1997, 1999a,b; Liao and Campo, 2002; Liao, 2003a,b, 2004, 2005, 2006, 2009, 2010a,b; Liao and Cheung, 2003; Liao and Tan, 2007; Li et al., 2010; Xu et al., 2010) is used to solve such a kind of nonlinear problem.

Consider a train of permanent deep-water waves propagating on a steady, non-uniform, shear current with a vertical distribution of vorticity. Assume that the water is inviscid and incompressible, and the flow is two dimensional, so that the stream function ψ exists. Besides, the magnitude of the vorticity varies exponentially with the magnitude of the stream function ψ , but remains constant along the same streamline. This mainly because the profile of the currents is mostly exponential over the depth in practice.

Mathematically, this problem is described by a nonlinear PDE with a nonlinear boundary condition on an unknown wave surface, as shown below. Due to its mathematical difficulty, earlier works either assume that the current is uniform and hence ignore the effects of the vorticity distribution, or that the waves are of small amplitude and the current is weak and hence simply neglect the nonlinear interaction

between them. In the former case, the wave is regarded to propagate on still water in a reference frame moving with the current at the same speed, so that the uniform current can be linearly added to the pure waves. This kind of problem was investigated by Fenton (1985) who derived Stokes' fifth-order solution for deep-water waves. For shallow water waves with a uniform current, Clamond (1999, 2003) and Clamond and Fructus (2003) used the renormalization method to obtain accurate approximations. Unfortunately, in practice, currents are mostly nonuniform and hence the vorticity appears. In this case, linear wave theory is commonly used, if the waves are of small amplitude or the current is weak. Contributions were made by Thompson (1949), Kishida and Sobey (1988) who investigated the linear shear current, Eastwood and Watson (1989) who examined the bilinear current. For details, please refer to Peregrine (1977), Tounmazi and Ahilan (1990) and Wang et al. (1997, 1999). But if neither the wave amplitude is small nor the current is weak, the above mentioned methods are too crude: it is found that they overestimate the waves in an aiding current but underestimate them in an opposing current.

The nonlinear theory of the wave-current interaction begins from Dalrymple (1977), who numerically studied the interaction of wave with a current in 1/7 power-law. Thomas (1990) numerically investigated the interaction of nonlinear waves and a current with exponential distribution of vorticity. Wang et al. (1997) applied the perturbation method to obtain a fifth-order approximation. However, it is well-known that perturbation approximations are often valid for weakly nonlinear problems in general.

In 2003, Liao and Cheung (2003) successfully applied the HAM (Liao, 1992, 1997, 1999a,b; Liao and Campo, 2002; Liao, 2003a,b, 2004, 2005, 2006, 2009, 2010a,b; Liao and Cheung, 2003; Liao and Tan, 2007; Li et al., 2010; Xu et al., 2010) to gain accurate analytic approximations of two-dimensional (2D) nonlinear progressive deep-water waves. In 2009, Cheng et al. (2009) further applied the HAM to investigate the interaction between the 2D nonlinear deep-water wave and exponential shear currents.

15.2 Mathematical modeling

15.2.1 Original boundary-value equation

Let (x, y) denote the Cartesian coordinates, with the horizontal x -axis pointing in the direction of wave propagation, the y -axis pointing vertically upwards, and the origin lying the mean water level, respectively. A moving reference frame $(x' - ct, y') \mapsto (x, y)$ is used, where c is the wave phase speed, (x', y') is the coordinate in the still reference frame, respectively, so that the problem is steady. Let $y = \eta(x)$ denote the free surface, H the wave height, λ the wave length, and $\mathbf{u} = \{u(x, y), v(x, y)\}$ the total velocity of the wave on the current, respectively.

Since the fluid is incompressible and the flow is two dimensional, there exists the stream function $\psi(x, y)$ defined by

$$\psi_y(x, y) = u(x, y) - c, \quad \psi_x(x, y) = -v(x, y). \quad (15.1)$$

Let $\Omega(\psi)$ denote the given vorticity distribution of the current. Due to the definition of vorticity, we have the governing equation

$$\psi_{xx} + \psi_{yy} = -\Omega(\psi), \quad (15.2)$$

where the subscript denotes the partial derivatives.

Here, we investigate a special case

$$\Omega(\psi) = \varepsilon \exp(-\psi) \quad (15.3)$$

indicated by Phillips (1977), corresponding to a current with exponentially decaying vorticity in the direction of water depth, where ε is a physical parameter determining the strength of the vorticity. When $\varepsilon < 0$, the current moves in the same direction of wave propagation, called the aiding current. When $\varepsilon > 0$, the current moves in the opposing direction of wave propagation, called opposing current. Note that the vorticity of all uniform currents is zero. So, a given distribution of vorticity corresponds to an infinite number of currents. To avoid this, we define that $\varepsilon = 0$ corresponds to the still water.

On the free surface, we have the kinematic condition

$$\psi = 0, \quad \text{on} \quad y = \eta(x), \quad (15.4)$$

and the dynamic condition

$$\frac{1}{2}(\psi_x^2 + \psi_y^2) + gy = Q, \quad \text{on} \quad y = \eta(x), \quad (15.5)$$

where Q is the so-called Bernoulli constant, which is unknown and will be determined later. Besides, in the infinite depth, we have the boundary condition

$$\psi_x \rightarrow 0 \quad \text{as} \quad y \rightarrow -\infty. \quad (15.6)$$

Given the vorticity distribution $\Omega(\psi) = \varepsilon \exp(-\psi)$, the above-mentioned nonlinear boundary-value problem contains four unknowns: the stream function $\psi(x, y)$, the free surface $\eta(x)$, the wave phase speed c , and the Bernoulli constant Q . Note that the partial differential equation (15.2) contains the *exponential* term $\Omega(\psi) = \varepsilon \exp(-\psi)$, which possesses a rather strong nonlinearity for the unknown stream-function ψ . Besides, the *nonlinear* dynamic condition (15.5) is satisfied on the *unknown* free surface $y = \eta(x)$. Mathematically, the strong nonlinearity of the governing equation (15.2) and the boundary condition (15.5), and especially the unknown free surface $y = \eta(x)$, bring us great difficulties to solve this nonlinear boundary-value problem.

15.2.2 Dubreil-Jacotin transformation

To handle the unknown free surface, the Dubreil-Jacotin transformation (Dubreil-Jacotin, 1934; Thomas, 1990; Drennan et al., 1992) is employed to reformulate the problem in a *known* domain. The strategy is to regard the stream function ψ as an independent variable (like x) and y as an unknown function dependent on both x and ψ , expressed by $y(x, \psi)$. Then, according to the fundamental theorems in calculus, we have from (15.1) the local velocity components

$$u - c = \frac{1}{y_\psi}, \quad v = \frac{y_x}{y_\psi}. \quad (15.7)$$

For simplicity, the following dimensionless variables

$$x = k x^*, \quad y = k y^*, \quad \psi = \frac{k}{c} \psi^*, \quad \Omega(\psi) = \frac{1}{k c} \Omega^*(\psi), \quad (15.8)$$

are introduced, where the variables with asterisk are dimensional, and $k = 2\pi/\lambda$ is the wave number, respectively. Thus, the problem governed by the original PDEs (15.2) to (15.5) in the *unknown* domain

$$-\infty < y \leq \eta(x), \quad -\infty < x < +\infty$$

is reformulated in the *known* domain

$$0 \leq \psi < +\infty, \quad -\infty < x < +\infty$$

by a complicated nonlinear PDE

$$y_{xx} y_\psi^2 - 2 y_x y_\psi y_{x\psi} + (1 + y_x^2) y_{\psi\psi} = y_\psi^3 \Omega(\psi), \quad (15.9)$$

subject to the boundary conditions

$$\mu (1 + y_x^2) + 2 (y - \gamma) y_\psi^2 = 0 \quad \text{on } \psi = 0, \quad (15.10)$$

$$y_x \rightarrow 0 \quad \text{as } \psi \rightarrow +\infty, \quad (15.11)$$

where both $\mu = k c^2/g$ and $\gamma = k Q/g$ are unknown constants.

Note that, by means of the Dubreil-Jacotin transformation (Dubreil-Jacotin, 1934; Thomas, 1990; Drennan et al., 1992), the original nonlinear PDE (15.2) with the exponentially nonlinear term $\Omega(\psi) = \varepsilon \exp(-\psi)$ becomes the nonlinear PDE (15.9). Although (15.9) looks much more complicated than (15.2), its nonlinearity is actually weaker, because the exponential term $\Omega(\psi) = \varepsilon \exp(-\psi)$ in (15.9) is now regarded as a *known* variable coefficient. More importantly, the free surface boundary conditions are now satisfied on the *known* boundary $\psi = 0$. In addition, the above PDEs contain only three unknowns: one unknown function $y(x, \psi)$ and two unknown constants μ and γ . So, compared to the original equations, the number of the unknowns is reduced by one, i.e. from four to three. Therefore, the use

of the Dubreil-Jacotin transformation (Dubreil-Jacotin, 1934; Thomas, 1990; Drennan et al., 1992) in fact simplifies the original problem, although (15.9) looks more complicated than the original PDE (15.2).

The free surface $\eta(x)$ corresponds to the stream line $\psi = 0$. Since the origin is defined in the mean water level, the free surface is given by

$$\eta(x) = y(x, 0) - \frac{1}{\pi} \int_0^\pi y(x, 0) dx. \quad (15.12)$$

The dimensionless wave height H (scaled by k^{-1}) is given by

$$H = y(0, 0) - y(\pi, 0). \quad (15.13)$$

Thus, it is straightforward to gain the free surface $y = \eta(x)$ and the wave height, as long as $y(x, \psi)$ is obtained.

15.3 Brief mathematical formulas

The HAM is employed to gain the accurate analytic approximations of the complicated nonlinear PDE (15.9) with the variable coefficient $\Omega(\psi) = \varepsilon \exp(-\psi)$, subject to the boundary conditions (15.10) and (15.11).

15.3.1 Solution expression

As mentioned in previous chapters, the so-called solution-expression plays an important role in the frame of the HAM. Mathematically, from the highly nonlinear PDE (15.9), it is hard to guess the expression form of $y(x, \psi)$. Fortunately, from the physical background of the problem, it is easy to find a proper solution expression of it. Physically speaking, in the moving reference frame, the solution is composed of three parts: a train of deep-water propagation wave, a uniform current caused by the motion of the reference frame, and a nonuniform current possessing an exponential distribution of vorticity. In case of the pure deep-water wave (i.e. $\Omega = 0$), the wave elevation is periodic, expressed by

$$y = \sum_{m=0}^{+\infty} a_m \cos(mx),$$

where a_m is a constant coefficient. Even if there exists a non-uniform current, the wave elevation is still periodic in the x direction, and therefore should be always expressed by the above expression. Besides, since (15.9) contains the term $\exp(-\psi)$, it is straightforward that the solution $y(x, \psi)$ should contain the terms $\exp(-n\psi)$, where $n \geq 1$ is an integer. Such kind of terms $\exp(-n\psi)$ ($n \geq 1$) are also neces-

sary for the boundary condition (15.11) as $\psi \rightarrow +\infty$. Furthermore, it is well-known that the uniform current due to the motion of the reference frame won't give rise to the wave-current interaction. Therefore, from the physical viewpoints, even without solving the nonlinear PDEs (15.9) to (15.13), it is clear that $y(x, \psi)$ should be expressed in the form

$$y = -\psi + \sum_{n=1}^{+\infty} \sum_{m=0}^{+\infty} \alpha_{n,m} \exp(-n\psi) \cos(mx), \quad (15.14)$$

where the first part results from the uniform current due to the motion of the reference frame, and the second part from the combined effect of the nonlinear wave and the non-uniform current. The above formula provides us the so-called *solution-expression* of $y(x, \psi)$, which plays an important role in the frame of the HAM, as shown below. Note that (15.14) automatically satisfies the boundary condition (15.11).

15.3.2 Zeroth-order deformation equation

Let $q \in [0, 1]$ denote the embedding parameter, and $y_0(x, \psi)$, μ_0 , γ_0 denote the initial guesses of $y(x, \psi)$, μ and γ , respectively. In the frame of the HAM, we first construct three continuous variations $Y(x, \psi; q)$, $\Delta(q)$ and $\Gamma(q)$ such that, as q increases from 0 to 1, $Y(x, \psi; q)$ varies continuously from the initial guess $y_0(x, \psi)$ to the wave elevation $y(x, \psi)$, so does $\Delta(q)$ from the initial guess μ_0 to μ , and $\Gamma(q)$ from γ_0 to γ , respectively. Such kind of continuous variations (or deformations) are defined by the zeroth-order deformation equation

$$(1 - q) \mathcal{L} [Y(x, \psi; q) - y_0(x, \psi)] = c_0 q \mathcal{N} [Y(x, \psi; q)], \quad (15.15)$$

subject to the boundary conditions

$$\begin{aligned} & (1 - q) \mathcal{L}_b [Y(x, \psi; q) - y_0(x, \psi)] \\ &= c_0 q \mathcal{N}_b [Y(x, \psi; q), \Delta(q), \Gamma(q)], \quad \text{on } \psi = 0, \end{aligned} \quad (15.16)$$

and

$$\frac{\partial Y(x, \psi; q)}{\partial x} = 0, \quad \text{as } \psi \rightarrow +\infty, \quad (15.17)$$

with the definition of the wave height

$$Y(0, 0; q) - Y(\pi, 0; q) = H, \quad (15.18)$$

where $c_0 \neq 0$ is the convergence-control parameter, $\mathcal{L}$ and $\mathcal{L}_b$ are two auxiliary linear operators, $\mathcal{N}$ and $\mathcal{N}_b$ are two nonlinear operators defined by

$$\mathcal{N} [Y(x, \psi; q)] = Y_{xx} Y_\psi^2 - 2 Y_x Y_{x\psi} Y_\psi + (1 + Y_x^2) Y_{\psi\psi} - \Omega Y_\psi^3, \quad (15.19)$$

$$\mathcal{N}_b [Y(x, \psi; q), \Delta(q), \Gamma(q)] = \Delta (1 + Y_x^2) + 2 (Y - \Gamma) Y_\psi^2, \quad (15.20)$$

corresponding to (15.9) and (15.10), respectively. The auxiliary linear operators $\mathcal{L}$ and $\mathcal{L}_b$ have the property $\mathcal{L}(0) = 0$ and $\mathcal{L}_b(0) = 0$, and will be chosen later.

When $q = 0$, since $\mathcal{L}(0) = 0$ and $\mathcal{L}_b(0) = 0$, the zeroth-order deformation equations (15.15) to (15.18) have the solution

$$Y(x, \psi; 0) = y_0(x, \psi), \quad \Delta(0) = \mu_0, \quad \Gamma(0) = \gamma_0. \quad (15.21)$$

When $q = 1$, since $c_0 \neq 0$, the zeroth-order deformation equations (15.15) to (15.18) are equivalent to the original PDEs (15.9) to (15.13), provided

$$Y(x, \psi; 1) = y(x, \psi), \quad \Delta(1) = \mu, \quad \Gamma(1) = \gamma. \quad (15.22)$$

Hence, as the embedding parameter q increases from 0 to 1, $Y(x, \psi; q)$, $\Delta(q)$ and $\Gamma(q)$ indeed vary from the initial guesses $y_0(x, \psi)$, μ_0 and γ_0 to the exact solutions $y(x, \psi)$, μ and γ , respectively. Mathematically speaking, the zeroth-order deformation equations (15.15) to (15.18) construct three homotopies:

$$Y(x, \psi; q) : y_0(x, \psi) \sim y(x, \psi), \quad \Delta(q) : \mu_0 \sim \mu, \quad \Gamma(q) : \gamma_0 \sim \gamma. \quad (15.23)$$

Using (15.21) and expanding $Y(x, y; q)$, $\Delta(q)$ and $\Gamma(q)$ into the Maclaurin series with respect to the embedding parameter q , we have the homotopy-Maclaurin series

$$Y(x, \psi; q) = y_0(x, \psi) + \sum_{n=1}^{+\infty} y_n(x, \psi) q^n, \quad (15.24)$$

$$\Delta(q) = \mu_0 + \sum_{n=1}^{+\infty} \mu_n q^n, \quad (15.25)$$

$$\Gamma(q) = \gamma_0 + \sum_{n=1}^{+\infty} \gamma_n q^n, \quad (15.26)$$

where

$$y_n(x, \psi) = \mathcal{D}_n [Y(x, \psi; q)] = \frac{1}{n!} \frac{\partial^n Y(x, \psi; q)}{\partial q^n} \Big|_{q=0},$$

and

$$\mu_n = \mathcal{D}_n [\Delta(q)], \quad \gamma_n = \mathcal{D}_n [\Gamma(q)]$$

are the n th-order homotopy-derivatives of $Y(x, \psi; q)$, $\Delta(q)$ and $\Gamma(q)$, respectively, and $\mathcal{D}_n$ is the n th-order homotopy-derivative operator. It is well-known that a power series has often a finite convergence radius. Fortunately, the zeroth-order deformation equations (15.15) and (15.16) contains the convergence-control parameter c_0 , which can be used to adjust and control the convergence radius of the series. Besides, we have extremely large freedom to choose the auxiliary linear operators $\mathcal{L}$ and $\mathcal{L}_b$. Assuming that all of them are so properly chosen that the above homotopy-Maclaurin series are absolutely convergent at $q = 1$, we have, according to (15.22),

the homotopy-series

$$y(x, \psi) = y_0(x, \psi) + \sum_{n=1}^{+\infty} y_n(x, \psi), \quad (15.27)$$

$$\mu = \mu_0 + \sum_{n=1}^{+\infty} \mu_n, \quad (15.28)$$

$$\gamma = \gamma_0 + \sum_{n=1}^{+\infty} \gamma_n. \quad (15.29)$$

15.3.3 High-order deformation equation

The governing equation and boundary conditions of the unknown $y_n(x, \psi)$ can be deduced uniquely from the zeroth-order deformation equations (15.15) to (15.18). Directly substituting the series (15.24) to (15.26) into the zeroth-order deformation equations (15.15) to (15.18), then equating the coefficients of the like-power of q , we have the n th-order deformation equations

$$\mathcal{L}[y_n(x, \psi) - \chi_n y_{n-1}(x, \psi)] = c_0 \delta_{n-1}(x, \psi), \quad (15.30)$$

subject to the boundary condition

$$\mathcal{L}_b(y_n - \chi_n y_{n-1}) = c_0 \delta_{n-1}^b(x) \quad \text{on } \psi = 0, \quad (15.31)$$

and

$$\frac{\partial y_n}{\partial x} = 0 \quad \text{as } \psi \rightarrow +\infty, \quad (15.32)$$

with the restriction

$$y_n(0, 0) - y_n(\pi, 0) = 0, \quad (15.33)$$

where

$$\chi_k = \begin{cases} 0, & k \leq 1, \\ 1, & k > 1, \end{cases} \quad (15.34)$$

and

$$\begin{aligned} \delta_k = & \sum_{i=0}^k \frac{\partial^2 y_{k-i}}{\partial x^2} \sum_{j=0}^i \frac{\partial y_{i-j}}{\partial \psi} \frac{\partial y_j}{\partial \psi} - 2 \sum_{i=0}^k \frac{\partial y_{k-i}}{\partial x} \sum_{j=0}^i \frac{\partial^2 y_{i-j}}{\partial x \partial \psi} \frac{\partial y_j}{\partial \psi} + \frac{\partial^2 y_k}{\partial \psi^2} \\ & + \sum_{i=0}^k \frac{\partial^2 y_{k-i}}{\partial \psi^2} \sum_{j=0}^i \frac{\partial y_{j-i}}{\partial x} \frac{\partial y_i}{\partial x} - \Omega(\psi) \sum_{i=0}^k \frac{\partial y_{k-i}}{\partial \psi} \sum_{j=0}^i \frac{\partial y_{j-i}}{\partial \psi} \frac{\partial y_i}{\partial \psi}, \end{aligned} \quad (15.35)$$

$$\delta_k^b = \mu_k + \sum_{i=0}^k \mu_{k-i} \sum_{j=0}^i \frac{\partial y_{i-j}}{\partial x} \frac{\partial y_j}{\partial x} + 2 \sum_{i=0}^k (y_{k-i} - \gamma_{k-i}) \sum_{j=0}^i \frac{\partial y_{i-j}}{\partial \psi} \frac{\partial y_j}{\partial \psi} \quad (15.36)$$

are gained by Theorem 4.1. Equations (15.30) and (15.31) can be gained directly by Theorem 4.15. For details, please refer to Chapter 4 and Cheng et al. (2009).

Note that the high-order deformation equation contains the two auxiliary linear operators $\mathcal{L}$ and $\mathcal{L}_b$. Fortunately, in the frame of the HAM, we have extremely large freedom to choose the auxiliary linear operators, as illustrated in Chapter 14. Note that the PDE (15.9) contains only one linear term $y_{\psi\psi}$. However, if we choose

$$\mathcal{L}[y(x, \psi)] = \frac{\partial^2 y(x, \psi)}{\partial \psi^2}$$

as the auxiliary linear operator, we obtain a solution $y(x, \psi)$ expressed in power series of ψ , which *disobeys* the solution-expression (15.14). Since a power series often has a finite radius of convergence, it is difficult for such kind of solution to satisfy the boundary condition (15.11) as $\psi \rightarrow +\infty$. So, if one follows the traditional ideas of perturbation methods, which have a high opinion of the linear terms of a nonlinear governing equation, this linear term $y_{\psi\psi}$ of the original PDE (15.9) might greatly mislead us. Fortunately, the HAM provides us extremely large freedom to choose the auxiliary linear operators, so that we can *completely* forget the linear term $y_{\psi\psi}$ in (15.9) and choose a proper auxiliary linear operator $\mathcal{L}$ mainly based on the solution-expression (15.14), which is gained under the physical considerations. Note that $u = \exp(-\psi) \cos(x)$ satisfies

$$\frac{\partial^2 u}{\partial \psi^2} + \frac{\partial^2 u}{\partial x^2} = 0.$$

So, to obey the solution expression (15.14), we choose the following auxiliary linear operator

$$\mathcal{L}(u) = \frac{\partial^2 u}{\partial \psi^2} + \frac{\partial^2 u}{\partial x^2}. \quad (15.37)$$

Note that the boundary condition (15.10) does not contain any linear terms, too. Similarly, to obey the solution-expression (15.14), we choose the following auxiliary linear operator

$$\mathcal{L}_b(u) = u + \frac{\partial u}{\partial \psi} \quad (15.38)$$

for the nonlinear boundary condition (15.10).

It should be emphasized that the above two auxiliary linear operators (15.37) and (15.38) have nearly no relationships with the governing equation (15.9) and the boundary condition (15.10), respectively! This is mainly because the HAM provides us extremely large freedom to choose the auxiliary linear operators $\mathcal{L}$ and $\mathcal{L}_b$ so that the solution-expression (15.14) is satisfied. As mentioned in previous chapters, this kind of freedom is an obvious advantage of the HAM over other analytic techniques, and can greatly simplify resolving some nonlinear problems.

Besides, the HAM also provides us great freedom to choose the initial guesses. The initial guess $y_0(x, \psi)$ should also satisfy the solution expression (15.14) and in addition the definition of wave height (15.13). So, it is straightforward for us to choose such an initial guess

$$y_0(x, \psi) = -\psi + \frac{H}{2} \exp(-\psi) \cos x. \quad (15.39)$$

Note that the initial solutions μ_0 and γ_0 are unknown up to now.

In the above n th-order deformation equations (15.30) to (15.33), the right hand-side terms $\delta_{n-1}(x, \psi)$ and $\delta_{n-1}^b(x)$ are only dependent upon $y_k(x, \psi)$, μ_k and γ_k , where $0 \leq k \leq n-1$, so that all of them are regarded as the known terms. Thus, according to the definitions (15.37) and (15.38) of the two auxiliary linear operators $\mathcal{L}$ and $\mathcal{L}_b$, the high-order deformation equation (15.30) is *linear*, subject to the two *linear* boundary conditions (15.31) and (15.32) on the *known* boundary $\psi = 0$ and $\psi \rightarrow +\infty$, respectively. Such kind of linear boundary-value problem in a fixed domain is much simpler, and thus easier to solve, than the original highly nonlinear PDE (15.9) with variable coefficient $\exp(-\psi)$.

15.3.4 Successive solution procedure

Note that the n th-order deformation equations (15.30) to (15.33) contain three unknowns, $y_n(x, \psi)$, μ_{n-1} and γ_{n-1} . Due to the definition (15.37) of the auxiliary linear operator $\mathcal{L}$, it holds for any positive integer M that

$$\mathcal{L} \left[\sum_{m=1}^M C_{n,m} \exp(-m\psi) \cos(mx) \right] = 0,$$

where $C_{n,m}$ is a constant. Thus, the general solution of the high-order deformation equation (15.30) reads

$$y_n(x, \psi) = \chi_n y_{n-1}(x, \psi) + y_n^*(x, \psi) + \sum_{m=1}^M C_{n,m} \exp(-m\psi) \cos(mx), \quad (15.40)$$

where M is a positive integer to be determined later, and

$$y_n^*(x, \psi) = c_0 \mathcal{L}^{-1} [\delta_{n-1}(x, \psi)] \quad (15.41)$$

is a special solution of (15.30). Here, $\mathcal{L}^{-1}$ is an inverse operator of $\mathcal{L}$, defined by

$$\mathcal{L}^{-1} [\exp(-m\psi) \cos(nx)] = \frac{\exp(-m\psi) \cos(nx)}{(m^2 - n^2)}, \quad m \neq n, \quad (15.42)$$

$$\mathcal{L}^{-1} [\exp(-m\psi) \sin(nx)] = \frac{\exp(-m\psi) \sin(nx)}{(m^2 - n^2)}, \quad m \neq n. \quad (15.43)$$

Using the above formulas, it is easy to gain the special solution $y_n^*(x, \psi)$ of (15.30), especially by means of the computer algebra system such as Mathematica, Maple and so on.

To determine the unknown μ_{n-1} , γ_{n-1} and $C_{n,m}$, we substitute the general solution (15.40) into the boundary condition (15.31), i.e.

$$\mathcal{L}_b \left[y_n^* + \sum_{m=1}^M C_{n,m} \exp(-m\psi) \cos(mx) \right] = c_0 \delta_{n-1}^b(x), \quad \text{on } \psi = 0, \quad (15.44)$$

where M is undetermined. The above equation can be rewritten in the form

$$\sum_{m=2}^M (1-m) C_{n,m} \cos(mx) = \sum_{m=0}^{2n+1} B_{n,m}(\gamma_{n-1}, \mu_{n-1}) \cos(mx), \quad (15.45)$$

where the coefficient $B_{n,m}(\gamma_{n-1}, \mu_{n-1})$ is determined by the special solution (15.41) and the right-hand side term $c_0 \delta_{n-1}^b(x)$ of (15.44). Balancing both sides of the above equation, we have

$$M = 2n + 1, \quad C_{n,m} = \frac{B_{n,m}(\gamma_{n-1}, \mu_{n-1})}{(1-m)}, \quad 1 < m \leq 2n + 1,$$

and

$$B_{n,0}(\gamma_{n-1}, \mu_{n-1}) = 0, \quad B_{n,1}(\gamma_{n-1}, \mu_{n-1}) = 0, \quad (15.46)$$

which exactly provide us the two algebraic equations to determine the unknown γ_{n-1} and μ_{n-1} .

Up to now, only the coefficient $C_{n,1}$ is unknown. Substituting (15.40) into (15.33) gives the algebraic equation

$$y_n^*(0, 0) - y_n^*(\pi, 0) + \sum_{m=1}^{2n+1} C_{n,m} - \sum_{m=1}^{2n+1} C_{n,m} \cos(m\pi) = 0 \quad (15.47)$$

for $C_{n,1}$, whose solution reads

$$C_{n,1} = -\frac{1}{2} \left\{ y_n^*(0, 0) - y_n^*(\pi, 0) + \sum_{m=2}^{2n+1} [1 - (-1)^m] C_{n,m} \right\}. \quad (15.48)$$

Hence, all unknowns are determined and thus the problem is closed.

In this way, we obtain $y_n(x, \psi)$, μ_{n-1} and γ_{n-1} successively in order $n = 1, 2, 3, \dots$ by means of only algebra computations. For example, solving the 1st-order deformation equations, we have

$$\gamma_0 = \frac{1}{320} [160 + 25 H^2 + \varepsilon (80 + 3 H^2)], \quad (15.49)$$

$$\mu_0 = \frac{1}{8 + H^2} \left[8 - \frac{3}{4} H^2 + \frac{5}{32} H^4 + \varepsilon \left(4 - \frac{1}{60} H^2 + \frac{3}{160} H^4 \right) \right] \quad (15.50)$$

and

$$y_1(x, \psi) = c_0 [A_{1,0}(\psi) + A_{1,1}(\psi) \cos x + A_{1,2}(\psi) \cos(2x) + A_{1,3}(\psi) \cos(3x)], \quad (15.51)$$

where

$$\begin{aligned} A_{1,0}(\psi) &= -\frac{H^2}{8} \exp(-2\psi) + \varepsilon \exp(-\psi) + \frac{\varepsilon H^2}{24} \exp(-3\psi), \\ A_{1,1}(\psi) &= \frac{3H^3}{64} \exp(-\psi) - \frac{H^3}{64} \exp(-3\psi) - \frac{\varepsilon H}{2} \exp(-\psi) \\ &\quad - \frac{9\varepsilon H^3}{2240} \exp(-\psi) + \frac{\varepsilon H}{2} \exp(-2\psi) + \frac{\varepsilon H^3}{160} \exp(-4\psi), \\ A_{1,2}(\psi) &= -\frac{H^2(256 + 40H^2 - 5H^4)}{128(8 + H^2)} \exp(-2\psi) \\ &\quad - \frac{\varepsilon H^2(384 + 136H^2 - 9H^4)}{1920(8 + H^2)} \exp(-2\psi) + \frac{3\varepsilon H^2}{40} \exp(-3\psi), \\ A_{1,3}(\psi) &= -\frac{H^3}{32} \exp(-3\psi) - \frac{3\varepsilon H^3}{448} \exp(-3\psi) + \frac{\varepsilon H^3}{224} \exp(-4\psi). \end{aligned}$$

Similarly, we can obtain $\gamma_1, \mu_1, y_2(x, \psi)$, and so on. It is found that all of our solutions $y_n(x, \psi)$ contain the term $\exp(-m\psi)$, where $m \geq 1$, so that the boundary condition (15.32) is automatically satisfied. As long as the convergent series solution of $y(x, \psi)$ is obtained, one can get the velocity field by (15.7) and the wave elevation by (15.12), respectively. Note that the dimensionless wave phase speed $k c^2/g$ is given by μ , and the energy on the free surface is given by γ , respectively.

It is found that $y(x, \psi)$ can be expressed by

$$y(x, \psi) = y_c(\psi) + y_w(x, \psi) + y_i(x, \psi),$$

where

$$y_c(\psi) = y(x, \psi)|_{H=0}$$

is related to the pure current without waves,

$$y_w(x, \psi) = y(x, \psi)|_{\varepsilon=0}$$

is related to the pure waves without the current, and $y_i(x, \psi)$ is related to the wave-current interaction, respectively.

For more details, please refer to Cheng et al. (2009).

15.4 Homotopy approximations

Based on the formulas mentioned above, a corresponding Mathematica code is developed, which is given in the Appendix 15.1 and free available as open resource (Accessed 25 Nov 2011, will be updated in the future) at

<http://numericaltank.sjtu.edu.cn/HAM.htm>

Note that the homotopy-series of $y(x, \psi)$ and $\mu = kc^2/g$, $\gamma = kQ/g$ contain the so-called convergence-control parameter c_0 , which provides us a convenient way to guarantee the convergence of the homotopy-series solution, as shown below.

The accuracy of the m th-order approximation is indicated by means of the averaged squared residual of the governing equation

$$E_m = \frac{1}{(1 + N_x)(1 + N_\psi)} \sum_{m=0}^{N_x} \sum_{j=0}^{N_\psi} \left\{ \mathcal{N} \left[\sum_{n=0}^m y_n(x_i, \psi_j) \right] \right\}^2 \quad (15.52)$$

and the averaged squared residual of the nonlinear boundary condition

$$E_m^b = \frac{1}{1 + N_x} \sum_{i=0}^{N_x} \left\{ \mathcal{N}_b \left[\sum_{n=0}^m y_n(x_i, 0), \sum_{n=0}^m \mu_n, \sum_{n=0}^m \gamma_n \right] \right\}^2, \quad (15.53)$$

where

$$x_i = i \left(\frac{\pi}{N_x} \right), \quad \psi_j = j \left(\frac{2\pi}{N_\psi} \right)$$

with $N_x = N_\psi = 10$. From the physical point of view, the residual error of the governing equation below one wave-length (2π) in water depth is rather small and thus is neglected, because it is well-known that the wave velocity field decays exponentially as the water depth increases.

Without loss of generality, let us first consider the case $H = 3/10$ and $\varepsilon = 1/5$. The averaged squared residual E_m and E_m^b of the governing equation and the nonlinear boundary condition versus the convergence-control parameter c_0 are as shown in Fig. 15.1 and Fig. 15.2, respectively. Note that both of E_m and E_m^b decreases in the interval $-0.6 \leq c_0 < 0$ as m increases. Besides, the optimal value of c_0 given by the minimum of E_m and E_m^b is about -0.55 , as shown in Table 15.1 and Table 15.2, respectively. This is indeed true: the homotopy-approximations when $H = 3/10$ and $\varepsilon = 1/5$ given by means of $c_0 = -11/20$ converges quickly, as shown in Table 15.3. Notice that the averaged squared residuals of the governing equation and the nonlinear boundary condition decrease monotonously to 1.1×10^{-21} and 5.2×10^{-22} at the 40th-order of approximation, respectively. Besides, we gain the convergent values $\mu = 0.81442133$ and $\gamma = 0.40556503$, respectively. By means of the homotopy-Padé acceleration technique, we obtain the more accurate convergent values of $\mu = 0.814421334285$ and $\gamma = 0.405565029876$, as shown in Table 15.4. Furthermore, it is found that even the 3rd-order homotopy approximation of the wave elevation is accurate enough, as shown in Fig. 15.3. All of these verify that we can indeed gain accurate analytic approximations of the complicated nonlinear PDE (15.9) subject to the nonlinear boundary condition (15.10) by means of the HAM as mentioned above. Thus, given physically reasonable values of H and ε , we can gain convergent homotopy-approximations in a similar way by means of the Mathematica code given in the Appendix 15.1.

Fig. 15.1 Averaged squared residual of the governing equation (15.9) versus c_0 when $H = 3/10$ and $\varepsilon = 1/5$. Dashed line: 5th-order homotopy approximation; Dash-dotted line: 8th-order homotopy-approximation; Solid line: 10th-order homotopy-approximation

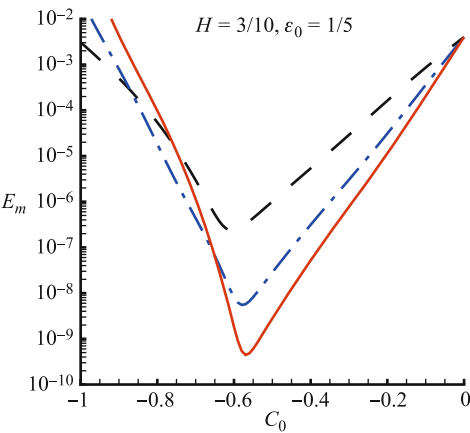


Fig. 15.2 Averaged squared residual of boundary-condition (15.10) versus c_0 when $H = 3/10$ and $\varepsilon = 1/5$. Dashed line: 5th-order homotopy approximation; Dash-dotted line: 8th-order homotopy-approximation; Solid line: 10th-order homotopy-approximation

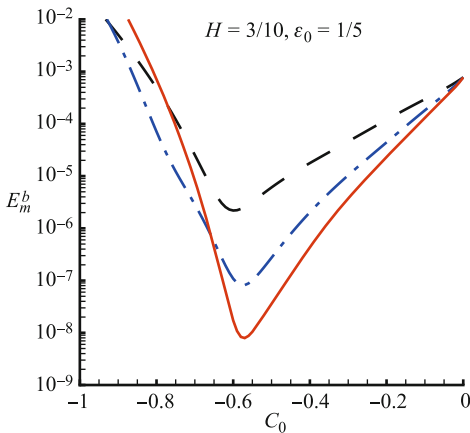


Table 15.1 Minimum of squared residual of the governing equation (15.9) and the corresponding optimal convergence-control parameter c_0 when $H = 3/10$ and $\varepsilon = 1/5$.

Order of approximation m	Minimum of squared residual E_m	Optimal c_0
5	2.3×10^{-7}	-0.6049
8	5.5×10^{-9}	-0.5775
10	4.4×10^{-10}	-0.5686
15	1.3×10^{-10}	-0.5569

Before discussing the wave-current interaction, let us consider the two limiting cases: the pure water wave, corresponding to the zero vorticity $\Omega = 0$, and the pure current, corresponding to the zero wave-height $H = 0$. When $\Omega = 0$, it is the Stokes deep-water wave model (Stokes, 1880). According to Stokes' theory (Marshall et al., 1992), a train of propagating deep-water waves breaks when the wave height arrives at its maximum value $H = 0.886$, corresponding to the maximum crest angle

Table 15.2 Minimum of squared residual of the boundary condition (15.10) and the corresponding optimal convergence-control parameter c_0 when $H = 3/10$ and $\varepsilon = 1/5$.

Order of approximation m	Minimum of squared residual E_m^b	Optimal c_0
5	2.2×10^{-6}	-0.5976
8	8.1×10^{-8}	-0.5749
10	7.9×10^{-9}	-0.5723
15	5.7×10^{-11}	-0.5707

Table 15.3 The m th-order homotopy-approximations of μ , γ and the corresponding squared residual E_m , E_m^b when $H = 3/10$, $\varepsilon = 1/5$ by means of $c_0 = -11/20$.

m	E_m	E_m^b	$\mu = k c^2/g$	$\gamma = k Q/g$
3	7.2×10^{-6}	3.6×10^{-5}	0.8167	0.4078
5	4.4×10^{-7}	3.2×10^{-6}	0.8131	0.4042
10	6.0×10^{-10}	1.0×10^{-8}	0.8142	0.4053
15	1.4×10^{-12}	8.3×10^{-11}	0.8144	0.4054
20	7.2×10^{-15}	7.4×10^{-13}	0.81442	0.40556
25	9.4×10^{-17}	5.6×10^{-15}	0.814421	0.405565
30	2.3×10^{-18}	3.5×10^{-17}	0.81442133	0.4055650
35	5.3×10^{-20}	1.6×10^{-19}	0.81442133	0.40556503
40	1.1×10^{-21}	5.2×10^{-22}	0.81442133	0.40556503

Table 15.4 The $[m, m]$ homotopy-Padé approximation of μ and γ when $H = 3/10$ and $\varepsilon = 1/5$.

m	$\mu = k c^2/g$	$\gamma = k Q/g$
2	0.8029	0.4035
4	0.8154	0.4056
6	0.814421	0.40556
8	0.814421	0.405565
10	0.81442133	0.40556502
12	0.814421334	0.405565029
14	0.81442133428	0.40556502987
16	0.814421334285	0.405565029876
18	0.814421334285	0.405565029876
20	0.814421334285	0.405565029876

$\alpha = 120$ degree. It is found that, even for the waves with large wave-amplitude close to the limiting wave, our $[m, m]$ homotopy-Padé approximations of $\mu = k c^2/g$ and the crest elevation $\eta_c = \eta(0)$ converge, as shown in Table 15.5 and Table 15.6, respectively. A comparative error of η_c with the linear estimated crest elevation $\eta_c \approx H/2$ is given at the last column in Table 15.6. It is found that the comparative error is around 25% for large amplitude waves. Besides, the homotopy-Padé approximations of the wave phase speed agrees well with the analytic results given by Schwartz (1974) and Longuet-Higgins and Tanaka (1997), as shown in Fig. 15.4. Note that, as shown in Table 15.5 and Table 15.6, the maximum wave height given by the

Fig. 15.3 Wave elevation when $H = 3/10$ and $\varepsilon = 1/5$ given by $c_0 = -11/20$. Solid line: 15th-order homotopy-approximation; Symbols: 3rd-order homotopy-approximation.

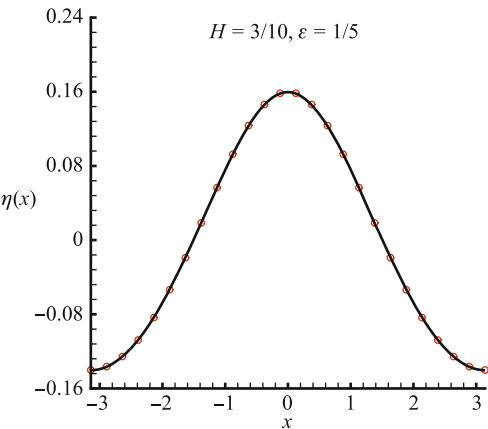


Fig. 15.4 Comparison of the dispersion relationship of the pure deep-water waves. Solid line: [23,23] homotopy-Padé approximation; Filled circles: Longuet-Higgins's result (Longuet-Higgins and Tanaka, 1997); Open circles: Schwartz's result (Schwartz, 1974).

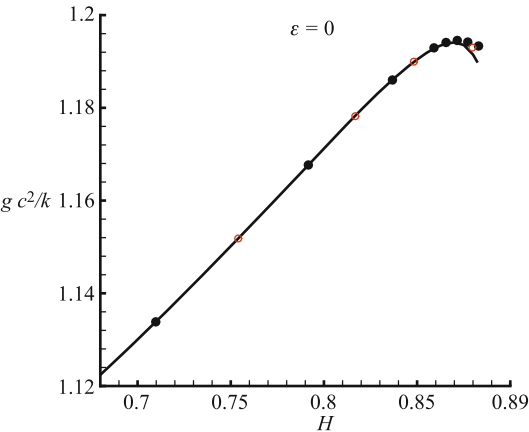


Table 15.5 The $[m, m]$ homotopy-Padé approximation of $\mu = k^2/g$ for the pure deep-water waves without current, corresponding to $\Omega = 0$.

H	$m = 15$	$m = 17$	$m = 19$	$m = 20$	$m = 21$
0.65	1.1113	1.1113	1.1113	1.1113	1.1113
0.70	1.1300	1.1300	1.1300	1.1300	1.1300
0.75	1.1502	1.1502	1.1502	1.1502	1.1502
0.80	1.1712	1.1712	1.1712	1.1712	1.1712
0.85	1.1905	1.1904	1.1905	1.1904	1.1904
0.86	1.1926	1.1929	1.1930	1.1930	1.1930
0.87	1.1939	1.1940	1.1941	1.1941	1.1941
0.875	1.1933	1.1932	1.1936	1.1936	1.1936
0.88	1.1915	1.1909	1.1917	1.1916	1.1916
0.882	1.1902	1.1892	1.1902	1.1902	1.1902
0.8825	1.1898	1.1887	1.1898	1.1898	1.1897

Table 15.6 The $[m, m]$ homotopy-Padé approximation of the crest elevation η_c for the pure deep-water waves without current, corresponding to $\Omega = 0$, compared with $H/2$.

H	$m = 15$	$m = 17$	$m = 19$	$m = 20$	$1 - H/2\eta_c$
0.65	0.3875	0.3875	0.3875	0.3875	16.13%
0.7	0.4250	0.4250	0.4250	0.4250	17.65%
0.75	0.4649	0.4649	0.4649	0.4649	19.34%
0.8	0.5080	0.5079	0.5079	0.5079	21.24%
0.85	0.5566	0.5566	0.5565	0.5565	23.63%
0.86	0.5675	0.5672	0.5672	0.5671	24.18%
0.87	0.5793	0.5805	0.5798	0.5800	25.00%
0.875	0.5855	0.5858	0.5856	0.5858	25.32%
0.88	0.5923	0.5922	0.5922	0.5922	25.70%
0.882	0.5952	0.5953	0.5949	0.5949	25.87%
0.8825	0.5956	0.5956	0.5956	0.5956	25.93%

HAM is about $H = 0.8825$, which is a little less than $H = 0.886$ given by Stokes theory (Marshall et al., 1992).

Note that $H = 0$ corresponds to the pure current without waves. When $c_0 = -1$, the fifth-order homotopy-approximation of the pure current reads

$$y(\psi) = -\psi - \varepsilon \exp(-\psi) + \frac{3}{4} \varepsilon^2 \exp(-2\psi) - \frac{5}{6} \varepsilon^3 \exp(-3\psi) + \frac{35}{32} \varepsilon^4 \exp(-4\psi) - \frac{63}{40} \varepsilon^5 \exp(-5\psi) + \text{const.} \quad (15.54)$$

Using (15.2), we have the velocity profile of the pure current in the still coordinates

$$u_c(\psi) = -\varepsilon \exp(-\psi) + \frac{1}{2} \varepsilon^2 \exp(-2\psi) - \frac{1}{2} \varepsilon^3 \exp(-3\psi) + \frac{5}{8} \varepsilon^4 \exp(-4\psi) - \frac{7}{8} \varepsilon^5 \exp(-5\psi) + \frac{21}{16} \varepsilon^6 \exp(-6\psi), \quad (15.55)$$

which is exactly the same as the perturbation solution (Wang et al., 1997, 1999). The above two limiting cases indicate the validity of the HAM for this kind of complicated nonlinear PDE.

The general cases of $\varepsilon \neq 0$ and $H \neq 0$ are of most interest. The $[2, 2]$ homotopy-Padé approximation of the wave phase speed reads

$$c^2/c_0^2 = \frac{\sum_{i=0}^8 \alpha_i(H) \varepsilon^i}{1 + \sum_{j=0}^8 \beta_j(H) \varepsilon^j}, \quad (15.56)$$

where

$$\alpha_0 = 1 - 18.3545H^2 - 19.0783H^4 - 9.5071H^6 - 1.6706H^8$$

$$\begin{aligned}
& -0.1686H^{10} - 0.3944H^{12} + 0.1497H^{14}, \\
\alpha_1 = & -19.8095H^{-2} + 94.9902 + 79.8629H^2 + 49.2679H^4 + 16.4719H^6 \\
& + 1.3127H^8 + 0.7249H^{10} - 0.5257H^{12}, \\
\alpha_2 = & -0.8127H^{-4} - 153.8798H^{-2} - 17.4780 + 8.4334H^2 - 5.1748H^4 \\
& - 4.1199H^6 - 2.9438H^{10} + 0.7047H^{10}, \\
\alpha_3 = & 6.6032H^{-4} - 161.7258H^{-2} - 95.2033 - 20.2378H^2 + 3.0971H^4 \\
& + 2.7794H^6 - 1.2655H^8 + 0.8404H^{10}, \\
\alpha_4 = & 5.2780H^{-4} - 79.0493H^{-2} - 48.4005 - 17.5988H^2 - 5.1302H^4 \\
& + 0.4472H^6 + 0.4421H^8 + 0.0306H^{10}, \\
\alpha_5 = & 5.9728H^{-4} - 28.6492H^{-2} - 22.03866 - 3.7271H^2 + 3.03746H^4 \\
& + 2.7544H^6 + 0.9785H^8 - 0.1480H^{10}, \\
\alpha_6 = & 0.4561H^{-4} - 4.9730H^{-2} - 6.5921 - 4.9442H^2 - 2.6846H^4 \\
& - 0.9577H^6 - 0.2226H^8 + 0.0393H^{10}, \\
\alpha_7 = & -1.0606H^{-4} - 2.26127H^{-2} - 2.5945 - 1.6613H^2 - 0.6098H^4 \\
& - 0.1298H^6 - 0.01597H^8 + 0.0014H^{10}, \\
\alpha_8 = & 0.4849H^{-4} + 1.2882H^{-2} + 1.1771 + 0.5258H^2 + 0.1341H^4 \\
& + 0.0232H^6 + 0.0034H^8 + 0.0004H^{10},
\end{aligned}$$

and

$$\begin{aligned}
\beta_0 = & -18.6152H^2 - 14.3027H^4 - 4.7494H^6 + 0.3693H^8 \\
& - 0.1756H^{10} - 0.6505H^{12} - 0.1882H^{14}, \\
\beta_1 = & -19.8095H^{-2} + 101.3902 + 30.9370H^2 + 5.9367H^4 - 1.9934H^6 \\
& + 0.5200H^8 + 3.2108H^{10} + 1.2336H^{12}, \\
\beta_2 = & -0.8127H^{-4} - 180.3940H^{-2} + 152.2704 + 118.3418H^2 + 26.6539H^4 \\
& - 5.0904H^6 - 9.6998H^8 - 3.3617H^{10}, \\
\beta_3 = & 5.5196H^{-4} - 364.0214H^{-2} - 98.3501 + 54.1467H^2 + 29.9964H^4 \\
& + 5.5684H^6 - 1.2260H^8 - 0.4686H^{10}, \\
\beta_4 = & 14.4435H^{-4} - 241.5237H^{-2} - 201.5196 - 46.7589H^2 + 12.6025H^4 \\
& + 11.9062H^6 + 2.9939H^8 + 0.1049H^{10}, \\
\beta_5 = & 10.2959H^{-4} - 54.8513H^{-2} - 80.1939 - 46.4665H^2 - 13.8617H^4 \\
& - 1.5235H^6 + 0.18215H^8 + 0.0065H^{10}, \\
\beta_6 = & 6.7965H^{-4} + 6.2383H^{-2} - 5.5078 - 9.7100H^2 - 5.4381H^4 \\
& - 1.4762H^6 - 0.17256H^8 - 0.0018H^{10}, \\
\beta_7 = & 1.28H^{-4} + 6.4458H^{-2} + 7.2395 + 3.6241H^2 + 0.9258H^4 \\
& + 0.1129H^6 + 0.0041H^8 - 0.0002H^{10}, \\
\beta_8 = & -1.2567H^{-4} - 1.6932H^{-2} - 0.9140 - 0.2487H^2 - 0.0366H^4
\end{aligned}$$

$$-0.0037H^6 - 0.0006H^8.$$

Fig. 15.5 The influence of the vorticity parameter ε on the phase speed $k c^2/g$ of deep-water waves when $H = 0.1$. Solid line: $[2, 2]$ homotopy-Padé approximation; Filled circles: $[8, 8]$ homotopy-Padé approximation; Open circle: the 5th-order perturbation solution (Wang et al., 1997).

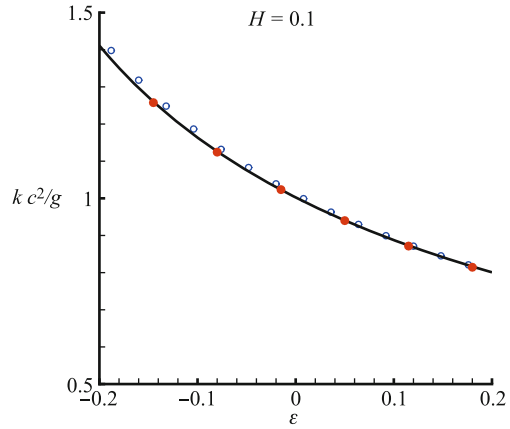
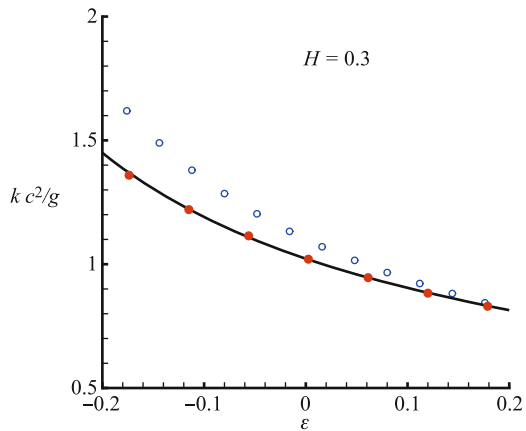


Fig. 15.6 The influence of the vorticity parameter ε on the phase speed $k c^2/g$ of deep-water waves when $H = 0.3$. Solid line: $[2, 2]$ homotopy-Padé approximation; Filled circles: $[8, 8]$ homotopy-Padé approximation; Open circle: the 5th-order perturbation solution (Wang et al., 1997).



As shown in Fig. 15.5 to Fig. 15.7, the above $[2, 2]$ homotopy-Padé approximation is accurate enough even for large wave-height H , which indicates not only the accuracy of the general expression (15.56) but also the convergence of the homotopy-series solution for waves with large amplitude. Note that the fifth-order perturbation results (Wang et al., 1997) are valid only for waves with small amplitude (such as $H = 0.1$) and become more and more inaccurate when the wave-height H increases, as shown in Fig. 15.5 to Fig. 15.7. Thus, for large wave-height ($H \geq 0.3$), perturbation approximations often overestimate the wave phase speed for both aiding and opposing currents.

Given ε , one can get the dispersion relationship in a similar way by means of homotopy-Padé technique. The curves $\mu = k c^2/g$ versus the wave-height H in case

Fig. 15.7 The influence of the vorticity parameter ε on the phase speed $k c^2/g$ of deep-water waves when $H = 0.5$. Solid line: $[2, 2]$ homotopy-Padé approximation; Filled circles: $[8, 8]$ homotopy-Padé approximation; Open circle: the 5th-order perturbation solution (Wang et al., 1997).

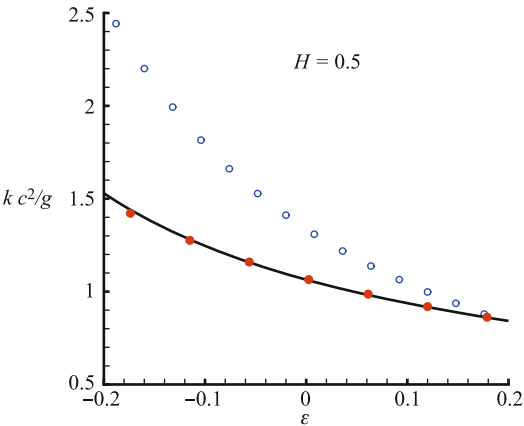


Fig. 15.8 The dispersion relationship of waves on a current with an exponential distribution of vorticity $\Omega = \varepsilon \exp(-\psi)$ in case of $\varepsilon = -0.25, -0.15, 0, 0.25, 0.5$. Solid line: $[12, 12]$ homotopy-Padé approximation; Symbols: $[8, 8]$ homotopy-Padé approximation.

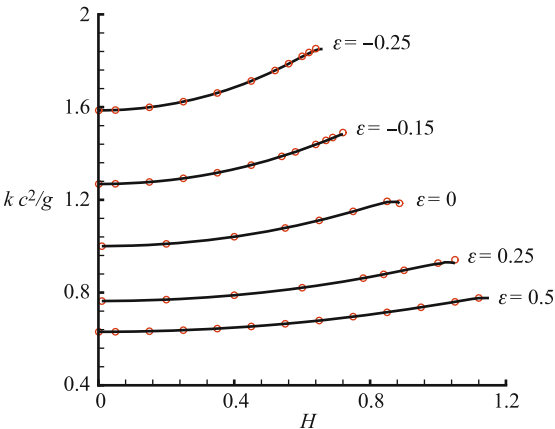


Table 15.7 Maximum wave-height of propagating waves on a current with the exponential distribution of vorticity $\Omega = \varepsilon \exp(-\psi)$.

ε	Maximum wave-height
-0.25	0.67
0	0.8825
0.25	1.04
0.50	1.17

of $\varepsilon = -0.25, -0.15, 0, 0.25$ and 0.5 are as shown in Fig. 15.8. Note that the $[8, 8]$ and $[12, 12]$ homotopy-Padé approximations agree quite well, except at a few points very close to the highest wave. In comparison with pure waves ($\varepsilon = 0$), the aiding exponential shear currents ($\varepsilon < 0$) tend to enlarge the wave phase velocity for given wave height H , but the opposing exponential shear currents ($\varepsilon > 0$) tend to shorten it. Especially, as shown in Table 15.7 and Fig. 15.9, the maximum wave-heights are strongly dependent upon the vorticity of the currents: the aiding exponential

Fig. 15.9 Maximum wave-height versus ε in case of shear currents with the exponential vorticity $\Omega = \varepsilon \exp(-\psi)$.

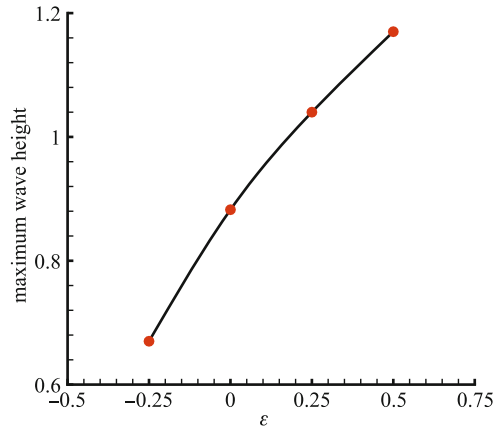
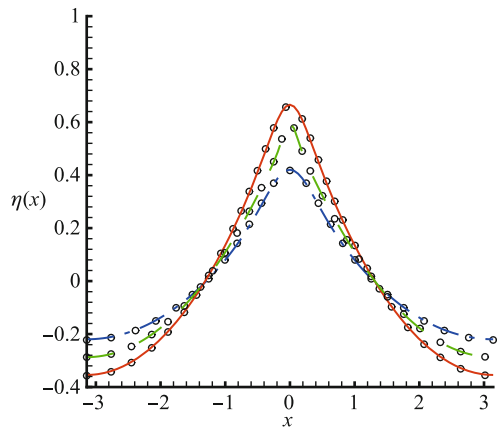
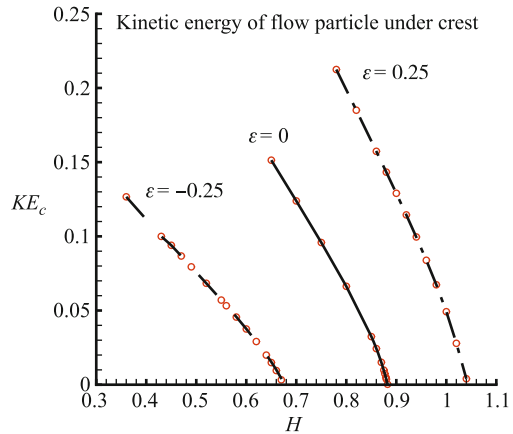


Fig. 15.10 Approximations of wave elevation by means of $c_0 = -1/2$. Solid line: 25th-order approximation when $H = 1.02$ and $\varepsilon = 0.25$; Dashed line: 45th-order approximation when $H = 0.8825$ and $\varepsilon = 0$; Dash-dotted line: 25th-order approximation when $H = 0.64$ and $\varepsilon = -0.25$; Circles: the 20th-order homotopy-approximations.



shear currents ($\varepsilon < 0$) tend to shorten the maximum wave-height, but the opposing exponential shear currents ($\varepsilon > 0$) tend to enlarge it. Note that, the maximum wave-height of the pure propagating waves without currents is about $H = 0.8825$. However, in case of the opposing exponential shear currents ($\varepsilon > 0$), the wave-height can be greater than the limiting value for pure waves, as shown in Table 15.7 and Fig. 15.9. For example, in case of the opposing exponential currents with $0.25 \leq \varepsilon \leq 0.5$, we obtain convergent series solutions when $H = 0.92$, which is even higher than $H = 0.886$ given by Stokes theory for pure waves on still water (see Marshall et al. (1992)). It should be emphasized that, according to Stokes theory, the propagating wave with $H = 0.92$ can *not* exist for pure waves on still water ($\varepsilon = 0$). However, in case of the opposing shear current with $\varepsilon = 0.25$, we obtain the convergent series solution of waves even with $H = 1.02$, as shown in Fig. 15.10. Thus, based on the homotopy-approximations, the maximum wave-height of propagating water waves on an opposing shear current ($\varepsilon > 0$) can be larger than that of pure waves on still water. Note that the wave shape in case of $H = 1.02$ and $\varepsilon = 0.25$ is steeper even

Fig. 15.11 The kinetic energy KE_c of flow particle at crest versus the wave height H . Filled circles: [11, 11] homotopy-Padé approximation; Solid line: [12, 12] homotopy-Padé approximation; Open circles: [15, 15] homotopy-Padé approximation; Dashed line: [20, 20] homotopy-Padé approximation.



than that of waves on still water with $H = 0.8825$, as shown in Fig. 15.10. For more physical discussions, please refer to Cheng et al. (2009).

As suggested by Stokes (1880, 1894), a train of propagating waves on still water breaks when the fluid velocity at crest is equal to the wave phase speed. To check this criterion for waves on a shear current, we also calculate the kinetic energy of the flow particle at crest in the reference frame moving with speed c , defined by

$$KE_c = \frac{1}{2} \mathbf{u}^2 = \frac{1}{\mu} [\gamma - y(0, 0)]. \quad (15.57)$$

The curves of KE_c versus the wave-height H in case of the aiding current ($\varepsilon = -0.25$), still water ($\varepsilon = 0$) and the opposing current ($\varepsilon = 0.25$) are as shown in Fig. 15.11. Note that $KE_c \approx 0$ when $H \approx 0.8825$ for the pure waves on still water without currents ($\varepsilon = 0$), which exactly corresponds to the highest wave given by Schwartz (1974) and Longuet-Higgins and Tanaka (1997). However, we have $KE_c \approx 0$ when $H \approx 0.67$ for waves on an aiding current ($\varepsilon = -0.25$), and when $H \approx 1.04$ for waves on an opposing current ($\varepsilon = 0.25$), respectively. So, it seems that the highest-wave corresponds to $KE_c = 0$, i.e. the fluid velocity at crest equals the wave phase speed. Note that the three curves of $KE_c \sim H$ in case of $\varepsilon = \pm 0.25$ and $\varepsilon = 0$ are parallel approximately. So, qualitatively speaking, an aiding shear current tends to shorten the maximum wave height, but an opposing shear current has an opposite effect. Therefore, according to the approximations given by the HAM, Stokes' criterion of wave breaking has general meanings and is still correct even for waves on a non-uniform current, i.e. a train of propagating waves on a shear current breaks as the fluid velocity at crest equals the wave phase speed. For more details, please refer to Cheng et al. (2009).

This is a good example to show that the HAM can be used as a helpful tool to solve some complicated nonlinear PDEs, so as to deepen and enrich our physical understandings about some complicated nonlinear phenomenon.

15.5 Concluding remarks

The HAM is applied to investigate the nonlinear interaction of the periodic traveling waves on a non-uniform current with exponential distribution of vorticity. By properly choosing a simple auxiliary linear operator, the original highly nonlinear PDE with variable coefficient is transferred into an infinite number of much simpler linear PDEs, which are rather easy to solve. The convergent series solutions are gained by means of optimal convergence-control parameter c_0 and homotopy-Padé technique so that the highest wave can be obtained for the given non-uniform currents. These analytic approximations reveal the basic characteristic of the nonlinear interaction between the periodic traveling waves and the non-uniform currents. They illustrate that the HAM can be used as an useful analytic tool to deepen our physical understanding about some complicated nonlinear phenomena.

Mathematically, it should be emphasized that the two auxiliary linear operators (15.37) and (15.38) have nearly no relationships with the original PDE (15.9) and the nonlinear boundary condition (15.10). Note that other analytic techniques, especially perturbation techniques, are strongly dependent upon linear operators in original governing equations. Unlike other analytic techniques, the HAM provides us extremely large freedom to choose the auxiliary linear operators according to the physical background of a given nonlinear problem. So, in the frame of the HAM, we need not spend too much time in the details of governing equations and boundary conditions of a given nonlinear problem, but should pay more attention on its physical background, so as to find out a proper solution expression. This is in essence quite different from other analytic techniques. Note that, it is the HAM that provides us such kind of extremely large freedom to greatly simplify resolving some nonlinear problems, as shown in this and previous chapters. Besides, as mentioned in previous chapters, such kind of extremely large freedom is based on the guarantee of the convergent series solution by means of the optimal convergence-control parameter c_0 : the freedom on the choice of the auxiliary linear operator has no meanings at all, if the corresponding series solution is divergent.

Physically, it is found that, for given wave amplitude, waves propagate faster on an aiding shear current but more slowly on an opposing one, compared with waves in still water. Besides, different from the uniform current, an aiding shear current tends to sharpen the crest but smoothen the trough, while an opposing shear current has the opposite effects. So, it is not the magnitude but the non-uniformity of the current that influences the wave shape. Especially, it is found that Stokes' criterion of wave breaking is still correct even for propagating waves on a non-uniform current, and the highest wave on an opposing shear current is even higher and steeper than that of waves on still water.

All of these verify the validity of the HAM for some complicated nonlinear PDEs. This example suggests that the HAM can be used as a useful tool to deepen our physical understandings about some complicated nonlinear phenomena.

Appendix 15.1 Mathematica code of wave-current interaction

The interaction of 2D nonlinear progress waves and non-uniform currents are solved by means of the HAM. This Mathematica code is free available at

<http://numericaltank.sjtu.edu.cn/HAM.htm>

Mathematica code of wave-current interaction

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August 2010

```
(*****
(*      Interaction of wave & nonuniform currents      *)
(* Governing equation:                                *)
(*       $y_{xx}(y,z)^2 - 2y_{xy}y_{xz}(y,z) + (1+y_x^2)y_{zz} = \omega$ , for  $z > 0$  *)
(* Boundary condition                                  *)
(*       $\mu(1+y_x^2) + 2y(y,z)^2 - 2kk(y,z)^2 == 0$ , for  $z = 0$  *)
(*       $y(x,z) \rightarrow 0$ , as  $z \rightarrow \text{Infinity}$  *)
(*       $H = y(0,0) - y(\pi,0)$  *)
(*****)

(*****
(* Define chi_[k] *)
(*****
chi[k_] := If[k<=1,0,1];

(*****
(* Define initial guess *)
(*****
y[0] = -z + H/2 Exp[-z] * (Exp[I*x]+Exp[-I*x])/2;

(*****
(* Define GetfAll[k] *)
(*****
GetAll[k_] := Module[{ },
yz[k]      = Expand[D[y[k],z]];
yzz[k]     = Expand[D[yz[k],z]];
yx[k]      = Expand[D[y[k],x]];
yxx[k]     = Expand[D[yx[k],x]];
ygamma[k]  = 2 (y[k] - gamma[k]) / Expand;
yxz[k]     = Expand[D[yx[k],z]];
yxyz[k]    = Expand[Sum[yx[i]*yz[k-i],{i,0,k}]];
yzyyz[k]   = Expand[Sum[yzyz[i]*yz[k-i],{i,0,k}]];
yzyz[k]    = Expand[Sum[yz[i]*yz[k-i],{i,0,k}]];
yxyxz[k]   = Expand[Sum[yx[i]*yxz[k-i],{i,0,k}]];
yxyx[k]    = Expand[Sum[yx[i]*yx[k-i],{i,0,k}]];
];
```

```

(*****
(* Define GetRHS[k] *)
(*****
GetRHS[k_] := Module[{temp, i, j, n},
temp[1] = Sum[yxx[k-1-i]*yzyz[i], {i, 0, k-1}];
temp[2] = Sum[yxyz[k-1-i]*yz[i], {i, 0, k-1}];
temp[3] = Sum[yzz[k-1-i]*yx[i], {i, 0, k-1}];
temp[4] = yzyzyz[k-1]*Omega;
delta[k] = Expand[temp[1]-2*temp[2]+yzz[k-1]+temp[3]-temp[4]];
RHS[k] = Expand[c0*delta[k]];
];

(*****
(* Define GetRHSb[k] *)
(*****
GetRHSb[k_] := Module[{temp, i, j, n},
temp[1] = Sum[mu[k-1-i]*yxyz[i], {i, 0, k-1}];
temp[2] = Sum[(gamma[k-1-i])*yzyz[i], {i, 0, k-1}];
temp[3] = mu[k-1];
deltaB[k] = Expand[temp[1] + temp[2] + temp[3]];
RHSb[k] = Expand[c0*deltaB[k]];
];

(*****
(* Define the the auxiliary linear operator L and Lb *)
(*****
L[f_] := Expand[D[f, {z, 2}] + D[f, {x, 2}]];
Lb[f_] := Expand[D[f, z] + f];

(*****
(* Define inverse operator of auxiliary linear operator Linv *)
(*****
Linv[Exp[n_.z]*Exp[m_.x]] := 1/(n^2+m^2)*Exp[n*z]*Exp[m*x];
Linv[Exp[m_.x]] := Exp[m*x]/m^2;
Linv[Exp[n_.z]] := Exp[n*z]/n^2;

(*****
(* The property of the inverse operator Linv *)
(*****
Linv[p_Plus] := Map[Linv, p];
Linv[c_*f_] := c*Linv[f] /; FreeQ[c, z] && FreeQ[c, x];

(*****
(* Define GetySpecial[k] *)
(*****
GetySpecial[k_] := Module[{temp},
temp[0] = Expand[RHS[k]];
ySpecial = Linv[temp[0]] //Expand;
];

(*****
(* Define two functions for common solution *)

```

```

(*****)
therest[f_,k_] := (-Lb[f] + RHSb[k])/ .z->0//ComplexExpand;
ccc[f_] := Select[f, FreeQ[#, x] &];

(*****)
(* Define Gety[k] *)
(*****)
Gety[k_] := Module[
{temp,sol,p0,p1,plist,clist,dlist,c1,property},
temp[0] = therest[ySpecial,k];
temp[1] = temp[0]//Expand;
p0 = ccc[temp[1]];
p1 = Coefficient[temp[1],Cos[x]];
temp[2] = temp[1] - p0 - p1*Cos[x];
sol = Solve[{p0 == 0,p1 == 0},{gamma[k-1],mu[k-1]}];
gamma[k-1] = gamma[k-1]/.sol[[1]]//Expand;
mu[k-1] = mu[k-1]/.sol[[1]]//Expand;
plist = Table[Coefficient[temp[2],Cos[i*x]],{i,2,2k+1}]
//Simplify;
clist = Table[plist[[i-1]]/(1-i),{i,2,2k+1}];
dlist = Table[clist[[i-1]]*(1-(-1)^i),{i,2,2k+1}];
temp[3] = Apply[Plus,dlist];
temp[4] = ySpecial /.{z->0,x->0 };
temp[5] = ySpecial /.{z->0,x->Pi};
temp[6] = Table[Exp[-i*z]*(Exp[i*I*x]+Exp[-i*I*x])/2,
{i,2,2k+1}];
c1 = -1/2*(temp[4]-temp[5]+temp[3]);
property = Dot[temp[6],clist]
+ c1*Exp[-z]*(Exp[I*x] + Exp[-I*x])/2;
y[k] = Expand[ySpecial + chi[k]*y[k-1] + property];
];

(*****)
(* Define GetYs[k] *)
(*****)
GetYs[k_] := Module[{temp},
temp[0] = Y[k] /. z->0;
temp[1] = Integrate[temp[0], {x,-Pi,Pi}]/2/Pi;
Ys[k] = temp[0] - temp[1]//Expand;
];

(*****)
(* Define GetErrB[k] *)
(* Gain squared residual of boundary condition *)
(*****)
GetErrB[k_] := Module[{temp,i,Yx,Yz,Nx,Np,dx,xx},
Yx = D[Y[k],x];
Yz = D[Y[k],z];
temp[1] = MU[k]*(1+Yx^2) + 2*(Y[k]-GAMMA[k])*Yz^2 /. z->0;
temp[2] = temp[1]^2;
Nx = 20;
dx = N[Pi/Nx,100];
sum = 0;
Np = 0;

```

```

For[i = 0, i <= Nx, i++,
  xx = i*dx;
  sum = sum + temp[2] /. x->xx;
  Np = Np + 1;
];
ErrB[k] = sum/Np;
If[NumberQ[ErrB[k]],
  ErrB[k] = Re[ErrB[k]];
  Print["k = ",k," Squared Residual of B.C. = ",ErrB[k]/N]
];

] ;

(*****
(* Define GetErr[k] *)
(* Gain squared residual of governing equation *)
(*****
GetErr[k_] := Module[
{temp,i,j,sum,xx,zz,Yx,Yz,Yxx,Yxz,Yzz,Nx,Nz,Np},
Yx = D[Y[k],x];
Yz = D[Y[k],z];
Yxx = D[Yx,x];
Yxz = D[Yx,z];
Yzz = D[Yz,z];
temp[1] = Yxx*Yz^2 - 2*Yx*Yz*Yxz + (1+Yx^2)*Yzz - Yz^3*Omega;
temp[2] = temp[1]^2;
Nx = 10;
Nz = 10;
dx = N[Pi/Nx,100];
dz = N[2*Pi/Nz,100];
sum = 0;
Np = 0;
For[i = 0, i <= Nx, i++,
  For[j = 0, j <= Nz, j++,
    xx = i*dx;
    zz = j*dz;
    sum = sum + temp[2] /. {x->xx,z->zz};
    Np = Np + 1;
  ];
];
Err[k] = sum/Np;
If[NumberQ[Err[k]],
  Err[k] = Re[Err[k]];
  Print["k = ",k," Squared Residual of G.E. = ",Err[k]/N]
];

] ;

(*****
(* Define hp[f_,m_,n_] *)
(* Gain [m,n] homotopy-Pade approximation *)
(*****
hp[f_,m_,n_] := Block[{k,i,df,res,q},
df[0] = f[0];
For[k = 1, k <= m+n, k++, df[k] = f[k] - f[k-1]//Expand];
res = df[0] + Sum[df[i]*q^i,{i,1,m+n}];

```

```

Pade[res, {q, 0, m, n}] /. q -> 1
];

(*****
(*                               Main Code                               *)
(*****
ham[m0_, m1_] := Module[{temp, k, n},
For[k = Max[1, m0], k <= m1, k = k + 1, Print[" k = ", k];
  GetAll[k - 1];
  GetRHS[k];
  GetRHSb[k];
  GetySpecial[k];
  Gety[k];
  Y[k] = Y[k - 1] + y[k] // ComplexExpand;
  If[k > 1,
    MU[k - 1] = MU[k - 2] + mu[k - 1] // Expand;
    GAMMA[k - 1] = GAMMA[k - 2] + gamma[k - 1] // Expand;
  ];
  Print[" mu = ", MU[k - 1] // N, " variation = ", mu[k - 1] // N];
  Print[" gamma = ", GAMMA[k - 1] // N, " variation = ",
    gamma[k - 1] // N];
];
Print[" Sucessful ! "];
];

(*****
(* Define initial guess u[0] and related functions *)
(*****
Y[0] = y[0];
GAMMA[0] = gamma[0];
MU[0] = mu[0];
ERR[0] = ComplexExpand[(Y[0] - GAMMA[0]) /. x -> 0 /. z -> 0];
Omega = epsilon * Exp[-z];

(* Physical and control parameters *)
epsilon = 1/5;
H = 3/10;
c0 = -11/20;

(* Print input data *)
Print["epsilon = ", epsilon];
Print[" H = ", H];
Print[" c0 = ", c0];

(* Gain the 10th-order approximation *)
ham[1, 11];

(* Gain squared residual of governing equation *)
For[k = 2, k <= 10, k = k + 2, GetErr[k]];

(* Gain squared residual of boundary condition *)
For[k = 2, k <= 10, k = k + 2, GetErrB[k]];

```

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Chapter 16

Resonance of Arbitrary Number of Periodic Traveling Water Waves

Abstract In this chapter, we verify the validity of the homotopy analysis method (HAM) for a rather complicated nonlinear PDE describing the nonlinear interaction of arbitrary number of traveling water waves. In the frame of the HAM, the wave-resonance criterion for arbitrary number of waves is gained, for the first time, which logically contains the famous Phillips' criterion for four small amplitude waves. Besides, it is found for the first time that, when the wave-resonance criterion is satisfied and the wave system is fully developed, there exist multiple steady-state resonant waves, whose amplitude might be much smaller than primary waves so that a resonant wave may contain much small percentage of the total wave energy. This example illustrates that the HAM can be used as a tool to deepen and enrich our understandings about some rather complicated nonlinear phenomena.

16.1 Introduction

In this chapter, we further illustrate that the homotopy analysis method (HAM) (Liao, 1992, 1997, 1999a,b; Liao and Campo, 2002; Liao, 2003a,b, 2004, 2005, 2006, 2009, 2010a,b, 2011; Liao and Tan, 2007) can be used to investigate some rather complicated nonlinear partial differential equations (PDEs) so as to deepen and enrich our understandings about some interesting nonlinear phenomena.

Let us consider here the nonlinear interaction of *arbitrary* number of periodic traveling gravity waves in deep water (Liao, 2011). Let z denote the vertical co-ordinate pointing upwards, x, y the horizontal co-ordinates, t the time, $z = \zeta(x, y, t)$ the free surface, respectively. The three axes of x, y, z are perpendicular to each other, with the unit vector $\mathbf{i}, \mathbf{j}, \mathbf{k}$, respectively, i.e. $\mathbf{i} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{k} = 0$, where $\cdot$ is the multiplication dot. Assume that the vorticity is negligible and there exists a potential $\varphi(x, y, z, t)$ that $\mathbf{u} = \nabla \varphi$ and

$$\nabla^2 \varphi = 0, \quad z \leq \zeta(x, y, t), \quad (16.1)$$

where

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}.$$

On the free surface $z = \zeta(x, y, t)$, the pressure is constant, which gives from Bernoulli's equation the dynamic boundary condition

$$g \zeta + \frac{\partial \varphi}{\partial t} + \frac{1}{2} \mathbf{u}^2 = 0, \quad \text{on } z = \zeta(x, y, t), \quad (16.2)$$

where g is the acceleration due to gravity. Besides, $z - \zeta$ vanishes following a particle, which gives the kinematic boundary condition

$$\frac{\partial \zeta}{\partial t} - \frac{\partial \varphi}{\partial z} + \left(\frac{\partial \varphi}{\partial x} \frac{\partial \zeta}{\partial x} + \frac{\partial \varphi}{\partial y} \frac{\partial \zeta}{\partial y} \right) = 0, \quad \text{on } z = \zeta(x, y, t). \quad (16.3)$$

Combining the above two equations gives the boundary condition

$$\frac{\partial^2 \varphi}{\partial t^2} + g \frac{\partial \varphi}{\partial z} + \frac{\partial(\mathbf{u}^2)}{\partial t} + \mathbf{u} \cdot \nabla \left(\frac{1}{2} \mathbf{u}^2 \right) = 0, \quad \text{on } z = \zeta(x, y, t), \quad (16.4)$$

where $\mathbf{u} = \nabla \varphi$ and $\mathbf{u}^2 = \nabla \varphi \cdot \nabla \varphi$. On the bottom, it holds

$$\frac{\partial \varphi}{\partial z} = 0, \quad \text{as } z \rightarrow -\infty. \quad (16.5)$$

For details, please refer to Phillips (1960) and Longuet-Higgins (1962).

In his pioneering work about nonlinear interaction of *four* gravity waves in deep water, Phillips (1960) gave the criterion of wave resonance

$$\mathbf{k}_1 \pm \mathbf{k}_2 \pm \mathbf{k}_3 \pm \mathbf{k}_4 = 0, \quad \sigma_1 \pm \sigma_2 \pm \sigma_3 \pm \sigma_4 = 0, \quad (16.6)$$

where $\sigma_i = \sqrt{gk_i}$ with $k_i = |\mathbf{k}_i|$ ($i = 1, 2, 3, 4$) is the wave angular frequency, and $\mathbf{k}_i$ denotes the wave number. It should be emphasized that Phillips' resonance criterion (16.6) works only for weakly nonlinear waves with small amplitudes, because $\sigma_i = \sqrt{gk_i}$ is the angular frequency of linear theory for a single gravity wave with small amplitude.

In a special case of $\mathbf{k}_4 = \mathbf{k}_1$, Phillips (1960) showed by means of perturbation methods that, if $\mathbf{k}_3 = 2\mathbf{k}_1 - \mathbf{k}_2$ and $\sigma_3 = 2\sigma_1 - \sigma_2$, then a steady-state solution for the triad did not exist, and the amplitude of the third wave, if initially zero, would grow linearly in time. This conclusion was confirmed by Longuet-Higgins (1962) via perturbation theory and supported by some experiments (Longuet-Higgins and Smith, 1966; McGoldrick et al., 1966). Besides, Benney (1962) solved the equations governing the time dependence of the resonant modes and studied the energy-sharing mechanism involved.

Although half century passed since Phillips' pioneering work (Phillips, 1960), there exist still some open questions about nonlinear interaction between gravity waves. First, Bretherton (1964) pointed out that the perturbation scheme used by

Phillips breaks down for large time, and then suggested by investigating a one-dimensional dispersive wave model that the amplitude of each wave component should be bounded. Besides, Phillips' wave resonance criterion (16.6) is valid for only *four* waves with *small* wave-amplitudes. What is the wave-resonance criterion for *arbitrary* number of waves with *large* amplitude? It seems difficult to apply the perturbation scheme used by Phillips (1960) and Longuet-Higgins (1962) to answer these questions, because the related algebra was daunting and "extraordinarily tedious", as mentioned by Phillips (1981).

In 2011, Liao (2011) successfully applied the HAM to give a wave-resonance criterion for *arbitrary* number of traveling waves by means of the so-called homotopy multiple-variable method, which keeps the clear physical meaning of the multiple-scales of perturbation techniques but completely abandons the small physical parameters. In this chapter, the nonlinear interaction of periodic traveling water waves is used as an example to describe the basic ideas of the homotopy multiple-variable method. By means of this method, the wave resonance criterion for arbitrary number of traveling waves with large amplitude is gained, which logically contains Phillips' criterion for four small-amplitude waves. The detailed mathematical derivation for the high-order deformation equations is given in the Appendix 16.1. For more details, please refer to Liao (2011).

16.2 Resonance criterion of two small-amplitude primary waves

16.2.1 Brief Mathematical formulas

Without loss of generality, let us first consider a fully-developed gravity wave system basically composed of two primary traveling waves in deep water with wave numbers $\mathbf{k}_1, \mathbf{k}_2$ and the corresponding angular frequencies σ_1, σ_2 , respectively, where $\mathbf{k}_1 \times \mathbf{k}_2 \neq 0$ (i.e. the two traveling waves are not collinear). Due to nonlinear interaction, this wave system contains an infinite number of wave components with the corresponding wave number $m\mathbf{k}_1 + n\mathbf{k}_2$, where m, n are integers. Since the wave system is assumed to be fully developed, each wave amplitude is independent of the time. Let α_1, α_2 denote the angles between the positive x -axis $\mathbf{i}$ and the wave number vectors $\mathbf{k}_1$ and $\mathbf{k}_2$, respectively, where $\mathbf{k}_1 \cdot \mathbf{k} = \mathbf{k}_2 \cdot \mathbf{k} = 0$, i.e. the z -axis is perpendicular to the wave numbers $\mathbf{k}_1, \mathbf{k}_2$. Then,

$$\mathbf{k}_1 = k_1 (\cos \alpha_1 \mathbf{i} + \sin \alpha_1 \mathbf{j}), \quad \mathbf{k}_2 = k_2 (\cos \alpha_2 \mathbf{i} + \sin \alpha_2 \mathbf{j}), \quad (16.7)$$

where $k_1 = |\mathbf{k}_1|$ and $k_2 = |\mathbf{k}_2|$.

Write $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$. According to the linear gravity wave theory, the two primary waves traveling with the wave numbers $\mathbf{k}_1$ and $\mathbf{k}_2$ are given by $a_1 \cos \xi_1, a_2 \cos \xi_2$, respectively, where

$$\xi_1 = \mathbf{k}_1 \cdot \mathbf{r} - \sigma_1 t, \quad \xi_2 = \mathbf{k}_2 \cdot \mathbf{r} - \sigma_2 t. \quad (16.8)$$

So, the above two variables have very clear physical meaning: the wave profile must be a periodic function of ξ_1 and ξ_2 . Mathematically, using these two variables, the time t should not appear *explicitly* for a fully developed wave system. In other words, one can express the potential function $\varphi(x, y, z, t) = \phi(\xi_1, \xi_2, z)$ and the wave surface $\zeta(x, y, t) = \eta(\xi_1, \xi_2)$, respectively, for the nonlinear interaction of a fully developed wave system composed of the two primary traveling waves.

Then,

$$\mathbf{u} = \nabla\varphi = \mathbf{i}\frac{\partial\varphi}{\partial x} + \mathbf{j}\frac{\partial\varphi}{\partial y} + \mathbf{k}\frac{\partial\varphi}{\partial z} = \mathbf{k}_1\frac{\partial\phi}{\partial\xi_1} + \mathbf{k}_2\frac{\partial\phi}{\partial\xi_2} + \mathbf{k}\frac{\partial\phi}{\partial z} = \hat{\nabla}\phi, \quad (16.9)$$

where

$$\hat{\nabla} = \mathbf{k}_1\frac{\partial}{\partial\xi_1} + \mathbf{k}_2\frac{\partial}{\partial\xi_2} + \mathbf{k}\frac{\partial}{\partial z}. \quad (16.10)$$

Thus,

$$\begin{aligned} \mathbf{u}^2 &= \nabla\varphi \cdot \nabla\varphi = \hat{\nabla}\phi \cdot \hat{\nabla}\phi \\ &= k_1^2 \left(\frac{\partial\phi}{\partial\xi_1} \right)^2 + 2\mathbf{k}_1 \cdot \mathbf{k}_2 \frac{\partial\phi}{\partial\xi_1} \frac{\partial\phi}{\partial\xi_2} + k_2^2 \left(\frac{\partial\phi}{\partial\xi_2} \right)^2 + \left(\frac{\partial\phi}{\partial z} \right)^2, \end{aligned} \quad (16.11)$$

where $\mathbf{k}_1 \cdot \mathbf{k} = \mathbf{k}_2 \cdot \mathbf{k} = 0$ is used, and $\mathbf{k}_1 \cdot \mathbf{k}_2 = k_1 k_2 \cos(\alpha_1 - \alpha_2)$. In general, it holds

$$\begin{aligned} \hat{\nabla}\phi \cdot \hat{\nabla}\psi &= k_1^2 \frac{\partial\phi}{\partial\xi_1} \frac{\partial\psi}{\partial\xi_1} + \mathbf{k}_1 \cdot \mathbf{k}_2 \left(\frac{\partial\phi}{\partial\xi_1} \frac{\partial\psi}{\partial\xi_2} + \frac{\partial\psi}{\partial\xi_1} \frac{\partial\phi}{\partial\xi_2} \right) \\ &\quad + k_2^2 \frac{\partial\phi}{\partial\xi_2} \frac{\partial\psi}{\partial\xi_2} + \frac{\partial\phi}{\partial z} \frac{\partial\psi}{\partial z} \end{aligned} \quad (16.12)$$

for arbitrary functions $\phi(\xi_1, \xi_2, z)$ and $\psi(\xi_1, \xi_2, z)$.

Similarly, we have

$$\nabla^2\varphi = \hat{\nabla}^2\phi = k_1^2 \frac{\partial^2\phi}{\partial\xi_1^2} + 2\mathbf{k}_1 \cdot \mathbf{k}_2 \frac{\partial^2\phi}{\partial\xi_1\partial\xi_2} + k_2^2 \frac{\partial^2\phi}{\partial\xi_2^2} + \frac{\partial^2\phi}{\partial z^2}. \quad (16.13)$$

Then, the original governing equation reads

$$\hat{\nabla}^2\phi = k_1^2 \frac{\partial^2\phi}{\partial\xi_1^2} + 2\mathbf{k}_1 \cdot \mathbf{k}_2 \frac{\partial^2\phi}{\partial\xi_1\partial\xi_2} + k_2^2 \frac{\partial^2\phi}{\partial\xi_2^2} + \frac{\partial^2\phi}{\partial z^2} = 0, \quad (16.14)$$

which has the general solution

$$\phi = [A \cos(m\xi_1 + n\xi_2) + B \sin(m\xi_1 + n\xi_2)] e^{|\mathbf{m}\mathbf{k}_1 + \mathbf{n}\mathbf{k}_2|z}, \quad (16.15)$$

where m, n are integers, A, B are integral constants, and $-\infty < z \leq \eta(\xi_1, \xi_2)$, respectively.

For the sake of simplicity, define

$$\begin{aligned}
 f &= \frac{1}{2} \hat{\nabla} \phi \cdot \hat{\nabla} \phi = \frac{\mathbf{u}^2}{2} \\
 &= \frac{1}{2} \left[k_1^2 \left(\frac{\partial \phi}{\partial \xi_1} \right)^2 + 2 \mathbf{k}_1 \cdot \mathbf{k}_2 \frac{\partial \phi}{\partial \xi_1} \frac{\partial \phi}{\partial \xi_2} + k_2^2 \left(\frac{\partial \phi}{\partial \xi_2} \right)^2 + \left(\frac{\partial \phi}{\partial z} \right)^2 \right]. \quad (16.16)
 \end{aligned}$$

Using the new variables ξ_1 and ξ_2 , the dynamic boundary condition (16.2) becomes

$$\eta = \frac{1}{g} \left(\sigma_1 \frac{\partial \phi}{\partial \xi_1} + \sigma_2 \frac{\partial \phi}{\partial \xi_2} - f \right), \quad \text{on } z = \eta(\xi_1, \xi_2). \quad (16.17)$$

On the free surface $z = \eta(\xi_1, \xi_2)$, the kinematic boundary condition (16.4) reads

$$\begin{aligned}
 &\sigma_1^2 \frac{\partial^2 \phi}{\partial \xi_1^2} + 2 \sigma_1 \sigma_2 \frac{\partial^2 \phi}{\partial \xi_1 \partial \xi_2} + \sigma_2^2 \frac{\partial^2 \phi}{\partial \xi_2^2} + g \frac{\partial \phi}{\partial z} \\
 &- 2 \left(\sigma_1 \frac{\partial f}{\partial \xi_1} + \sigma_2 \frac{\partial f}{\partial \xi_2} \right) + \hat{\nabla} \phi \cdot \hat{\nabla} f = 0, \quad (16.18)
 \end{aligned}$$

where

$$\frac{\partial f}{\partial \xi_1} = \hat{\nabla} \phi \cdot \hat{\nabla} \left(\frac{\partial \phi}{\partial \xi_1} \right), \quad \frac{\partial f}{\partial \xi_2} = \hat{\nabla} \phi \cdot \hat{\nabla} \left(\frac{\partial \phi}{\partial \xi_2} \right) \quad (16.19)$$

and

$$\hat{\nabla} \phi \cdot \hat{\nabla} f = k_1^2 \frac{\partial \phi}{\partial \xi_1} \frac{\partial f}{\partial \xi_1} + \mathbf{k}_1 \cdot \mathbf{k}_2 \left(\frac{\partial \phi}{\partial \xi_1} \frac{\partial f}{\partial \xi_2} + \frac{\partial f}{\partial \xi_1} \frac{\partial \phi}{\partial \xi_2} \right) + k_2^2 \frac{\partial \phi}{\partial \xi_2} \frac{\partial f}{\partial \xi_2} + \frac{\partial \phi}{\partial z} \frac{\partial f}{\partial z}.$$

On the bottom, it holds

$$\frac{\partial \phi}{\partial z} = 0, \quad \text{as } z \rightarrow -\infty. \quad (16.20)$$

Given two physically reasonable angular frequencies σ_1 and σ_2 , our aim is to find out the corresponding unknown potential function $\phi(\xi_1, \xi_2, z)$ and the unknown free surface $\eta(\xi_1, \xi_2)$, which are governed by the linear PDE (16.14) subject to two *nonlinear* boundary conditions (16.17) and (16.18) on the *unknown* free surface $z = \eta(\xi_1, \xi_2)$, and one linear boundary condition (16.20) on the bottom. Here, it should be emphasized that, by means of the two independent variables ξ_1 and ξ_2 , the time t does not appear *explicitly* in the unknown potential function and wave surface. This greatly simplifies resolving the problem, as shown below.

As mentioned before, the two variables ξ_1 and ξ_2 have clear physical meanings and the solutions of this problem should be periodic functions of ξ_1 and ξ_2 . From physical points of view, the wave surface should be in the form

$$\eta(\xi_1, \xi_2) = \sum_{m=0}^{+\infty} \sum_{n=-\infty}^{+\infty} a_{m,n} \cos(m\xi_1 + n\xi_2), \quad (16.21)$$

where $a_{m,n}$ is the amplitude of the wave component $\cos(m\xi_1 + n\xi_2)$. Note that (16.15) is the general solution of the governing equation (16.14). So, the corre-

sponding potential function should be in the form

$$\phi(\xi_1, \xi_2, z) = \sum_{m=0}^{+\infty} \sum_{n=-\infty}^{+\infty} b_{m,n} \Psi_{m,n}(\xi_1, \xi_2, z) \quad (16.22)$$

where

$$\Psi_{m,n}(\xi_1, \xi_2, z) = \sin(m\xi_1 + n\xi_2) e^{i(m\mathbf{k}_1 + n\mathbf{k}_2)z}, \quad (16.23)$$

and $b_{m,n}$ is unknown coefficient independent of ξ_1, ξ_2, z . Note that the potential function $\phi(\xi_1, \xi_2, z)$ defined by (16.22) *automatically* satisfies the governing equation (16.14) and the bottom condition (16.20). The above expressions (16.21) and (16.22) are called the *solution-expressions* of η and ϕ , respectively, which play important roles in the frame of the HAM, as shown below.

For simplicity, define a nonlinear operator

$$\begin{aligned} \mathcal{N}[\phi(\xi_1, \xi_2, z)] = & \sigma_1^2 \frac{\partial^2 \phi}{\partial \xi_1^2} + 2\sigma_1 \sigma_2 \frac{\partial^2 \phi}{\partial \xi_1 \partial \xi_2} + \sigma_2^2 \frac{\partial^2 \phi}{\partial \xi_2^2} + g \frac{\partial \phi}{\partial z} \\ & - 2 \left(\sigma_1 \frac{\partial f}{\partial \xi_1} + \sigma_2 \frac{\partial f}{\partial \xi_2} \right) + \hat{\nabla} \phi \cdot \hat{\nabla} f, \end{aligned} \quad (16.24)$$

corresponding to the kinematic boundary condition (16.18), where the angular frequencies σ_1, σ_2 are given. Note that it contains a linear operator

$$\mathcal{L}_0(\phi) = \sigma_1^2 \frac{\partial^2 \phi}{\partial \xi_1^2} + 2\sigma_1 \sigma_2 \frac{\partial^2 \phi}{\partial \xi_1 \partial \xi_2} + \sigma_2^2 \frac{\partial^2 \phi}{\partial \xi_2^2} + g \frac{\partial \phi}{\partial z}. \quad (16.25)$$

Let $\mathcal{L}$ denote an auxiliary linear operator with the property $\mathcal{L}(0) = 0$. As mentioned in previous chapters, the HAM provides us the extremely large freedom to choose the auxiliary linear operator. Based on the results of the linear wave theory, i.e.

$$\sigma_1 \approx \sqrt{g k_1} = \bar{\sigma}_1, \quad \sigma_2 \approx \sqrt{g k_2} = \bar{\sigma}_2, \quad (16.26)$$

we choose such an auxiliary linear operator

$$\mathcal{L}(\phi) = \bar{\sigma}_1^2 \frac{\partial^2 \phi}{\partial \xi_1^2} + 2\bar{\sigma}_1 \bar{\sigma}_2 \frac{\partial^2 \phi}{\partial \xi_1 \partial \xi_2} + \bar{\sigma}_2^2 \frac{\partial^2 \phi}{\partial \xi_2^2} + g \frac{\partial \phi}{\partial z}. \quad (16.27)$$

Then, let $q \in [0, 1]$ denote the embedding parameter, $c_0 \neq 0$ the convergence-control parameter, $\phi_0(\xi_1, \xi_2, z)$ an initial guess of the potential function satisfying $\hat{\nabla}^2 \phi_0 = 0$ and the bottom condition (16.20), respectively. We construct the zeroth-order deformation equation

$$\hat{\nabla}^2 \check{\phi}(\xi_1, \xi_2, z; q) = 0, \quad -\infty < z \leq \check{\eta}(\xi_1, \xi_2; q), \quad (16.28)$$

subject to the two boundary conditions on $z = \check{\eta}(\xi_1, \xi_2; q)$:

$$(1 - q) \mathcal{L}[\check{\phi}(\xi_1, \xi_2, z; q) - \phi_0(\xi_1, \xi_2, z)] = q c_0 \mathcal{N}[\check{\phi}(\xi_1, \xi_2, z; q)], \quad (16.29)$$

and

$$(1-q)\check{\eta}(\xi_1, \xi_2; q) = q c_0 \left\{ \check{\eta}(\xi_1, \xi_2; q) - \frac{1}{g} \left[\sigma_1 \frac{\partial \check{\phi}(\xi_1, \xi_2, z; q)}{\partial \xi_1} + \sigma_2 \frac{\partial \check{\phi}(\xi_1, \xi_2, z; q)}{\partial \xi_2} - \check{f} \right] \right\}, \quad (16.30)$$

where

$$\check{f} = \frac{1}{2} \hat{\nabla} \check{\phi}(\xi_1, \xi_2, z; q) \cdot \hat{\nabla} \check{\phi}(\xi_1, \xi_2, z; q).$$

Besides, at the bottom, it holds

$$\frac{\partial \check{\phi}(\xi_1, \xi_2, z; q)}{\partial z} = 0, \quad \text{as } z \rightarrow -\infty. \quad (16.31)$$

When $q = 0$, since the initial guess $\phi_0(\xi_1, \xi_2, z)$ satisfies the governing equation (16.14) and the bottom condition (16.20), we have due to the property $\mathcal{L}(0) = 0$ of the auxiliary linear operator (16.27) that

$$\check{\phi}(\xi_1, \xi_2, z; 0) = \phi_0(\xi_1, \xi_2, z), \quad (16.32)$$

and

$$\check{\eta}(\xi_1, \xi_2; 0) = 0, \quad (16.33)$$

which provide us the initial approximations of the potential function $\phi(\xi_1, \xi_2, z)$ and the free surface $\eta(\xi_1, \xi_2)$. When $q = 1$, since $c_0 \neq 0$, (16.28) to (16.31) are equivalent to the original equations (16.14), (16.17), (16.18) and (16.20), respectively, provided

$$\check{\phi}(\xi_1, \xi_2, z; 1) = \phi(\xi_1, \xi_2, z), \quad \check{\eta}(\xi_1, \xi_2; 1) = \eta(\xi_1, \xi_2). \quad (16.34)$$

So, as $q \in [0, 1]$ increases from 0 to 1, $\check{\phi}(\xi_1, \xi_2, z; q)$ deforms (or varies) *continuously* from the initial approximation $\phi_0(\xi_1, \xi_2, z)$ to the unknown potential function $\phi(\xi_1, \xi_2, z)$, so does $\check{\eta}(\xi_1, \xi_2; q)$ from 0 to the unknown wave profile $\eta(\xi_1, \xi_2)$, respectively. Mathematically speaking, (16.28) to (16.31) define two homotopies:

$$\begin{aligned} \check{\phi}(\xi_1, \xi_2, z; q) &: \phi_0(\xi_1, \xi_2, z) \sim \phi(\xi_1, \xi_2, z), \\ \check{\eta}(\xi_1, \xi_2; q) &: 0 \sim \eta(\xi_1, \xi_2), \end{aligned}$$

Note that we have freedom to choose the convergence-control parameter c_0 . Assuming that c_0 is so properly chosen that the homotopy-Maclaurin series

$$\check{\phi}(\xi_1, \xi_2, z; q) = \phi_0(\xi_1, \xi_2, z) + \sum_{n=1}^{+\infty} \phi_n(\xi_1, \xi_2, z) q^n, \quad (16.35)$$

$$\check{\eta}(\xi_1, \xi_2; q) = \sum_{n=1}^{+\infty} \eta_n(\xi_1, \xi_2) q^n, \quad (16.36)$$

exist and absolutely converge at $q = 1$, we have due to (16.34) the homotopy-series solution

$$\phi(\xi_1, \xi_2, z) = \phi_0(\xi_1, \xi_2, z) + \sum_{n=1}^{+\infty} \phi_n(\xi_1, \xi_2, z), \quad (16.37)$$

$$\eta(\xi_1, \xi_2) = \sum_{n=1}^{+\infty} \eta_n(\xi_1, \xi_2), \quad (16.38)$$

where

$$\begin{aligned} \phi_n(\xi_1, \xi_2, z) &= \frac{1}{n!} \frac{\partial^n \check{\phi}(\xi_1, \xi_2, z; q)}{\partial q^n} \Big|_{q=0} = \mathcal{D}_n [\check{\phi}(\xi_1, \xi_2, z; q)], \\ \eta_n(\xi_1, \xi_2) &= \frac{1}{n!} \frac{\partial^n \check{\eta}(\xi_1, \xi_2; q)}{\partial q^n} \Big|_{q=0} = \mathcal{D}_n [\check{\eta}(\xi_1, \xi_2; q)], \end{aligned}$$

are the n th-order homotopy-derivatives, $\mathcal{D}_n$ is the n th-order homotopy-derivative operator. The m th-order homotopy-approximations read

$$\begin{aligned} \phi(\xi_1, \xi_2, z) &\approx \phi_0(\xi_1, \xi_2, z) + \sum_{n=1}^m \phi_n(\xi_1, \xi_2, z), \\ \eta(\xi_1, \xi_2) &\approx \sum_{n=1}^m \eta_n(\xi_1, \xi_2). \end{aligned}$$

The PDEs for the unknown $\phi_n(\xi_1, \xi_2, z)$ and $\eta_n(\xi_1, \xi_2)$ can be derived directly from the zeroth-order deformation equations (16.28) to (16.31). Substituting the homotopy-Maclaurin series (16.35) into the governing equation (16.28) and the boundary condition (16.31) at bottom, and equating the like-power of the embedding parameter q , we have

$$\hat{\mathbf{V}}^2 \phi_m(\xi_1, \xi_2, z) = 0, \quad (16.39)$$

subject to the boundary condition at bottom

$$\frac{\partial \phi_m(\xi_1, \xi_2, z)}{\partial z} = 0, \quad \text{as } z \rightarrow -\infty, \quad (16.40)$$

where $m \geq 1$. It should be emphasized that the two boundary conditions (16.29) and (16.30) are satisfied on the unknown boundary $z = \check{\eta}(\xi_1, \xi_2; q)$, which itself is dependent upon the embedding parameter q , too. So, it is relatively more complicated to deduce the corresponding equations. Briefly speaking, substituting the series (16.35) and (16.36) into the boundary condition (16.29) and (16.30) with $z = \check{\eta}(\xi_1, \xi_2; q)$, then equating the like-power of q , we have two linear boundary conditions on $z = 0$:

$$\bar{\mathcal{L}}(\phi_m) = c_0 \Delta_{m-1}^\phi + \chi_m S_{m-1} - \bar{S}_m, \quad m \geq 1, \quad (16.41)$$

and

$$\eta_m(\xi_1, \xi_2) = c_0 \Delta_{m-1}^\eta + \chi_m \eta_{m-1}, \quad (16.42)$$

where

$$\Delta_{m-1}^\eta = \eta_{m-1} - \frac{1}{g} \left[\left(\sigma_1 \bar{\phi}_{m-1}^{1,0} + \sigma_2 \bar{\phi}_{m-1}^{0,1} \right) - \Gamma_{m-1,0} \right] \quad (16.43)$$

and

$$\mathcal{L}(\phi_m) = \left(\bar{\sigma}_1^2 \frac{\partial^2 \phi_m}{\partial \xi_1^2} + 2 \bar{\sigma}_1 \bar{\sigma}_2 \frac{\partial^2 \phi_m}{\partial \xi_1 \partial \xi_2} + \bar{\sigma}_2^2 \frac{\partial^2 \phi_m}{\partial \xi_2^2} + g \frac{\partial \phi_m}{\partial z} \right) \Big|_{z=0}. \quad (16.44)$$

The detailed derivation of the above equations and the definitions of Δ_{m-1}^ϕ , S_{m-1} , $\bar{S}_m$, χ_m , $\Gamma_{m-1,0}$, $\bar{\phi}_{m-1}^{1,0}$, $\bar{\phi}_{m-1}^{0,1}$ are given in the Appendix 16.1. Note that, the sub-problems for ϕ_m and η_m are not only *linear* but also *decoupled*: given ϕ_{m-1} and η_{m-1} , it is straightforward to get η_m directly, and then ϕ_m is obtained by solving the linear Laplace equation (16.39) with two linear boundary conditions (16.40) and (16.41). Thus, the high-order deformation equations can be easily solved, especially by means of the computer algebra system such as Mathematica.

In a similar way as shown in Chapter 2, we can prove the following theorem:

Convergence Theorem *The homotopy-series solution (16.37) and (16.38) satisfy the original governing equation (16.14) and the boundary conditions (16.17), (16.18) and (16.20), provided that*

$$\sum_{m=0}^{+\infty} \Delta_m^\phi = 0, \quad \sum_{m=0}^{+\infty} \Delta_m^\eta = 0, \quad (16.45)$$

where $\Delta_m^\phi, \Delta_m^\eta$ are defined by (16.121) and (16.43), respectively.

Because the terms Δ_m^ϕ and Δ_m^η are by-products in solving the high-order deformation equations, the above theorem provides us a convenient way to check the convergence and accuracy of the homotopy-series solution. For this reason, we define the squared residuals

$$E_m^\phi = \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \left(\sum_{n=0}^m \Delta_n^\phi \right)^2 d\xi_1 d\xi_2, \quad (16.46)$$

$$E_m^\eta = \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \left(\sum_{n=0}^m \Delta_n^\eta \right)^2 d\xi_1 d\xi_2, \quad (16.47)$$

for the m th-order approximations of ϕ and η , respectively. According to the above-mentioned convergence theorem, if $E_m^\phi \rightarrow 0$ and $E_m^\eta \rightarrow 0$ as $m \rightarrow +\infty$, then the homotopy-series solution (16.37) and (16.38) satisfy the original equation and all

boundary conditions. Besides, the values of E_m^ϕ and E_m^η also indicate the accuracy of the m th-order homotopy-approximation of ϕ and η , respectively.

Note that the auxiliary linear operator (16.27) has the property

$$\mathcal{L}(\Psi_{m,n}) = [g|m\mathbf{k}_1 + n\mathbf{k}_2| - (m\bar{\sigma}_1 + n\bar{\sigma}_2)^2] \Psi_{m,n}, \quad (16.48)$$

where $\Psi_{m,n}$ is given by (16.23), which automatically satisfies the governing equation (16.39) and the bottom condition (16.40). Thus, mathematically speaking, the linear m th-order deformation equation has an infinite number of eigenfunctions $\Psi_{m,n}$ and the corresponding eigenvalue

$$\lambda_{m,n} = g|m\mathbf{k}_1 + n\mathbf{k}_2| - (m\bar{\sigma}_1 + n\bar{\sigma}_2)^2 \quad (16.49)$$

with the property

$$\mathcal{L}(\Psi_{m,n}) = \lambda_{m,n} \Psi_{m,n}.$$

Therefore, its inverse operator $\mathcal{L}^{-1}$ is defined by

$$\mathcal{L}^{-1}(\Psi_{m,n}) = \frac{\Psi_{m,n}}{\lambda_{m,n}}, \quad \lambda_{m,n} \neq 0. \quad (16.50)$$

Note that the inverse operator $\mathcal{L}^{-1}$ has definition *only* for non-zero eigenvalue $\lambda_{m,n} \neq 0$. When $\lambda_{m,n} = 0$, we have

$$g|m\mathbf{k}_1 + n\mathbf{k}_2| = (m\bar{\sigma}_1 + n\bar{\sigma}_2)^2, \quad (16.51)$$

which is exactly the criterion of the so-called “wave resonance” given by Phillips and Longuet-Higgins. Let N_λ denote the number of eigenfunctions whose eigenvalues are zero. As shown below, the value of N_λ is a key of this problem.

When $n = 0$ in (16.51), it holds

$$\lambda_{m,0} = g k_1 (|m| - m^2),$$

which equals to zero only when $|m| = 1$ (note that $m = n = 0$ corresponds to $\phi = 0$ and thus is not considered here). Similarly, when $m = 0$, the eigenvalue $\lambda_{0,n}$ equals to zero only when $|n| = 1$. So, there exist at least two eigenfunctions $\Psi_{1,0} = e^{k_1 z} \sin \xi_1$ and $\Psi_{0,1} = e^{k_2 z} \sin \xi_2$ whose eigenvalues are zero, i.e.

$$\mathcal{L}(C_1 e^{k_1 z} \sin \xi_1 + C_2 e^{k_2 z} \sin \xi_2) = 0 \quad (16.52)$$

for any constants C_1 and C_2 . Therefore, it holds $N_\lambda \geq 2$ in case of two primary waves. As mentioned by Phillips (1960) and Longuet-Higgins (1962), the criterion (16.51) of wave resonance can be satisfied for some special wave numbers and angular frequencies. Thus, when the criterion (16.51) is satisfied in case of $m = m'$ and $n = n'$, where m' and n' are integers with $m'^2 + n'^2 \neq 1$, there exist three eigenfunctions $\Psi_{1,0} = e^{k_1 z} \sin \xi_1$, $\Psi_{0,1} = e^{k_2 z} \sin \xi_2$ and

$$\Psi_{m',n'} = e^{|m'\mathbf{k}_1+n'\mathbf{k}_2|z} \sin(m'\xi_1 + n'\xi_2)$$

whose eigenvalues are zero, i.e.

$$\mathcal{L} \left[C_1 e^{k_1 z} \sin \xi_1 + C_2 e^{k_2 z} \sin \xi_2 + C_3 e^{|m'\mathbf{k}_1+n'\mathbf{k}_2|z} \sin(m'\xi_1 + n'\xi_2) \right] = 0 \quad (16.53)$$

for any constants C_1, C_2 and C_3 . Without loss of generality, Longuet-Higgins (1962) discussed a special case $m' = 2$ and $n' = -1$, corresponding to the eigenfunction

$$\Psi_{2,-1} = \exp(|2\mathbf{k}_1 - \mathbf{k}_2|z) \sin(2\xi_1 - \xi_2).$$

In this section, it implies $m' = 2$ and $n' = -1$ when there exist three zero eigenvalues ($N_\lambda = 3$) in case of two primary waves, if not mentioned.

According to the definitions (16.27) and (16.44), it holds

$$\bar{\mathcal{L}}(\phi) = \mathcal{L}(\phi)|_{z=0}.$$

Thus,

$$\bar{\mathcal{L}}(\Psi_{m,n}) = \lambda_{m,n} \Psi_{m,n}|_{z=0} = \lambda_{m,n} \sin(m\xi_1 + n\xi_2),$$

which gives the definition of the linear inverse operator

$$\bar{\mathcal{L}}^{-1}[\sin(m\xi_1 + n\xi_2)] = \frac{\Psi_{m,n}}{\lambda_{m,n}}, \quad \lambda_{m,n} \neq 0. \quad (16.54)$$

Using this inverse operator, it is easy to solve the linear Laplace equation (16.39) with two linear boundary conditions (16.40) and (16.41), as illustrated below. Here, we emphasize that the above inverse operator has definition only for non-zero eigenvalue $\lambda_{m,n} \neq 0$. For more details, please refer to Liao (2011).

16.2.2 Non-resonant waves

First, let us consider the case that there exist only two eigenfunctions

$$\Psi_{1,0} = \exp(k_1 z) \sin \xi_1, \quad \Psi_{0,1} = \exp(k_2 z) \sin \xi_2$$

whose eigenvalues are zero, i.e. $\lambda_{1,0} = \lambda_{0,1} = 0$. Using these two eigenfunctions and according to the linear wave theory, we construct the initial approximation of the potential function

$$\phi_0(\xi_1, \xi_2, z) = A_0 \sqrt{\frac{g}{k_1}} \Psi_{1,0} + B_0 \sqrt{\frac{g}{k_2}} \Psi_{0,1}, \quad (16.55)$$

where A_0 and B_0 are unknown constants.

The corresponding first-order deformation equation about the potential function $\phi_1(\xi_1, \xi_2, z)$ reads

$$\hat{\nabla}^2 \phi_1(\xi_1, \xi_2, z) = 0, \quad (16.56)$$

subject to the boundary condition on $z = 0$:

$$\begin{aligned} \bar{\mathcal{L}}(\phi_1) = & b_1^{1,0} \sin \xi_1 + b_1^{0,1} \sin \xi_2 + b_1^{1,1} \sin(\xi_1 + \xi_2) + d_1^{1,1} \sin(\xi_1 - \xi_2) \\ & + b_1^{2,1} \sin(2\xi_1 + \xi_2) + d_1^{2,1} \sin(2\xi_1 - \xi_2) \\ & + b_1^{1,2} \sin(\xi_1 + 2\xi_2) + d_1^{1,2} \sin(\xi_1 - 2\xi_2), \end{aligned} \quad (16.57)$$

and the boundary condition on the bottom:

$$\left. \frac{\partial \phi_1}{\partial z} \right|_{z=0} = 0, \quad (16.58)$$

where $\bar{\mathcal{L}}$ is defined by (16.44), and $b_1^{i,j}, d_1^{i,j}$ are constants. Especially, we have

$$\begin{aligned} b_1^{1,0} &= c_0 A_0 \sqrt{\frac{g}{k_1}} \left[gk_1 - \sigma_1^2 + gk_1(A_0 k_1)^2 + 2gk_1(B_0 k_2)^2 + \frac{gB_0^2 k_1^2 k_2}{2} \sin^2(\alpha_1 - \alpha_2) \right], \\ b_1^{0,1} &= c_0 B_0 \sqrt{\frac{g}{k_2}} \left[gk_2 - \sigma_2^2 + gk_2(B_0 k_2)^2 + 2gk_2(A_0 k_1)^2 + \frac{gA_0^2 k_2^2 k_1}{2} \sin^2(\alpha_1 - \alpha_2) \right]. \end{aligned}$$

Since $\lambda_{1,0} = \lambda_{0,1} = 0$, according to (16.54), it must hold

$$b_1^{1,0} = b_1^{0,1} = 0$$

so as to avoid the so-called “secular” terms $\xi_1 \sin \xi_1$ and $\xi_2 \sin \xi_2$. This provides us the following algebraic equations

$$(A_0 k_1)^2 + \left[2 + \frac{k_1}{2k_2} \sin^2(\alpha_1 - \alpha_2) \right] (B_0 k_2)^2 = \frac{\sigma_1^2}{gk_1} - 1, \quad (16.59)$$

$$\left[2 + \frac{k_2}{2k_1} \sin^2(\alpha_1 - \alpha_2) \right] (A_0 k_1)^2 + (B_0 k_2)^2 = \frac{\sigma_2^2}{gk_2} - 1, \quad (16.60)$$

whose solutions are

$$A_0 = \pm \frac{\varepsilon}{k_1} \sqrt{\left(\frac{\sigma_2^2}{gk_2} - 1 \right) \left[2 + \frac{k_1}{2k_2} \sin^2(\alpha_1 - \alpha_2) \right] - \left(\frac{\sigma_1^2}{gk_1} - 1 \right)}, \quad (16.61)$$

$$B_0 = \pm \frac{\varepsilon}{k_2} \sqrt{\left(\frac{\sigma_1^2}{gk_1} - 1 \right) \left[2 + \frac{k_2}{2k_1} \sin^2(\alpha_1 - \alpha_2) \right] - \left(\frac{\sigma_2^2}{gk_2} - 1 \right)}, \quad (16.62)$$

where

$$\varepsilon = \left\{ \left[2 + \frac{k_1}{2k_2} \sin^2(\alpha_1 - \alpha_2) \right] \left[2 + \frac{k_2}{2k_1} \sin^2(\alpha_1 - \alpha_2) \right] - 1 \right\}^{-1/2}.$$

Note that A_0 and B_0 have multiple values: they can be either positive or negative.

Then, by means of the linear inverse operator $\mathcal{L}^{-1}$ defined by (16.54), the common solution of $\phi_1(\xi_1, \xi_2, z)$ reads

$$\begin{aligned} \phi_1 = & A_1 \sqrt{\frac{g}{k_1}} \Psi_{1,0} + B_1 \sqrt{\frac{g}{k_2}} \Psi_{0,1} + b_1^{1,1} \left(\frac{\Psi_{1,1}}{\lambda_{1,1}} \right) + d_1^{1,1} \left(\frac{\Psi_{1,-1}}{\lambda_{1,-1}} \right) \\ & + b_1^{2,1} \left(\frac{\Psi_{2,1}}{\lambda_{2,1}} \right) + d_1^{2,1} \left(\frac{\Psi_{2,-1}}{\lambda_{2,-1}} \right) + b_1^{1,2} \left(\frac{\Psi_{1,2}}{\lambda_{1,2}} \right) + d_1^{1,2} \left(\frac{\Psi_{1,-2}}{\lambda_{1,-2}} \right), \end{aligned} \quad (16.63)$$

where A_1 and B_1 are unknown coefficients, the eigenfunction $\Psi_{m,n}$ and eigenvalue $\lambda_{m,n}$ are defined by (16.23) and (16.49), respectively. In other words, ϕ_1 is a sum (or linear combination) of eigenfunctions. Note that ϕ_1 automatically satisfies the Laplace equation (16.39) and the bottom condition (16.40) for *any* constants A_1 and B_1 . On the other hand, given the initial guess ϕ_0 , it is straightforward to calculate $\eta_1(\xi_1, \xi_2)$ directly by means of the formula (16.42).

The above approach has general meaning. In a similar way, we can obtain $\eta_m(\xi_1, \xi_2)$ and $\phi_m(\xi_1, \xi_2, z)$, successively, in the order of $m = 1, 2, 3$, and so on. Note that, the two unknown coefficients A_m and B_m ($m \geq 1$) can be determined exactly in the same way like A_0 and B_0 by means of avoiding the “secular” terms $\xi_1 \sin \xi_1$ and $\xi_2 \sin \xi_2$. Note that only fundamental operations are needed in the above approach so that it is convenient to gain high-order approximations by means of computer algebraic system such as Mathematica.

Without loss of generality, let us consider here such a special case of the two primary waves that

$$\frac{\sigma_1}{\sqrt{gk_1}} = \frac{\sigma_2}{\sqrt{gk_2}} = 1.0003, \quad \alpha_1 = 0, \quad \alpha_2 = \frac{\pi}{36}, \quad k_2 = \frac{\pi}{5}, \quad (16.64)$$

with different ratios of k_1/k_2 . Here, the number 1.0003 is chosen so that the perturbation theory is valid with high accuracy and thus we can compare our results with those given by Phillips (1960) and Longuet-Higgins (1962), who suggested that the wave resonance (with amplitude growing in time) occurs if the resonance criterion (16.51) is satisfied, i.e.

$$\frac{k_2}{k_1} \approx 0.8925, \quad \frac{\sigma_1}{\sigma_2} = \sqrt{\frac{k_1}{k_2}} \approx 1.0585$$

in the current case. As mentioned before, there are three eigenfunctions ($N_\lambda = 3$) whose eigenvalues are zero when the above criterion is satisfied. To avoid this, we consider here the non-resonant waves with different wave number k_1 except $k_2/k_1 = 0.8925$.

Let us first consider the case $k_2/k_1 = 1$. Without loss of generality, we first use the negative values of A_0 and B_0 given by (16.61) and (16.62), respectively. It is

Fig. 16.1 Squared residual E_m^ϕ versus c_0 in case of (16.64) with $k_2/k_1 = 1$. Solid line: 1st-order approximation; Dashed line: 3rd-order approximation; Dash-dotted line: 5th-order approximation.

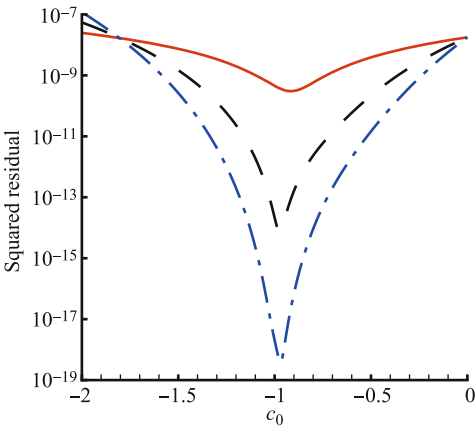


Table 16.1 Squared residuals of the two nonlinear boundary conditions in case of (16.64) when $k_2/k_1 = 1$.

Order of approximation m	E_m^ϕ	E_m^η
1	1.9×10^{-8}	5.1×10^{-4}
3	3.5×10^{-12}	1.2×10^{-9}
5	2.0×10^{-16}	4.3×10^{-14}
8	2.8×10^{-22}	3.8×10^{-20}
10	4.7×10^{-26}	6.4×10^{-24}

found that, as the order of approximation m increases, the squared residual E_m^ϕ decreases in the interval $-1.8 \leq c_0 < 0$, and the optimal value of c_0 is close to -1 , as shown in Fig. 16.1. Indeed, when $c_0 = -1$, the corresponding squared residuals of the two boundary conditions decrease rather quickly to the level 10^{-24} at the 10th-order approximation, as listed in Table 16.1. According to the Convergence Theorem mentioned above, the corresponding homotopy-series (16.37) and (16.38) must be the solutions.

Let $a_{1,0}$, $a_{0,1}$ and $a_{2,-1}$ denote the amplitudes of wave components $\cos \xi_1$, $\cos \xi_2$ and $\cos(2\xi_1 - \xi_2)$, respectively. Obviously, $a_{1,0} = a_{0,1}$ in case of $k_1 = k_2$. As shown in Table 16.2, each wave component converges rather quickly, which agree well with those obtained by the homotopy-Padé method (Liao and Tan, 2007; Liao, 2003b), given in Table 16.3. All of these indicate the validity of the HAM-based analytic approach mentioned above.

Similarly, we can get convergent series solutions in case of (16.64) with different ratio of k_2/k_1 , as shown in Table 16.4. Note that, as the ratio of k_2/k_1 decreases, the wave amplitude of the component $\cos(2\xi_1 - \xi_2)$ first increases monotonically to the maximum value at $k_2/k_1 = 0.8925$ (the corresponding result is given by the approach described in Sect.16.2.3), and then decreases monotonically. Note that the variation of $a_{1,0}$ and $a_{0,1}$ is not monotonic.

Table 16.2 Analytic approximations of some components of wave-amplitude in case of (16.64) when $k_2/k_1 = 1$.

Order of approximation m	$a_{1,0} = a_{0,1}$	$a_{2,-1}$
1	-0.022502	0
2	-0.022846	0.00059739
3	-0.022814	0.00057198
4	-0.022816	0.00057208
6	-0.022816	0.00057226
8	-0.022816	0.00057226
10	-0.022816	0.00057226

Table 16.3 Analytical approximations of some components of wave-amplitude given by the $[m, m]$ homotopy-Padé method in case of (16.64) when $k_2/k_1 = 1$.

m	$a_{1,0} = a_{0,1}$	$a_{2,-1}$
2	-0.022816	0.00057211
3	-0.022816	0.00057226
4	-0.022816	0.00057226
5	-0.022816	0.00057226

Table 16.4 Components of wave-amplitude in case of (16.64) with different values of k_2/k_1 .

k_2/k_1	$a_{1,0}$	$a_{0,1}$	$a_{2,-1}$
1.00	-0.0228	-0.0228	0.00057
0.95	-0.0229	-0.0218	0.00084
0.93	-0.0230	-0.0215	0.00116
0.92	-0.0231	-0.0214	0.00150
0.91	-0.0231	-0.0215	0.00219
0.905	-0.0232	-0.0216	0.00290
0.90	-0.023	-0.022	0.00433
0.8925	-0.0205	-0.0232	0.00898
0.88	-0.0235	-0.0179	0.00307
0.86	-0.0221	-0.0189	0.00120
0.85	-0.0226	-0.0187	0.00087
0.83	-0.0229	-0.0182	0.00054
0.80	-0.0232	-0.0172	0.00033
0.70	-0.0239	-0.0121	0.00010

Phillips (1960) pointed out that, when the resonance criterion (16.51) is satisfied, wave resonance occurs so that the amplitude of wave component $\cos(2\xi_1 - \xi_2)$ grows in time, say, it contains more and more wave energy. However, as shown in Table 16.4, $|a_{1,0}|$ and $|a_{0,1}|$ are much *larger* than $|a_{2,1}|$ for *all* values of k_2/k_1 , even if the resonance criterion (16.51) is satisfied. Physically, it means that the resonant wave is *unnecessary* to have most of wave energy! We will discuss this very interesting phenomena later in details.

16.2.3 Resonant waves

Now, let us further consider the case of (16.64) with a special ratio $k_2/k_1 = 0.8925$, i.e. the resonance criterion (16.51) is satisfied. It was suggested first by Phillips (1960) and then confirmed by Longuet-Higgins (1962) that the so-called wave resonance occurs in this case so that the amplitude of wave component $\cos(2\xi_1 - \xi_2)$ grows in time, if $a_{2,-1}$ is initially zero.

In this case, the criterion (16.51) is satisfied, i.e.

$$g|2\mathbf{k}_1 - \mathbf{k}_2| = (2\bar{\sigma}_1 - \bar{\sigma}_2)^2.$$

According to (16.64), we have

$$k_1 = 0.703998, \quad k_2 = 0.628319, \quad k_3 = |2\mathbf{k}_1 - \mathbf{k}_2| = 0.783981.$$

Thus, according to (16.49), we have an additional zero eigenvalue $\lambda_{2,-1} = 0$ with the corresponding eigenfunction

$$\Psi_{2,-1} = e^{i2\mathbf{k}_1 - \mathbf{k}_2 \cdot \mathbf{z}} \sin(2\xi_1 - \xi_2).$$

So, we have now *three* eigenfunctions $\Psi_{1,0}$, $\Psi_{0,1}$ and $\Psi_{2,-1}$ whose eigenvalues are zero, i.e. $\lambda_{1,0} = 0$, $\lambda_{0,1} = 0$ and $\lambda_{2,-1} = 0$. Here, it should be emphasized that the wave resonance criterion given by Phillips (1960) is mathematically equivalent to

$$\lambda_{m,n} = 0, \quad m^2 + n^2 > 1.$$

Note that the additional zero eigenvalue $\lambda_{2,-1}$ breaks down the approach mentioned in Sect. 16.2.2, because the term $\Psi_{2,-1}/\lambda_{2,-1}$ in (16.63) becomes infinite: this is exactly the reason why Phillips (1960) and Longuet-Higgins (1962) suggested the existence of the so-called wave-resonance with amplitude growing in time, which, however, is physically possible only for small time, since the wave energy can not be infinite and besides the wave breaking occurs for large enough amplitude wave.

Note that, in case of non-resonant waves investigated in Sect. 16.2.2, the initial guess (16.55) is a linear combination of the *two* eigenfunctions $\Psi_{1,0}$ and $\Psi_{0,1}$ whose eigenvalues $\lambda_{1,0}$ and $\lambda_{0,1}$ are zero. In the current case of wave resonance, the only difference is that we have an additional eigenfunction $\Psi_{2,-1}$ whose eigenvalue $\lambda_{2,-1}$ is zero, too. Note that the HAM provides us with great freedom to choose the initial guess. With such kind of freedom, why not use these three eigenfunctions (with zero eigenvalue) to express the initial guess ϕ_0 ? In other words, we can express the initial guess ϕ_0 by *all* eigenfunctions whose eigenvalues are zero, i.e.

$$\phi_0(\xi_1, \xi_2, z) = \bar{A}_0 \sqrt{\frac{g}{k_1}} \Psi_{1,0} + \bar{B}_0 \sqrt{\frac{g}{k_2}} \Psi_{0,1} + \bar{C}_0 \sqrt{\frac{g}{k_3}} \Psi_{2,-1}, \quad (16.65)$$

where $\bar{A}_0, \bar{B}_0, \bar{C}_0$ are *unknown* constants. Similarly, substituting the above expression into the deformation equations (16.39) to (16.41), we have the same first-order

deformation equation (16.56) with the same boundary condition (16.58) at bottom, but a more complicated equation corresponding to the boundary condition on $z = 0$, i.e.

$$\begin{aligned} \bar{\mathcal{L}}(\phi_1) = & \bar{b}_1^{1,0} \sin \xi_1 + \bar{b}_1^{0,1} \sin \xi_2 + \bar{b}_1^{2,0} \sin(2\xi_1) + \bar{b}_1^{3,0} \sin(3\xi_1) \\ & + \bar{b}_1^{1,1} \sin(\xi_1 + \xi_2) + \bar{d}_1^{1,1} \sin(\xi_1 - \xi_2) \\ & + \bar{b}_1^{2,1} \sin(2\xi_1 + \xi_2) + \bar{d}_1^{2,1} \sin(2\xi_1 - \xi_2) \\ & + \bar{b}_1^{1,2} \sin(\xi_1 + 2\xi_2) + \bar{d}_1^{1,2} \sin(\xi_1 - 2\xi_2) \\ & + \bar{d}_1^{2,2} \sin(2\xi_1 - 2\xi_2) + \bar{d}_1^{2,3} \sin(2\xi_1 - 3\xi_2) \\ & + \bar{d}_1^{3,1} \sin(3\xi_1 - \xi_2) + \bar{d}_1^{3,2} \sin(3\xi_1 - 2\xi_2) \\ & + \bar{d}_1^{4,1} \sin(4\xi_1 - \xi_2) + \bar{d}_1^{4,2} \sin(4\xi_1 - 2\xi_2) + \bar{d}_1^{4,3} \sin(4\xi_1 - 3\xi_2) \\ & + \bar{d}_1^{5,2} \sin(5\xi_1 - 2\xi_2) + \bar{d}_1^{6,3} \sin(6\xi_1 - 3\xi_2), \end{aligned} \quad (16.66)$$

where $\bar{b}_1^{m,n}, \bar{d}_1^{m,n}$ are constant coefficients, and the linear operator $\bar{\mathcal{L}}$ is defined by (16.44). Note that there exist now *three* zero eigenvalues, i.e. $\lambda_{1,0} = 0$, $\lambda_{0,1} = 0$ and $\lambda_{2,-1} = 0$. Therefore, according to the definition (16.54) of the inverse operator $\bar{\mathcal{L}}^{-1}$, *not only* the two coefficients $\bar{b}_1^{1,0}$ and $\bar{b}_1^{0,1}$ *but also* the additional coefficient $\bar{d}_1^{2,1}$ must be zero, so that the secular terms can be avoided. Enforcing

$$\bar{b}_1^{1,0} = 0, \quad \bar{b}_1^{0,1} = 0, \quad \bar{d}_1^{2,1} = 0$$

gives a set of nonlinear algebraic equations

$$\begin{cases} 12.7576\bar{A}_0^2 + 20.3675\bar{B}_0^2 + 31.6768\bar{C}_0^2 + 25.6718\bar{B}_0\bar{C}_0 = 0.01545, \\ 24.1456\bar{A}_0^2 + 9.6004\bar{B}_0^2 + 30.0398\bar{C}_0^2 + 14.9558\bar{A}_0^2\bar{C}_0/\bar{B}_0 = 0.01459, \\ 26.9621\bar{A}_0^2 + 21.6116\bar{B}_0^2 + 16.6956\bar{C}_0^2 + 10.7158\bar{A}_0^2\bar{B}_0/\bar{C}_0 = 0.01630, \end{cases} \quad (16.67)$$

for the special case mentioned above. The set of these nonlinear algebraic equations has four complex and twelve real solutions. Because the complex solutions have no physical meanings, we list only its twelve real roots in Table 16.5. It is found that the twelve roots fall into three groups, and different groups give different solutions, as shown later. After solving this set of nonlinear algebraic equations, the initial guess $\phi_0(\xi_1, \xi_2, z)$ is known and therefore it is straightforward to get $\eta_1(\xi_1, \xi_2)$ directly by means of (16.42). More importantly, on the right-hand side of (16.66), the terms $\sin \xi_1$, $\sin \xi_2$ and especially $\sin(2\xi_1 - \xi_2)$ disappear now. Then, using the inverse operator (16.54), it is straightforward to get the common solution of the first-order approximation

$$\begin{aligned} \phi_1 = & \bar{A}_1 \sqrt{\frac{g}{k_1}} \Psi_{1,0} + \bar{B}_1 \sqrt{\frac{g}{k_2}} \Psi_{0,1} + \bar{C}_1 \sqrt{\frac{g}{k_3}} \Psi_{2,-1} \\ & + \bar{b}_1^{2,0} \left(\frac{\Psi_{2,0}}{\lambda_{2,0}} \right) + \bar{b}_1^{3,0} \left(\frac{\Psi_{3,0}}{\lambda_{3,0}} \right) + \bar{b}_1^{1,1} \left(\frac{\Psi_{1,1}}{\lambda_{1,1}} \right) + \bar{d}_1^{1,1} \left(\frac{\Psi_{1,-1}}{\lambda_{1,-1}} \right) \end{aligned} \quad (16.68)$$

$$\begin{aligned}
& + \bar{b}_1^{2,1} \left(\frac{\Psi_{2,1}}{\lambda_{2,1}} \right) + \bar{b}_1^{1,2} \left(\frac{\Psi_{1,2}}{\lambda_{1,2}} \right) + \bar{d}_1^{1,2} \left(\frac{\Psi_{1,-2}}{\lambda_{1,-2}} \right) + \bar{d}_1^{2,2} \left(\frac{\Psi_{2,-2}}{\lambda_{2,-2}} \right) \\
& + \bar{d}_1^{2,3} \left(\frac{\Psi_{2,-3}}{\lambda_{2,-3}} \right) + \bar{d}_1^{3,1} \left(\frac{\Psi_{3,-1}}{\lambda_{3,-1}} \right) + \bar{d}_1^{3,2} \left(\frac{\Psi_{3,-2}}{\lambda_{3,-2}} \right) + \bar{d}_1^{4,1} \left(\frac{\Psi_{4,-1}}{\lambda_{4,-1}} \right) \\
& + \bar{d}_1^{4,2} \left(\frac{\Psi_{4,-2}}{\lambda_{4,-2}} \right) + \bar{d}_1^{4,3} \left(\frac{\Psi_{4,-3}}{\lambda_{4,-3}} \right) + \bar{d}_1^{5,2} \left(\frac{\Psi_{5,-2}}{\lambda_{5,-2}} \right) + \bar{d}_1^{6,3} \left(\frac{\Psi_{6,-3}}{\lambda_{6,-3}} \right).
\end{aligned}$$

Table 16.5 Roots of (16.67)

Series number of roots, K	$\bar{A}_0$	$\bar{B}_0$	$\bar{C}_0$
(Group-I)			
1	-0.0156112	0.0282054	-0.0084973
2	-0.0156112	-0.0282054	0.0084973
3	0.0156112	0.0282054	-0.0084973
4	0.0156112	-0.0282054	0.0084973
(Group-II)			
5	-0.0155774	-0.0141927	-0.0113800
6	-0.0155774	0.0141927	0.0113800
7	0.0155774	-0.0141927	-0.0113800
8	0.0155774	0.0141927	0.0113800
(Group-III)			
9	-0.0155626	0.0106109	-0.0226353
10	-0.0155626	-0.0106109	0.0226353
11	0.0155626	0.0106109	-0.0226353
12	0.0155626	-0.0106109	0.0226353

It should be emphasized that all eigenvalues $\lambda_{m,n}$ in the above expression are nonzero so that $\phi_1(\xi_1, \xi_2, z)$ is finite. Besides, all coefficients in the above expression are independent of the time so that the corresponding wave profile does *not* grow in time. Like the initial guess $\phi_0(\xi_1, \xi_2, z)$ defined by (16.65), the common solution $\phi_1(\xi_1, \xi_2, z)$ given by (16.68) has three unknown coefficients $\bar{A}_1, \bar{B}_1$ and $\bar{C}_1$, which can be determined similarly by avoiding the “secular” terms in $\phi_2(\xi_1, \xi_2, z)$. So, the above approach has general meanings. Therefore, in a similar way, we can get $\eta_m(\xi_1, \xi_2)$ and $\phi_m(\xi_1, \xi_2, z)$ successively, in the order of $m = 1, 2, 3$ and so on.

Note that the wave amplitude components $a_{1,0}$ and $a_{0,1}$ in Table 16.4 are negative. To calculate the corresponding wave amplitude components in case of (16.64) with $k_2/k_1 = 0.8925$, we choose the 2nd root in Group-I, corresponding to $K = 2$, i.e.

$$\bar{A}_0 = -0.0156112, \quad \bar{B}_0 = -0.0282054, \quad \bar{C}_0 = 0.00849726.$$

Similarly, the corresponding squared residual E_m^ϕ versus c_0 is given in Fig. 16.2, which indicates that the optimal value of c_0 is close to -1 . Indeed, by means of $c_0 = -1$, the squared residuals E_m^ϕ and E_m^η of the two boundary conditions decrease rapidly to the level 10^{-19} at the 20th-order homotopy-approximation, as

Fig. 16.2 Squared residual E_m^ϕ versus c_0 in case of (16.64) with $k_2/k_1 = 0.8925$. Solid line: 1st-order approximation; Dashed line: 3rd-order approximation; Dash-dotted line: 5th-order approximation.

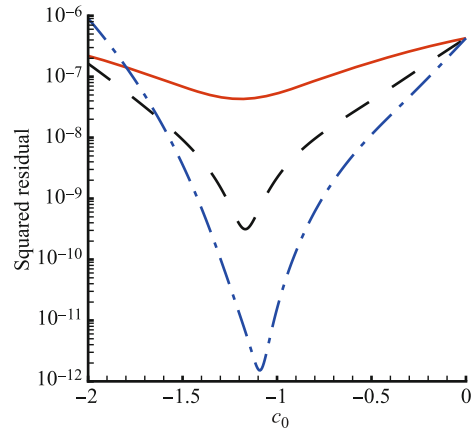


Table 16.6 Squared residual of the two boundary conditions in case of (16.64) when $k_2/k_1 = 0.8925$.

Order of approximation m	E_m^ϕ	E_m^η
1	4.0×10^{-7}	4.8×10^{-4}
3	1.8×10^{-8}	1.0×10^{-6}
5	2.4×10^{-10}	1.2×10^{-8}
8	1.2×10^{-13}	1.5×10^{-11}
10	1.3×10^{-14}	1.2×10^{-12}
15	1.2×10^{-17}	3.4×10^{-16}
20	1.2×10^{-20}	1.9×10^{-19}

shown in Table 16.6. According to the Convergence Theorem mentioned above, the homotopy-series (16.37) and (16.38) satisfy the original governing equation (16.14) and all boundary conditions (16.17), (16.18) and (16.20). Besides, it is found that the corresponding wave amplitude components $a_{1,0}$, $a_{0,1}$ and $a_{2,-1}$ converge to -0.02051 , -0.023212 , 0.0089752 , respectively, as shown in Table 16.7. To confirm the convergence, we further employ the homotopy-Padé technique (Liao and Tan, 2007; Liao, 2003b) to accelerate the convergence and obtain the same convergent wave amplitude components

$$a_{1,0} = -0.0205119, \quad a_{0,1} = -0.0232118, \quad a_{2,-1} = 0.0089752,$$

as shown in Table 16.8. Therefore, we indeed get convergent series solution of resonant waves with *constant* amplitudes even when the wave-resonance criterion (16.51) is exactly satisfied.

Combining the above result with those listed in Table 16.4, we obtain the whole pattern of the dimensionless wave amplitude component $k_3(a_{2,-1})$ versus k_2/k_1 in case of (16.64), as shown in Fig. 16.3. It is true that, as the ratio k_2/k_1 goes to 0.8925, corresponding to the criterion (16.51) of wave resonance, the dimensionless wave amplitude component $k_3 a_{2,-1}$ increases to its maximum. Note that the am-

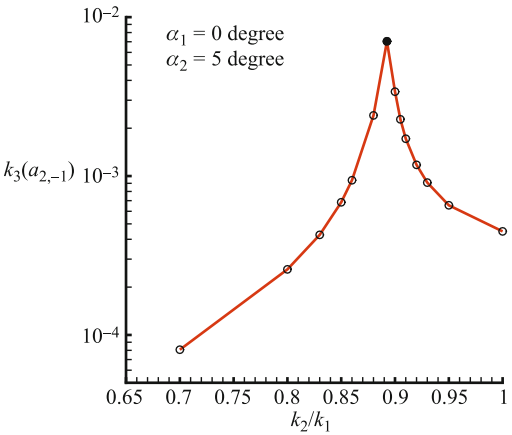
Table 16.7 Homotopy-approximations of wave-amplitude components in case of (16.64) when $k_2/k_1 = 0.8925$.

Order of approximation m	$a_{1,0}$	$a_{0,1}$	$a_{2,-1}$
1	-0.015616	-0.028214	0.0084999
3	-0.020587	-0.023469	0.0094143
5	-0.020533	-0.023231	0.0090083
7	-0.020504	-0.023220	0.0089755
9	-0.020511	-0.023213	0.0089757
11	-0.020512	-0.023212	0.0089753
13	-0.020512	-0.023212	0.0089752
15	-0.020512	-0.023212	0.0089752
18	-0.020512	-0.023212	0.0089752

Table 16.8 Homotopy-approximations of wave-amplitude components given by the $[m,m]$ homotopy-Padé method in case of (16.64) with $k_2/k_1 = 0.8925$.

m	$a_{1,0}$	$a_{0,1}$	$a_{2,-1}$
2	-0.0206010	-0.0232058	0.0089660
3	-0.0204911	-0.0232251	0.0089568
4	-0.0205102	-0.0232172	0.0089780
5	-0.0205122	-0.0232118	0.0089752
6	-0.0205119	-0.0232118	0.0089752
7	-0.0205119	-0.0232118	0.0089752
8	-0.0205119	-0.0232118	0.0089752
9	-0.0205119	-0.0232118	0.0089752

Fig. 16.3 The dimensionless wave amplitude component $k_3(a_{2,-1})$ versus k_2/k_1 in case of (16.64). Filled circle: result when $k_2/k_1 = 0.8925$; Open circles: results when $k_2/k_1 \neq 0.8925$.



plitude $|a_{2,-1}|$ of the wave component $\cos(2\xi_1 - \xi_2)$ is a *finite* constant, even if the resonance criterion (16.51) is satisfied exactly. Besides, it is very interesting that, in case of $k_2/k_1 = 0.8925$, the amplitude $|a_{2,-1}|$ of the resonant wave component

is even much *smaller* than the wave amplitudes $|a_{1,0}|$ and $|a_{0,1}|$ of the two primary waves!

The above results are obtained by means of the 2nd root of Group I in Table 16.5, corresponding to $K = 2$. Similarly, using different roots in Table 16.5, we can search for the corresponding convergent series solutions. It is found that the four different roots of each group in Table 16.5 give the same absolute values $|a_{1,0}|$, $|a_{0,1}|$ and $|a_{2,-1}|$ of wave components, as listed in Table 16.9. Considering the fact that the wave energy spectrum is determined by amplitude square of wave components, we regard the four different solutions in each group as the same. Thus, in case of (16.64) with $k_2/k_1 = 0.8925$, there are *three* different resonant-wave patterns with *different* wave energy spectrums. Here, it should be emphasized that the amplitude $|a_{2,-1}|$ of the resonant wave is the *smallest* in Group I, and is the middle in Group II, although it is the largest in Group III. So, the amplitude $|a_{2,-1}|$ of the resonant wave is *not* special at all: it is just normal as the wave amplitude components $|a_{1,0}|$ and $|a_{0,1}|$ of the two primary waves! It is very interesting that, for a fully developed wave system, there exist *multiple* steady-state solutions even if Phillips' resonance criterion is exactly satisfied, and besides the resonant wave amplitude may be much *smaller* than primary wave amplitudes. To the best of our knowledge, these interesting results has *never* been reported by other numerical and analytic methods. This illustrates that the HAM can be used as a useful tool to solve complicated nonlinear PDEs so as to deepen our physical understanding about some nonlinear phenomena.

Let Π denote the sum of amplitude square of all wave components. Write

$$\Pi_0 = a_{1,0}^2 + a_{0,1}^2 + a_{2,-1}^2.$$

In case of (16.64) with $k_2/k_1 = 0.8925$, corresponding to the occurrence of wave resonance, it is found that Π_0/Π equals to 98.82%, 98.27% and 99.76% for Group I, II and III, respectively. This is mainly because amplitudes of other wave components such as $a_{1,-2}$ are much smaller, as shown in Table 16.9. Thus, these three wave components nearly contain the whole wave energy. Note that, given two primary traveling waves with wave numbers $\mathbf{k}_1$ and $\mathbf{k}_2$, there exist an infinite number of different wave components $a_{m,n} \cos(m\xi_2 + n\xi_2)$ with the wave number $m\mathbf{k}_1 + n\mathbf{k}_2$, where m and n are arbitrary integers. Let

$$\mathbf{K} = \{m\mathbf{k}_1 + n\mathbf{k}_2 | m, n \text{ are integers}\}$$

denote a set of all these wave numbers. Each wave number in $\mathbf{K}$ corresponds to an eigenfunction defined by (16.23) with an eigenvalue defined by (16.49). Our com-

Table 16.9 Multiple amplitudes of resonant waves in case of (16.64) with $k_2/k_1 = 0.8925$ given by different roots of (16.67) listed in Table 16.5.

	$ a_{1,0} $	$ a_{0,1} $	$ a_{2,-1} $	$ a_{1,-2} $
Group I	0.02051186921	0.02321179687	0.00897520547	0.00022907754
Group II	0.01475607438	0.01002488089	0.01464333477	0.00096077706
Group III	0.00971236473	0.01032248128	0.02576462018	0.00059968416

putations suggest that, for a fully developed wave system composed of two traveling primary waves with *small* amplitudes, the main of wave energy focuses on the wave components whose eigenvalues are zero (or close to zero). This provides us an alternative explanation for the so-called “wave resonance”. According to this explanation, the resonant wave $a_{2,-1} \cos(2\xi_1 - \xi_2)$ is just as important as the two

Table 16.10 Wave energy distribution of the resonant wave (Group I) in case of $\alpha_1 = 0, \alpha_2 = \pi/36, k_2/k_1 = 0.8925$ with different ratios of $\sigma_1/\sqrt{gk_1} = \sigma_2/\sqrt{gk_2}$.

$\sigma_1/\sqrt{gk_1} = \sigma_2/\sqrt{gk_2}$	$a_{1,0}^2/\Pi$	$a_{0,1}^2/\Pi$	$a_{2,-1}^2/\Pi$	Π_0/Π
1.0001	36.97%	53.95%	8.99%	99.91%
1.0002	38.40%	52.74%	8.42%	99.56%
1.0003	39.97%	51.19%	7.65%	98.82%
1.0004	41.59%	49.28%	6.70%	97.57%
1.0005	43.10%	47.09%	5.60%	95.78%
1.0006	44.38%	44.75%	4.43%	93.56%
1.0007	45.65%	42.73%	3.58%	91.95%
1.0008	46.23%	39.07%	2.22%	87.52%

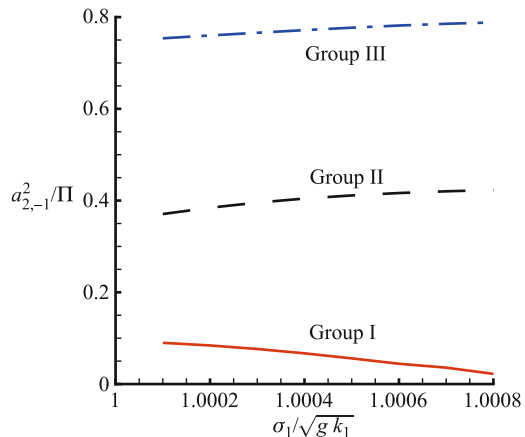
Table 16.11 Wave energy distribution of the resonant wave (Group II) in case of $\alpha_1 = 0, \alpha_2 = \pi/36, k_2/k_1 = 0.8925$ with different ratios of $\sigma_1/\sqrt{gk_1} = \sigma_2/\sqrt{gk_2}$.

$\sigma_1/\sqrt{gk_1} = \sigma_2/\sqrt{gk_2}$	$a_{1,0}^2/\Pi$	$a_{0,1}^2/\Pi$	$a_{2,-1}^2/\Pi$	Π_0/Π
1.0001	42.30%	20.43%	37.07%	99.79%
1.0002	41.26%	19.50%	38.44%	99.20%
1.0003	40.17%	18.54%	39.56%	98.27%
1.0004	39.05%	17.56%	40.44%	97.06%
1.0005	37.93%	16.56%	41.12%	95.61%
1.0006	36.79%	15.54%	41.64%	93.97%
1.0007	35.65%	14.51%	42.02%	92.18%
1.0008	34.49%	13.46%	42.29%	90.24%

Table 16.12 Wave energy distribution of the resonant wave (Group III) in case of $\alpha_1 = 0, \alpha_2 = \pi/36, k_2/k_1 = 0.8925$ with different ratios of $\sigma_1/\sqrt{gk_1} = \sigma_2/\sqrt{gk_2}$.

$\sigma_1/\sqrt{gk_1} = \sigma_2/\sqrt{gk_2}$	$a_{1,0}^2/\Pi$	$a_{0,1}^2/\Pi$	$a_{2,-1}^2/\Pi$	Π_0/Π
1.0001	12.13%	12.47%	75.37%	99.97%
1.0002	11.53%	12.38%	75.99%	99.90%
1.0003	10.88%	12.29%	76.58%	99.76%
1.0004	10.20%	12.20%	77.15%	99.55%
1.0005	9.47%	12.11%	77.68%	99.26%
1.0006	8.72%	12.01%	78.15%	98.88%
1.0007	7.94%	11.90%	78.53%	98.38%
1.0008	7.15%	11.78%	78.82%	97.75%

Fig. 16.4 $a_{2,-1}^2/\Pi$ versus $\sigma_1/\sqrt{gk_1}$ in case of (16.64) when $k_2/k_1 = 0.8925$ and $\sigma_1/\sqrt{gk_1} = \sigma_2/\sqrt{gk_2}$ for different group of solutions. Solid line: Group I; Dashed line: Group II; Dash-dotted line: Group III.



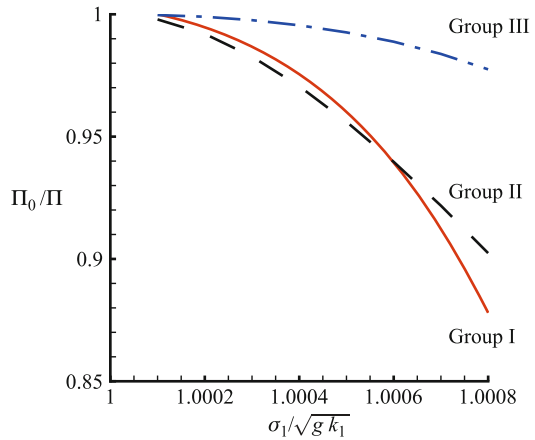
primary waves $a_{1,0} \cos \xi_1$ and $a_{0,1} \cos \xi_2$, and therefore is *not* very special. This viewpoint can well explain why the amplitude of resonant wave is often smaller than those of non-resonant waves.

The wave energy distribution of the fully-developed wave system composed of the two traveling primary waves (corresponding to Group I) in case of $\alpha_1 = 0$, $\alpha_2 = \pi/36$, $k_2/k_1 = 0.8925$ with different ratios of $\sigma_1/\sqrt{gk_1} = \sigma_2/\sqrt{gk_2}$ is given in Table 16.10. It is found that the two primary waves contain most of the total wave energy. Besides, as the ratio of $\sigma_1/\sqrt{gk_1} = \sigma_2/\sqrt{gk_2}$ increases, the resonant wave component $a_{2,-1} \cos(2\xi_1 - \xi_2)$ contains less and less percentage of the total wave energy. Especially, when $\sigma_1/\sqrt{gk_1} = \sigma_2/\sqrt{gk_2} = 1.0008$, the resonant wave of Group I contains only 2.22% of the total wave energy. This result is very interesting but surprising, because, according to traditional theory, the resonance wave is supposed to have a large wave amplitude and thus to contain the main of the total wave energy. Similarly, the wave energy distribution of the fully-developed wave system (corresponding to Group II) is given in Table 16.11. It is found that the resonant wave of Group II has the comparable wave amplitudes with the comparable percentage of the total wave energy to one of primary waves. Among three groups, there exists only one group (i.e. Group III) such that the resonant wave has the largest wave amplitude and thus contain the main part of the total wave energy, as shown in Fig. 16.4 and Table 16.12.

Note that, the primary and resonant waves as a whole contain most of the total wave energy, especially when all wave amplitude components are small, as shown in Fig. 16.5. However, as wave amplitudes increase, they contain less and less percentage of the total wave energy, as shown in Fig. 16.5. This also suggests that Phillips' wave resonance criterion (16.6) is valid only for small-amplitude traveling waves.

Phillips (1960) considered two small-amplitude primary waves $a_{1,0} \cos \xi_1$ and $a_{0,1} \cos \xi_2$ at $t = 0$, and investigated the nonlinear interaction between them. Phillips (1960) revealed that, when the wave component $a_{m,n} \cos(m\xi_1 + n\xi_2)$ is initially zero, i.e. $a_{m,n} = 0$, the wave-amplitude component $a_{m,n}$ increases linearly in time,

Fig. 16.5 Π_0/Π versus $\sigma_1/\sqrt{gk_1}$ in case of (16.64) when $k_2/k_1 = 0.8925$ and $\sigma_1/\sqrt{gk_1} = \sigma_2/\sqrt{gk_2}$ for different group of solutions. Solid line: Group I; Dashed line: Group II; Dash-dotted line: Group III.



if the resonance criterion (16.51) is satisfied. So, Phillips' resonance criterion is valid for the evaluation of wave system in case of small time, say, $t \ll 1$. Since it is impossible in physics for the wave amplitude to increase to infinity, it is unclear whether or not the resonant wave component $a_{m,n} \cos(m\xi_1 + n\xi_2)$ contains the main of the total wave energy as $t \rightarrow +\infty$. Different from Phillips (1960), we consider here the “steady-state” nonlinear interaction of two small-amplitude primary waves, corresponding to the case of $t \rightarrow +\infty$ for the evaluation of waves, say, a fully-developed wave system. Our computations suggest that, if all wave components are fully developed so that they are in equilibrium each other, and if Phillips' resonance criterion (16.51) is satisfied, there exist *multiple* resonant wave component $a_{m,n} \cos(m\xi_1 + n\xi_2)$, and besides the amplitude $a_{m,n}$ of the resonant wave component may be much smaller than the primary ones $a_{1,0}$ and $a_{0,1}$, i.e. the resonant wave component is *not* necessary to have most of the total wave energy. Therefore, Phillips' resonance criterion is a *not* sufficient condition for a large resonant wave: even if Phillips' resonance criterion is satisfied, the amplitude of the resonant wave may be much smaller than the two primary ones.

The above results have general meaning, although they are obtained in a special case (16.64). They strongly suggest that, for a fully developed system composed of two traveling primary waves, there exist steady-state multiple “resonant waves” even if Phillips' wave resonance criterion is exactly satisfied, besides the amplitude of the resonant wave component may be much smaller than those of the primary waves. These conclusions are still true for *arbitrary* number of traveling primary waves, as shown below. For more details, please refer to Liao (2011).

Currently, by means of DNS (direct numerical simulation) of the evolution of nonlinear random water wave fields with a continuous spectrum, Annenkov and Shrira (2006) investigated the role of exactly resonant, nearly resonant and non-resonant wave interactions, and their results indicate that the amplitudes of wave packets tend to constants. Their results, although obtained for a continuous wave spectrum, support some of our conclusions mentioned above.

16.3 Resonance criterion of arbitrary number of primary waves

In the following part of this chapter, the wave-resonance criterion in Sect. 16.2 is generalized in such a way that it is valid for *arbitrary* number of primary traveling waves.

16.3.1 Resonance criterion of small-amplitude waves

We verify in Sect. 16.2 by means of the HAM that, for a fully developed wave system composed of two primary traveling waves, Phillips' resonance criterion (16.51) for small-amplitude waves is mathematically equivalent to a zero eigenvalue, say,

$$\lambda_{m,n} = g |\mathbf{m}\mathbf{k}_1 + \mathbf{n}\mathbf{k}_2| - (m\bar{\sigma}_1 + n\bar{\sigma}_2)^2 = 0, \quad (16.69)$$

where m, n are integers and $\bar{\sigma}_i = \sqrt{g |\mathbf{k}_i|}$ for $i = 1, 2$. This conclusion has general meanings, and can be easily expanded in the frame of the HAM to give a resonance criterion for *arbitrary* number of travel waves with small amplitude. The key is to give an explicit expression of the eigenvalue in case of arbitrary number of traveling waves.

Let us consider the nonlinear interaction of κ periodic traveling waves with small amplitudes, where $2 \leq \kappa < +\infty$ is an arbitrary integer. Assume that the wave system is fully developed, i.e. the amplitude of each wave component is independent of the time. Let $\mathbf{k}_n$ and σ_n ($1 \leq n \leq \kappa$) denote the wave number and angular frequency of the n th periodic traveling waves in deep water. Define the variable

$$\xi_n = \mathbf{k}_n \cdot \mathbf{r} - \sigma_n t.$$

Then, we have the velocity potential and wave elevation

$$\varphi(x, y, z, t) = \phi(\xi_1, \xi_2, \dots, \xi_\kappa, z), \quad \zeta(x, y, t) = \eta(\xi_1, \xi_2, \dots, \xi_\kappa),$$

respectively. Similarly, we have

$$\begin{aligned} \frac{\partial^2 \varphi}{\partial t^2} &= \sum_{m=1}^{\kappa} \sum_{n=1}^{\kappa} \sigma_m \sigma_n \frac{\partial^2 \phi}{\partial \xi_m \partial \xi_n}, \\ \nabla \varphi &= \left(\sum_{m=1}^{\kappa} \mathbf{k}_m \frac{\partial \phi}{\partial \xi_m} \right) + \mathbf{k} \frac{\partial \phi}{\partial z} = \mathbf{u} = \hat{\nabla} \phi, \\ \nabla^2 \varphi &= \left(\sum_{m=1}^{\kappa} \sum_{n=1}^{\kappa} \mathbf{k}_m \cdot \mathbf{k}_n \frac{\partial^2 \phi}{\partial \xi_m \partial \xi_n} \right) + \frac{\partial^2 \phi}{\partial z^2} = \hat{\nabla}^2 \phi, \\ \hat{\nabla} \phi \cdot \hat{\nabla} \phi &= \left(\sum_{m=1}^{\kappa} \sum_{n=1}^{\kappa} \mathbf{k}_m \cdot \mathbf{k}_n \frac{\partial \phi}{\partial \xi_m} \frac{\partial \phi}{\partial \xi_n} \right) + \left(\frac{\partial \phi}{\partial z} \right)^2 = \mathbf{u}^2, \end{aligned}$$

$$\hat{\nabla}\phi \cdot \hat{\nabla}\psi = \left(\sum_{m=1}^{\kappa} \sum_{n=1}^{\kappa} \mathbf{k}_m \cdot \mathbf{k}_n \frac{\partial \phi}{\partial \xi_m} \frac{\partial \psi}{\partial \xi_n} \right) + \frac{\partial \phi}{\partial z} \frac{\partial \psi}{\partial z}.$$

Note that

$$\Psi_{m_1, m_2, \dots, m_{\kappa}, z} = \exp \left(\left| \sum_{n=1}^{\kappa} m_n \mathbf{k}_n \right| z \right) \sin \left(\sum_{n=1}^{\kappa} m_n \xi_n \right) \quad (16.70)$$

satisfies the Laplace equation $\hat{\nabla}\phi = 0$, i.e.

$$\hat{\nabla}^2 \Psi_{m_1, m_2, \dots, m_{\kappa}} = 0$$

and the boundary condition (16.5) at the bottom. The two nonlinear boundary conditions on the free surface can be written by means of the operators defined above in a similar way. Certainly, the corresponding PDE and boundary conditions are much more complicated than those in Sect. 16.2. Even so, in a rather similar way as mentioned above, we can construct the zeroth-order deformation equations and deduce the corresponding high-order deformation equations.

Similarly, we choose such an auxiliary linear operator

$$\mathcal{L}(\phi) = \left(\sum_{m=1}^{\kappa} \sum_{n=1}^{\kappa} \bar{\sigma}_m \bar{\sigma}_n \frac{\partial^2 \phi}{\partial \xi_m \partial \xi_n} \right) + g \frac{\partial^2 \phi}{\partial z^2}, \quad (16.71)$$

where $\bar{\sigma}_m = \sqrt{g |\mathbf{k}_m|}$ is based on the linear theory for small-amplitude waves. According to (16.70) and (16.71), it holds

$$\mathcal{L}(\Psi_{m_1, m_2, \dots, m_{\kappa}}) = \lambda_{m_1, m_2, \dots, m_{\kappa}} \Psi_{m_1, m_2, \dots, m_{\kappa}}, \quad (16.72)$$

where

$$\lambda_{m_1, m_2, \dots, m_{\kappa}} = g \left| \sum_{n=1}^{\kappa} m_n \mathbf{k}_n \right|^2 - \left(\sum_{n=1}^{\kappa} m_n \bar{\sigma}_n \right)^2 \quad (16.73)$$

is the eigenvalue and $\Psi_{m_1, m_2, \dots, m_{\kappa}}$ defined by (16.70) is the eigenfunction of the corresponding linear high-order deformation equation. Similarly, the inverse operator of (16.71) satisfies

$$\mathcal{L}^{-1}(\Psi_{m_1, m_2, \dots, m_{\kappa}}) = \frac{\Psi_{m_1, m_2, \dots, m_{\kappa}}}{\lambda_{m_1, m_2, \dots, m_{\kappa}}}, \quad \lambda_{m_1, m_2, \dots, m_{\kappa}} \neq 0. \quad (16.74)$$

Note that the inverse operator (16.74) is valid only for the non-zero eigenvalue $\lambda_{m_1, m_2, \dots, m_{\kappa}} \neq 0$. Besides, the eigenvalue of each primary traveling waves is zero, i.e.

$$\lambda_{m_1, m_2, \dots, m_{\kappa}} = 0, \quad \text{when } \sum_{n=1}^{\kappa} m_n^2 = 1.$$

Therefore, there exist at least κ zero eigenvalues for κ primary waves. Thus, the so-called wave resonance occurs when there are more than κ zero eigenvalues, i.e.

$$\lambda_{m_1, m_2, \dots, m_\kappa} = 0, \quad \text{when } \sum_{n=1}^{\kappa} m_n^2 > 1.$$

Then, according to (16.73), we have the generalized wave-resonance criterion

$$g \left| \sum_{n=1}^{\kappa} m_n \mathbf{k}_n \right| = \left(\sum_{n=1}^{\kappa} m_n \bar{\sigma}_n \right)^2, \quad \text{when } \sum_{n=1}^{\kappa} m_n^2 > 1, \quad (16.75)$$

where $\bar{\sigma}_n = \sqrt{g |\mathbf{k}_n|}$ is based on the linear theory for small-amplitude waves.

Note that (16.51) is a special case of the above generalized wave-resonance criterion. Besides, it contains the resonance criterion given by Phillips (1960). Thus, it is more general. Substituting $\bar{\sigma}_n = \sqrt{g |\mathbf{k}_n|}$ into (16.75) gives the generalized wave-resonance criterion

$$\left| \sum_{n=1}^{\kappa} m_n \mathbf{k}_n \right| = \left(\sum_{n=1}^{\kappa} m_n \sqrt{g |\mathbf{k}_n|} \right)^2, \quad \text{when } \sum_{n=1}^{\kappa} m_n^2 > 1, \quad (16.76)$$

for waves with small-amplitude, which is independent of the acceleration of gravity.

Assume that, for given κ primary traveling waves, there are $N_\lambda \geq \kappa$ eigenfunctions whose eigenvalues are zero. When $N_\lambda = \kappa$, there is no wave resonance. However, when $N_\lambda > \kappa$, there exist the so-called resonant wave. For simplicity, let Ψ_m^* ($1 \leq m \leq N_\lambda$) denote the m th eigenfunction with zero eigenvalue. According to (16.72), it holds

$$\mathcal{L} \left(\sum_{m=1}^{N_\lambda} A_m \Psi_m^* \right) = 0, \quad N_\lambda \geq \kappa \quad (16.77)$$

for any constant A_m . So, we can always choose such an initial guess that

$$\phi_0 = \sum_{m=1}^{N_\lambda} B_{0,m} \Psi_m^*, \quad (16.78)$$

where $B_{0,m}$ is unknown. Similarly, the N_λ unknown constants $B_{0,m}$ ($1 \leq m \leq N_\lambda$) are determined by avoiding the “secular” terms in ϕ_1 . Besides, according to (16.77), the common solution of ϕ_1 contains N_λ unknown constants $B_{1,m}$, which are similarly determined by avoiding the “secular” terms in ϕ_2 . In this way, one can solve the corresponding high-order deformation equations successively. Similarly, an optimal convergence-control parameter c_0 can be found to guarantee the convergence of the homotopy-series. The above approach is general, and works for *arbitrary* number of periodic traveling primary waves with small amplitudes. It provides us a way to investigate the fully-developed wave system composed of *arbitrary* number of primary traveling waves with small amplitudes.

16.3.2 Resonance criterion of large-amplitude waves

Note that the generalized wave resonance criterion (16.75) holds when $\bar{\sigma}_n = \sqrt{g|\mathbf{k}_n|}$ only, corresponding to small-amplitude gravity waves. What is the resonance criterion for *arbitrary* number of traveling gravity waves with *large* amplitude?

To answer this question, let us consider the physical meanings of (16.75). In general, the so-called resonance of a dynamic system occurs when the frequency of an external force (or disturbance) equals to the “natural” frequency of the dynamic system. For example, let us consider the resonance of a simple pendulum, as shown in Fig. 16.6, where $\mathbf{F} = A \cos(\omega t + \alpha)$ is the external force with the frequency ω and the phase difference α . Let ω_0 denote the natural frequency of the simple pendulum. When the maximum angle of oscillation $\theta_{\max}$ is small so that $\sin \theta \approx \theta$, $\omega_0 \approx \sqrt{g/l}$ is a very good approximation. So, if the frequency ω of the external force $\mathbf{F}$ is equal to the natural frequency ω_0 of the simple pendulum, i.e. $\omega = \sqrt{g/l}$, the total energy of the pendulum (and therefore $\theta_{\max}$) quickly increases in case of the phase difference $\alpha = 0$ (or decreases in case of $\alpha = \pi$): the so-called resonance occurs. However, $\omega_0 \approx \sqrt{g/l}$ is only valid for small $\theta_{\max}$: the natural frequency ω_0 increases as $\theta_{\max}$ becomes larger. So, as $\theta_{\max}$ becomes larger and larger so that the natural frequency ω_0 departs more and more from the frequency $\omega = \sqrt{g/l}$ of the external force $\mathbf{F}$, then the simple pendulum gains less and less energy from the external force: the maximum angle of oscillation $\theta_{\max}$ stops increasing when the simple pendulum can not gain energy from $\mathbf{F}$ any more in a period of oscillation.

The phenomenon of gravity wave resonance is physically similar to it in essence. For a single traveling wave with the wave number $\mathbf{k}'$ and the “natural” angular frequency σ'_0 , the resonance occurs when there exists an “external” periodic disturbance with the same angular frequency σ' , i.e. $\sigma' = \sigma'_0$. As mentioned above, this resonance mechanism is physically reasonable even for large wave amplitude.

Let us consider κ primary traveling waves with wave number $\mathbf{k}_n$ and angular frequency σ_n , where $1 \leq n \leq \kappa$. Due to nonlinear interaction, there exist an infinite number of wave components

$$\cos \left(\sum_{n=1}^{\kappa} m_n \xi_n \right),$$

where m_n is an integer that can be negative, zero, or positive. Note that

$$\sum_{n=1}^{\kappa} m_n \xi_n = \left(\sum_{n=1}^{\kappa} m_n \mathbf{k}_n \right) \cdot \mathbf{r} - \left(\sum_{n=1}^{\kappa} m_n \sigma_n \right) t.$$

So,

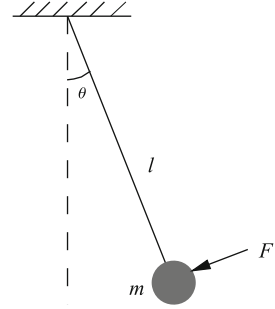
$$\mathbf{k}' = \sum_{n=1}^{\kappa} m_n \mathbf{k}_n \quad (16.79)$$

is the wavenumber and

$$\sigma' = \left| \sum_{n=1}^{\kappa} m_n \sigma_n \right| \quad (16.80)$$

is the corresponding angular frequency of the nonlinear-interaction wave. For the sake of simplicity, we call $\mathbf{k}'$ the nonlinear-interaction wavenumber and σ' the nonlinear-interaction angular frequency, respectively, where $\sum_{n=1}^{\kappa} m_n^2 > 1$.

Fig. 16.6 The resonance of a simple pendulum.



For small-amplitude waves, we have $\sigma_n \approx \sqrt{g k_n} = \bar{\sigma}_n$, which leads to

$$\left| \sum_{n=1}^{\kappa} m_n \bar{\sigma}_n \right| \approx \left| \sum_{n=1}^{\kappa} m_n \sigma_n \right| = \sigma' \quad (16.81)$$

Substituting the above expression and (16.79) into (16.75), we have the resonance criterion (for κ small-amplitude primary waves) in the form:

$$g|\mathbf{k}'| = \sigma'^2, \quad (16.82)$$

i.e.

$$\sigma' = \sqrt{g|\mathbf{k}'|}. \quad (16.83)$$

The above resonance criterion clearly reveals the physical relationship between the nonlinear-interaction wavenumber $\mathbf{k}' = \sum_{n=1}^{\kappa} m_n \mathbf{k}_n$ and the nonlinear-interaction an-

gular frequency $\sigma' = \left| \sum_{n=1}^{\kappa} m_n \sigma_n \right|$.

Let σ'_0 denote the “natural” angular frequency of a single traveling wave with the wavenumber $\mathbf{k}'$ and the wave amplitude a' . In case of small wave amplitudes, according to the linear theory, we have the “natural” angular frequency $\sigma'_0 \approx \sqrt{g|\mathbf{k}'|}$. Then, the above wave-resonance criterion becomes

$$\sigma' = \sigma'_0, \quad (16.84)$$

i.e.

$$\left| \sum_{n=1}^{\kappa} m_n \sigma_n \right| = \sigma'_0, \quad \sum_{n=1}^{\kappa} m_n^2 > 1. \quad (16.85)$$

Physically speaking, the wave resonance occurs when the nonlinear-interaction angular frequency $\sigma' = \left| \sum_{n=1}^{\kappa} m_n \sigma_n \right|$ of the corresponding nonlinear-interaction wave with wavenumber $\mathbf{k}' = \sum_{n=1}^{\kappa} m_n \mathbf{k}_n$ equals to its “natural” angular frequency σ'_0 . Note that, different from the nonlinear-interaction angular frequency σ' that is a kind of sum of angular frequencies of primary waves, the “natural” angular frequency σ'_0 of the corresponding wave component with wave number $\mathbf{k}'$ depends only upon the wavenumber $\mathbf{k}'$ and its amplitude a' , but has nothing to do with the angular frequencies of primary waves. Thus, in general, the nonlinear-interaction angular frequency σ' is not equal to the “natural” angular frequency σ'_0 of the nonlinear-interaction wave with wavenumber $\mathbf{k}'$. So, the wave resonance criterion is indeed rather special. This physical explanation agrees well with the traditional resonance theory. Thus, (16.84) and (16.85) reveal the physical essence of the gravity wave resonance.

Although (16.85) is derived from the wave-resonance criterion (16.75) for *small* wave amplitudes, this physical mechanism of gravity wave resonance has general meanings and holds for large wave amplitudes even if $\sigma_n \approx \sqrt{gk_n}$ is not a good approximation. So, (16.85) is the generalized wave-resonance criterion for *arbitrary* number of periodic traveling primary waves with *large* amplitudes. It should be emphasized that (16.85) logically contains the resonance criterion (16.75) for arbitrary number of small-amplitude waves, and besides Phillips’ resonance criterion (16.6) for four small-amplitude waves. Thus, it is rather general.

When the wave resonance criterion (16.85) is satisfied so that the wave energy transfers from the primary waves to a resonant one, the amplitudes of primary waves decreases and the amplitude of the resonant wave increases. Then, the angular frequency σ_n of each primary wave decreases but the “natural” angular frequency of the resonant wave increases so that the resonance criterion (16.85) does not hold any more. As a result, the “natural” frequency σ'_0 departs more and more from the nonlinear-interaction frequency σ' , so that the nonlinear-interaction wave gains less and less energy from the primary waves, until the whole wave system is in equilibrium, i.e. fully developed. This explains why a resonant gravity wave has finite value of amplitude.

As mentioned above, a resonant simple pendulum acted by an external force with the phase difference π , as shown in Fig. 16.6, *loses* its energy so that the maximum angle of oscillation $\theta_{\max}$ decreases. Similarly, when the wave resonance criterion (16.85) is satisfied, it is also possible that the wave energy transfers from the resonant wave to primary ones so that the amplitude of resonant wave decreases and the amplitudes of primary waves increase: this well explains why the amplitude of a resonant wave may be much smaller than primary ones, as shown in Table 16.10.

For example, consider two primary waves denoted by the wave number $\mathbf{k}_1$ and $\mathbf{k}_2$, and a resonant wave $2\mathbf{k}_1 - \mathbf{k}_2$, whose amplitudes are denoted by $a_{1,0}, a_{0,1}$ and $a_{2,-1}$, respectively. Assume that Phillips’ wave resonance criterion is satisfied. If $a_{2,-1}$ is initially zero, then $a_{2,-1}$ increases linearly in time for small time $t \ll 1$. This is the case investigated by Phillips (1960) and Longuet-Higgins (1962). However, if $a_{2,-1}$ is initially not equal to zero, i.e. $a_{2,-1} \neq 0$ at $t = 0$, then $a_{2,-1}$ may either

increase, i.e. the wave energy transfers from the primary waves to the resonant one, or decrease, i.e. the wave energy transfers from the resonant wave to the primary ones, depending on the initial condition at $t = 0$. From this viewpoint, it is easy to understand why a resonant wave may have a very smaller amplitude than primary waves in a fully developed wave system.

For more discussions, please refer to Liao (2011).

16.4 Concluding remark and discussions

In this chapter we verify the validity of the HAM for a rather complicated nonlinear PDE about the nonlinear interaction of arbitrary number of traveling water waves. In the frame of the HAM, the wave-resonance criterion for *arbitrary* number of small amplitude waves is gained, for the first time, which logically contains the famous Phillips' criterion for four small amplitude waves. Besides, it is found for the first time that, when the wave-resonance criterion is satisfied and the wave system is fully developed, there exist *multiple* steady-state resonant waves, whose amplitude might be much *smaller* than primary waves so that a resonant wave may contain much *small* percentage of the total wave energy. Thus, Phillips' resonance criterion is not sufficient to guarantee a large resonant wave amplitude. In addition, from the physical viewpoints, the above-mentioned wave-resonance criterion is further generalized for *arbitrary* number of *large* amplitude waves, which opens a new way to study the strongly nonlinear interactions of arbitrary number of primary traveling gravity waves with large amplitudes. This example illustrates that the HAM can be used as a tool to deepen and enrich our understandings about some rather complicated nonlinear phenomena.

Note that the wave-resonance criterion (16.85) for arbitrary number of traveling water waves with large amplitudes is given only from the physical view-points of resonance. Although it explains well why the amplitude of a resonant wave is finite and why it may be much smaller than primary ones, physical experiments and/or analytical/numerical results are needed to support it. Besides, it is interesting to study the evolution of a system of arbitrary number of traveling water waves far from equilibrium, especially the wave energy transfer between different wave components.

In this chapter, the so-called homotopy multiple-variable technique (Liao, 2011) is employed. Different from perturbation techniques used by Phillips (1960) and Longuet-Higgins (1962), it does not depend upon any small physical parameters, and besides provides us a convenient way to guarantee the convergence of solution series. Like multiple-scale perturbation method, its solution has clear physical meanings. For example, using the homotopy multiple-variable technique (Liao, 2011), the time t does not explicitly appear for a fully developed wave system: this not only greatly simplifies resolving the problem mathematically, but also contributes a lot to revealing the physical meanings¹ clearly, as shown in this chapter.

¹ Some researchers solved nonlinear wave-type PDEs by simply expanding the solution in Taylor series with respect to the time t . Unfortunately, this often leads to very complicated solution expressions with rather little physical meanings.

The homotopy multiple-variable method (Liao, 2011) is more general than the famous multiple-scale perturbation technique. By means of the multiple-scale perturbation technique, one often rewrites an unknown solution $u(t)$ in the form

$$u(t) = u_0(T_0, T_1, T_2) + u_1(T_0, T_1, T_2) \varepsilon + u_2(T_0, T_1, T_2) \varepsilon^2 + \cdots,$$

where

$$T_0 = t, \quad T_1 = \varepsilon t, \quad T_2 = \varepsilon^2 t$$

denote different timescales with the small physical parameter ε , and then transfers a nonlinear problem into a sequence of linear perturbed problems via the small physical parameter ε . Using the homotopy multiple-variable method (Liao, 2011), we can also rewrite $u(t)$ by $\tilde{u}(\xi_0, \xi_1, \xi_2)$ with the same definition

$$\xi_n = \varepsilon^n t.$$

However, different from the multiple-scale perturbation techniques, we need *not* any small physical parameters. Furthermore, it is easier to gain high-order approximations, and especially, if the multiple-variables are properly defined with clear physical meanings, it gives results with important physical meanings, as illustrated in this chapter. Thus, the homotopy multiple-variable method has general meanings. Considering the fact that the multiple-scale perturbation technique has been widely employed, the homotopy multiple-variable technique may be also applied to solve many types of nonlinear problems in science and engineering, such as the famous natural phenomena about freak wave (Kharif and Pelinovsky, 2003; Gibbs and Taylor, 2005; Adcock and Taylor, 2009), the nonlinear interaction of arbitrary number of traveling water waves far from equilibrium, and so on, although further modifications are needed in future.

In Part III of this book, we verify the validity of the HAM for some nonlinear PDEs. In Chapter 13, the HAM is employed to solve a famous problem in finance, i.e. the American input option. Explicit analytic approximations of the optimal exercise boundary $B(\tau)$ are gained in polynomials of $\sqrt{\tau}$ to $o(\tau^M)$, which are often valid a couple of dozen years prior to expiry, whereas other asymptotic/perturbation formulas only a couple of days or weeks. Such kind of formula valid in so long time has never been reported and is helpful for businessmen. In Chapter 14, the 2nd-order 2D (or 3D) nonlinear Gelfand equation is solved in a rather easy way by means of transforming it into an infinite number of 4th (or 6th) order linear PDEs. Such kind of transform has never been used by other numerical and analytic methods. It also suggests that we human being might have much larger freedom to solve nonlinear problems than we traditionally thought and believed. All of these show the originality and flexibility of the HAM. In Chapter 15, the HAM is applied successfully to solve a complicated nonlinear PDE about nonlinear interaction between water wave and an exponential shear current. It is found, for the first time, that the traditional wave break criterion is still valid even when an exponential shear current exists. In the current chapter, the HAM is employed to give, for the first time, the wave-resonance criterion for arbitrary number of traveling water waves. All of these

illustrate that the HAM can be applied to solve some complicated nonlinear PDEs so as to deepen and enrich our physical understandings for some interesting nonlinear phenomena.

It must be emphasized that the HAM is *not* valid for *all* nonlinear problems, since our aim is to develop a analytic approach valid for *as many* nonlinear problems *as possible* only. It is even an open question if there exists an analytic approximation method valid for all nonlinear problems. In essence, it is rather hard to understand complicated nonlinear phenomena, especially those related to chaos and turbulence. “*The small truth has words which are clear; the great truth has great silence*”, as pointed out by Rabindranth Tagore (1861—1941). However, although nonlinear ODEs and PDEs are still more difficult to solve than linear ones, the HAM provides us an useful, alternative tool to investigate them.

Appendix 16.1 Detailed derivation of high-order equation

Write

$$\left(\sum_{i=1}^{+\infty} \eta_i q^i \right)^m = \sum_{n=m}^{+\infty} \mu_{m,n} q^n, \quad (16.86)$$

with the definition

$$\mu_{1,n}(\xi_1, \xi_2) = \eta_n(\xi_1, \xi_2), \quad n \geq 1. \quad (16.87)$$

Then,

$$\left(\sum_{i=1}^{+\infty} \eta_i q^i \right)^{m+1} = \left(\sum_{n=m}^{+\infty} \mu_{m,n} q^n \right) \left(\sum_{i=1}^{+\infty} \eta_i q^i \right) = \sum_{n=m+1}^{+\infty} \mu_{m+1,n} q^n, \quad (16.88)$$

which gives

$$\mu_{m,n}(\xi_1, \xi_2) = \sum_{i=m-1}^{n-1} \mu_{m-1,i}(\xi_1, \xi_2) \eta_{n-i}(\xi_1, \xi_2), \quad m \geq 2, n \geq m. \quad (16.89)$$

Define

$$\psi_{i,j}^{n,m}(\xi_1, \xi_2) = \frac{\partial^{i+j}}{\partial \xi_1^i \partial \xi_2^j} \left(\frac{1}{m!} \frac{\partial^m \phi_n}{\partial z^m} \Big|_{z=0} \right).$$

By Taylor series, we have for any z that

$$\phi_n(\xi_1, \xi_2, z) = \sum_{m=0}^{+\infty} \left(\frac{1}{m!} \frac{\partial^m \phi_n}{\partial z^m} \Big|_{z=0} \right) z^m = \sum_{m=0}^{+\infty} \psi_{0,0}^{n,m} z^m \quad (16.90)$$

and

$$\frac{\partial^{i+j} \phi_n}{\partial \xi_1^i \partial \xi_2^j} = \sum_{m=0}^{+\infty} \frac{\partial^{i+j}}{\partial \xi_1^i \partial \xi_2^j} \left(\frac{1}{m!} \frac{\partial^m \phi_n}{\partial z^m} \Big|_{z=0} \right) z^m = \sum_{m=0}^{+\infty} \psi_{i,j}^{n,m} z^m. \quad (16.91)$$

Then, on $z = \check{\eta}(\xi_1, \xi_2; q)$, we have using (16.86) that

$$\begin{aligned} \frac{\partial^{i+j} \phi_n}{\partial \xi_1^i \partial \xi_2^j} &= \sum_{m=0}^{+\infty} \psi_{i,j}^{n,m} \left(\sum_{s=1}^{+\infty} \eta_s q^s \right)^m = \psi_{i,j}^{n,0} + \sum_{m=1}^{+\infty} \psi_{i,j}^{n,m} \left(\sum_{s=m}^{+\infty} \mu_{m,s} q^s \right) \\ &= \sum_{m=0}^{+\infty} \beta_{i,j}^{n,m}(\xi_1, \xi_2) q^m, \end{aligned} \quad (16.92)$$

where

$$\beta_{i,j}^{n,0} = \psi_{i,j}^{n,0}, \quad (16.93)$$

$$\beta_{i,j}^{n,m} = \sum_{s=1}^m \psi_{i,j}^{n,s} \mu_{s,m}, \quad m \geq 1. \quad (16.94)$$

Similarly, on $z = \check{\eta}(\xi_1, \xi_2; q)$, it holds

$$\frac{\partial^{i+j}}{\partial \xi_1^i \partial \xi_2^j} \left(\frac{\partial \phi_n}{\partial z} \right) = \sum_{m=0}^{+\infty} \gamma_{i,j}^{n,m}(\xi_1, \xi_2) q^m, \quad (16.95)$$

$$\frac{\partial^{i+j}}{\partial \xi_1^i \partial \xi_2^j} \left(\frac{\partial^2 \phi_n}{\partial z^2} \right) = \sum_{m=0}^{+\infty} \delta_{i,j}^{n,m}(\xi_1, \xi_2) q^m, \quad (16.96)$$

where

$$\gamma_{i,j}^{n,0} = \psi_{i,j}^{n,1}, \quad (16.97)$$

$$\gamma_{i,j}^{n,m} = \sum_{s=1}^m (s+1) \psi_{i,j}^{n,s+1} \mu_{s,m}, \quad m \geq 1, \quad (16.98)$$

$$\delta_{i,j}^{n,0} = 2\psi_{i,j}^{n,2}, \quad (16.99)$$

$$\delta_{i,j}^{n,m} = \sum_{s=1}^m (s+1)(s+2) \psi_{i,j}^{n,s+2} \mu_{s,m}, \quad m \geq 1. \quad (16.100)$$

Then, on $z = \check{\eta}(\xi_1, \xi_2; q)$, it holds using (16.92) that

$$\begin{aligned} \check{\phi}(\xi_1, \xi_2, \check{\eta}; q) &= \sum_{n=0}^{+\infty} \phi_n(\xi_1, \xi_2, \check{\eta}) q^n = \sum_{n=0}^{+\infty} q^n \left[\sum_{m=0}^{+\infty} \beta_{0,0}^{n,m}(\xi_1, \xi_2) q^m \right] \\ &= \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} \beta_{0,0}^{n,m}(\xi_1, \xi_2) q^{m+n} = \sum_{s=0}^{+\infty} q^s \left[\sum_{m=0}^s \beta_{0,0}^{s-m,m}(\xi_1, \xi_2) \right] \\ &= \sum_{n=0}^{+\infty} \bar{\phi}_n^{0,0}(\xi_1, \xi_2) q^n, \end{aligned} \quad (16.101)$$

where

$$\bar{\phi}_n^{0,0}(\xi_1, \xi_2) = \sum_{m=0}^n \beta_{0,0}^{n-m,m}. \quad (16.102)$$

Similarly, we have

$$\frac{\partial^{i+j}\check{\phi}}{\partial \xi_1^i \partial \xi_2^j} = \sum_{n=0}^{+\infty} \bar{\phi}_n^{i,j}(\xi_1, \xi_2) q^n, \quad (16.103)$$

$$\frac{\partial^{i+j}}{\partial \xi_1^i \partial \xi_2^j} \left(\frac{\partial \check{\phi}}{\partial z} \right) = \sum_{n=0}^{+\infty} \bar{\phi}_{z,n}^{i,j}(\xi_1, \xi_2) q^n, \quad (16.104)$$

$$\frac{\partial^{i+j}}{\partial \xi_1^i \partial \xi_2^j} \left(\frac{\partial^2 \check{\phi}}{\partial z^2} \right) = \sum_{n=0}^{+\infty} \bar{\phi}_{zz,n}^{i,j}(\xi_1, \xi_2) q^n, \quad (16.105)$$

where

$$\bar{\phi}_n^{i,j}(\xi_1, \xi_2) = \sum_{m=0}^n \beta_{i,j}^{n-m,m}. \quad (16.106)$$

$$\bar{\phi}_{z,n}^{i,j}(\xi_1, \xi_2) = \sum_{m=0}^n \gamma_{i,j}^{n-m,m}, \quad (16.107)$$

$$\bar{\phi}_{zz,n}^{i,j}(\xi_1, \xi_2) = \sum_{m=0}^n \delta_{i,j}^{n-m,m}. \quad (16.108)$$

Then, on $z = \check{\eta}(\xi_1, \xi_2; q)$, it holds using (16.103) and (16.104) that

$$\begin{aligned} \check{f} &= \frac{1}{2} \hat{\nabla} \check{\phi} \cdot \hat{\nabla} \check{\phi} \\ &= \frac{k_1^2}{2} \left(\frac{\partial \check{\phi}}{\partial \xi_1} \right)^2 + \mathbf{k}_1 \cdot \mathbf{k}_2 \frac{\partial \check{\phi}}{\partial \xi_1} \frac{\partial \check{\phi}}{\partial \xi_2} + \frac{k_2^2}{2} \left(\frac{\partial \check{\phi}}{\partial \xi_2} \right)^2 + \frac{1}{2} \left(\frac{\partial \check{\phi}}{\partial z} \right)^2 \\ &= \sum_{m=0}^{+\infty} \Gamma_{m,0}(\xi_1, \xi_2) q^m, \end{aligned} \quad (16.109)$$

where

$$\begin{aligned} \Gamma_{m,0}(\xi_1, \xi_2) &= \frac{k_1^2}{2} \sum_{n=0}^m \bar{\phi}_n^{1,0} \bar{\phi}_{m-n}^{1,0} + \mathbf{k}_1 \cdot \mathbf{k}_2 \sum_{n=0}^m \bar{\phi}_n^{1,0} \bar{\phi}_{m-n}^{0,1} \\ &\quad + \frac{k_2^2}{2} \sum_{n=0}^m \bar{\phi}_n^{0,1} \bar{\phi}_{m-n}^{0,1} + \frac{1}{2} \sum_{n=0}^m \bar{\phi}_{z,n}^{0,0} \bar{\phi}_{z,m-n}^{0,0}. \end{aligned} \quad (16.110)$$

Similarly, it holds on $z = \check{\eta}(\xi_1, \xi_2; q)$ that

$$\frac{\partial \check{f}}{\partial \xi_1} = \hat{\nabla} \check{\phi} \cdot \hat{\nabla} \left(\frac{\partial \check{\phi}}{\partial \xi_1} \right)$$

$$\begin{aligned}
&= k_1^2 \frac{\partial \check{\phi}}{\partial \xi_1} \frac{\partial^2 \check{\phi}}{\partial \xi_1^2} + k_2^2 \frac{\partial \check{\phi}}{\partial \xi_2} \frac{\partial^2 \check{\phi}}{\partial \xi_1 \partial \xi_2} + \frac{\partial \check{\phi}}{\partial z} \frac{\partial}{\partial \xi_1} \left(\frac{\partial \check{\phi}}{\partial z} \right) \\
&\quad + \mathbf{k}_1 \cdot \mathbf{k}_2 \left(\frac{\partial \check{\phi}}{\partial \xi_1} \frac{\partial^2 \check{\phi}}{\partial \xi_1 \partial \xi_2} + \frac{\partial \check{\phi}}{\partial \xi_2} \frac{\partial^2 \check{\phi}}{\partial \xi_1^2} \right) \\
&= \sum_{m=0}^{+\infty} \Gamma_{m,1}(\xi_1, \xi_2) q^m, \tag{16.111}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \check{f}}{\partial \xi_2} &= \hat{\nabla} \check{\phi} \cdot \hat{\nabla} \left(\frac{\partial \check{\phi}}{\partial \xi_2} \right) \\
&= k_1^2 \frac{\partial \check{\phi}}{\partial \xi_1} \frac{\partial^2 \check{\phi}}{\partial \xi_1 \partial \xi_2} + k_2^2 \frac{\partial \check{\phi}}{\partial \xi_2} \frac{\partial^2 \check{\phi}}{\partial \xi_2^2} + \frac{\partial \check{\phi}}{\partial z} \frac{\partial}{\partial \xi_2} \left(\frac{\partial \check{\phi}}{\partial z} \right) \\
&\quad + \mathbf{k}_1 \cdot \mathbf{k}_2 \left(\frac{\partial \check{\phi}}{\partial \xi_1} \frac{\partial^2 \check{\phi}}{\partial \xi_2^2} + \frac{\partial \check{\phi}}{\partial \xi_2} \frac{\partial^2 \check{\phi}}{\partial \xi_1 \partial \xi_2} \right) \\
&= \sum_{m=0}^{+\infty} \Gamma_{m,2}(\xi_1, \xi_2) q^m, \tag{16.112}
\end{aligned}$$

where

$$\begin{aligned}
\Gamma_{m,1}(\xi_1, \xi_2) &= \sum_{n=0}^m \left(k_1^2 \bar{\phi}_n^{1,0} \bar{\phi}_{m-n}^{2,0} + k_2^2 \bar{\phi}_n^{0,1} \bar{\phi}_{m-n}^{1,1} + \bar{\phi}_{z,n}^{0,0} \bar{\phi}_{z,m-n}^{1,0} \right) \\
&\quad + \mathbf{k}_1 \cdot \mathbf{k}_2 \sum_{n=0}^m \left(\bar{\phi}_n^{1,0} \bar{\phi}_{m-n}^{1,1} + \bar{\phi}_n^{2,0} \bar{\phi}_{m-n}^{0,1} \right), \tag{16.113}
\end{aligned}$$

$$\begin{aligned}
\Gamma_{m,2}(\xi_1, \xi_2) &= \sum_{n=0}^m \left(k_1^2 \bar{\phi}_n^{1,0} \bar{\phi}_{m-n}^{1,1} + k_2^2 \bar{\phi}_n^{0,1} \bar{\phi}_{m-n}^{0,2} + \bar{\phi}_{z,n}^{0,0} \bar{\phi}_{z,m-n}^{0,1} \right) \\
&\quad + \mathbf{k}_1 \cdot \mathbf{k}_2 \sum_{n=0}^m \left(\bar{\phi}_n^{1,0} \bar{\phi}_{m-n}^{0,2} + \bar{\phi}_n^{0,1} \bar{\phi}_{m-n}^{1,1} \right). \tag{16.114}
\end{aligned}$$

Besides, on $z = \check{\eta}(\xi_1, \xi_2; q)$, we have by means of (16.103), (16.104) and (16.105) that

$$\begin{aligned}
\frac{\partial \check{f}}{\partial z} &= \hat{\nabla} \check{\phi} \cdot \hat{\nabla} \left(\frac{\partial \check{\phi}}{\partial z} \right) \\
&= k_1^2 \frac{\partial \check{\phi}}{\partial \xi_1} \frac{\partial}{\partial \xi_1} \left(\frac{\partial \check{\phi}}{\partial z} \right) + k_2^2 \frac{\partial \check{\phi}}{\partial \xi_2} \frac{\partial}{\partial \xi_2} \left(\frac{\partial \check{\phi}}{\partial z} \right) + \frac{\partial \check{\phi}}{\partial z} \frac{\partial^2 \check{\phi}}{\partial z^2} \\
&\quad + \mathbf{k}_1 \cdot \mathbf{k}_2 \left[\frac{\partial \check{\phi}}{\partial \xi_1} \frac{\partial}{\partial \xi_2} \left(\frac{\partial \check{\phi}}{\partial z} \right) + \frac{\partial \check{\phi}}{\partial \xi_2} \frac{\partial}{\partial \xi_1} \left(\frac{\partial \check{\phi}}{\partial z} \right) \right] \\
&= \sum_{m=0}^{+\infty} \Gamma_{m,3}(\xi_1, \xi_2) q^m, \tag{16.115}
\end{aligned}$$

where

$$\begin{aligned} \Gamma_{m,3}(\xi_1, \xi_2) = & \sum_{n=0}^m \left(k_1^2 \bar{\phi}_n^{1,0} \bar{\phi}_{z,m-n}^{1,0} + k_2^2 \bar{\phi}_n^{0,1} \bar{\phi}_{z,m-n}^{0,1} + \bar{\phi}_{z,n}^{0,0} \bar{\phi}_{zz,m-n}^{0,0} \right) \\ & + \mathbf{k}_1 \cdot \mathbf{k}_2 \sum_{n=0}^m \left(\bar{\phi}_n^{1,0} \bar{\phi}_{z,m-n}^{0,1} + \bar{\phi}_n^{0,1} \bar{\phi}_{z,m-n}^{1,0} \right). \end{aligned} \quad (16.116)$$

Furthermore, using (16.103), (16.111), (16.112) and (16.115), we have

$$\begin{aligned} \hat{\nabla} \check{\phi} \cdot \hat{\nabla} \check{f} &= k_1^2 \frac{\partial \check{\phi}}{\partial \xi_1} \frac{\partial \check{f}}{\partial \xi_1} + k_2^2 \frac{\partial \check{\phi}}{\partial \xi_2} \frac{\partial \check{f}}{\partial \xi_2} + \frac{\partial \check{\phi}}{\partial z} \frac{\partial \check{f}}{\partial z} \\ &+ \mathbf{k}_1 \cdot \mathbf{k}_2 \left(\frac{\partial \check{\phi}}{\partial \xi_1} \frac{\partial \check{f}}{\partial \xi_2} + \frac{\partial \check{\phi}}{\partial \xi_2} \frac{\partial \check{f}}{\partial \xi_1} \right) \\ &= \sum_{m=0}^{+\infty} \Lambda_m(\xi_1, \xi_2) q^m, \end{aligned} \quad (16.117)$$

where

$$\begin{aligned} \Lambda_m(\xi_1, \xi_2) = & \sum_{n=0}^m \left(k_1^2 \bar{\phi}_n^{1,0} \Gamma_{m-n,1} + k_2^2 \bar{\phi}_n^{0,1} \Gamma_{m-n,2} + \bar{\phi}_{z,n}^{0,0} \Gamma_{m-n,3} \right) \\ & + \mathbf{k}_1 \cdot \mathbf{k}_2 \sum_{n=0}^m \left(\bar{\phi}_n^{1,0} \Gamma_{m-n,2} + \bar{\phi}_n^{0,1} \Gamma_{m-n,1} \right). \end{aligned} \quad (16.118)$$

Then, using (16.103), (16.104), (16.111), (16.112) and (16.117), we have on $z = \check{\eta}(\xi_1, \xi_2; q)$ that

$$\begin{aligned} & \mathcal{N} [\check{\phi}(\xi_1, \xi_2, z; q)] \\ &= \sigma_1^2 \frac{\partial^2 \check{\phi}}{\partial \xi_1^2} + 2\sigma_1 \sigma_2 \frac{\partial^2 \check{\phi}}{\partial \xi_1 \partial \xi_2} + \sigma_2^2 \frac{\partial^2 \check{\phi}}{\partial \xi_2^2} + g \frac{\partial \check{\phi}}{\partial z} \\ & \quad - 2 \left(\sigma_1 \frac{\partial \check{f}}{\partial \xi_1} + \sigma_2 \frac{\partial \check{f}}{\partial \xi_2} \right) + \hat{\nabla} \check{\phi} \cdot \hat{\nabla} \check{f} \\ &= \sum_{m=0}^{+\infty} \Delta_m^\phi(\xi_1, \xi_2) q^m, \end{aligned} \quad (16.119)$$

where

$$\begin{aligned} \Delta_m^\phi(\xi_1, \xi_2) = & \sigma_1^2 \bar{\phi}_m^{2,0} + 2\sigma_1 \sigma_2 \bar{\phi}_m^{1,1} + \sigma_2^2 \bar{\phi}_m^{0,2} + g \bar{\phi}_{z,m}^{0,0} \\ & - 2(\sigma_1 \Gamma_{m,1} + \sigma_2 \Gamma_{m,2}) + \Lambda_m \end{aligned} \quad (16.120)$$

for $m \geq 0$.

Using (16.35) and (16.92), we have on $z = \check{\eta}(\xi_1, \xi_2; q)$ that

$$\check{\phi} - \phi_0 = \sum_{n=1}^{+\infty} \phi_n(\xi_1, \xi_2, \check{\eta}) q^n = \sum_{n=1}^{+\infty} q^n \left(\sum_{m=0}^{+\infty} \beta_{0,0}^{n,m} q^m \right)$$

$$= \sum_{n=1}^{+\infty} q^n \left(\sum_{m=0}^{n-1} \beta_{0,0}^{n-m,m} \right) \quad (16.121)$$

and similarly

$$\begin{aligned} \frac{\partial}{\partial z} (\check{\phi} - \phi_0) &= \sum_{n=1}^{+\infty} \frac{\partial \phi_n}{\partial z} q^n = \sum_{n=1}^{+\infty} q^n \left(\sum_{m=0}^{+\infty} \gamma_{0,0}^{n,m} q^m \right) \\ &= \sum_{n=1}^{+\infty} q^n \left(\sum_{m=0}^{n-1} \gamma_{0,0}^{n-m,m} \right), \end{aligned} \quad (16.122)$$

respectively. Then, on $z = \check{\eta}(\xi_1, \xi_2; q)$, it holds due to the linear property of the operator (16.27) that

$$\mathcal{L}(\check{\phi} - \phi_0) = \sum_{n=1}^{+\infty} S_n(\xi_1, \xi_2) q^n, \quad (16.123)$$

where

$$\begin{aligned} &S_n(\xi_1, \xi_2) \\ &= \sum_{m=0}^{n-1} \left(\bar{\sigma}_1^2 \beta_{2,0}^{n-m,m} + 2\bar{\sigma}_1 \bar{\sigma}_2 \beta_{1,1}^{n-m,m} + \bar{\sigma}_2^2 \beta_{0,2}^{n-m,m} + g \gamma_{0,0}^{n-m,m} \right). \end{aligned} \quad (16.124)$$

Then, on $z = \check{\eta}(\xi_1, \xi_2; q)$, it holds

$$(1-q)\mathcal{L}(\check{\phi} - \phi_0) = (1-q) \sum_{n=1}^{+\infty} S_n q^n = \sum_{n=1}^{+\infty} (S_n - \chi_n S_{n-1}) q^n, \quad (16.125)$$

where

$$\chi_n = \begin{cases} 0, & \text{when } n \leq 1, \\ 1, & \text{when } n > 1. \end{cases} \quad (16.126)$$

Substituting (16.125), (16.119) into (16.28) and equating the like-power of q , we have the boundary condition:

$$S_m(\xi_1, \xi_2) - \chi_m S_{m-1}(\xi_1, \xi_2) = c_0 \Delta_{m-1}^\phi(\xi_1, \xi_2), \quad m \geq 1. \quad (16.127)$$

Define

$$\begin{aligned} &\bar{S}_n(\xi_1, \xi_2) \\ &= \sum_{m=1}^{n-1} \left(\bar{\sigma}_1^2 \beta_{2,0}^{n-m,m} + 2\bar{\sigma}_1 \bar{\sigma}_2 \beta_{1,1}^{n-m,m} + \bar{\sigma}_2^2 \beta_{0,2}^{n-m,m} + g \gamma_{0,0}^{n-m,m} \right). \end{aligned} \quad (16.128)$$

Then,

$$S_n = \left(\bar{\sigma}_1^2 \beta_{2,0}^{n,0} + 2\bar{\sigma}_1 \bar{\sigma}_2 \beta_{1,1}^{n,0} + \bar{\sigma}_2^2 \beta_{0,2}^{n,0} + g \gamma_{0,0}^{n,0} \right) + \bar{S}_n$$

$$= \left(\bar{\sigma}_1^2 \frac{\partial^2 \phi_n}{\partial \xi_1^2} + 2\bar{\sigma}_1 \bar{\sigma}_2 \frac{\partial^2 \phi_n}{\partial \xi_1 \partial \xi_2} + \bar{\sigma}_2^2 \frac{\partial^2 \phi_n}{\partial \xi_2^2} + g \frac{\partial \phi_n}{\partial z} \right) \Big|_{z=0} + \bar{S}_n. \quad (16.129)$$

Substituting the above expression into (16.127) gives the boundary condition on $z = 0$:

$$\bar{\mathcal{L}}(\phi_m) = c_0 \Delta_{m-1}^\phi + \chi_m S_{m-1} - \bar{S}_m, \quad m \geq 1, \quad (16.130)$$

where $\bar{\mathcal{L}}$ is defined by (16.44).

Substituting the series (16.36), (16.103) and (16.109) into (16.30), equating the like-power of q , we have

$$\eta_m(\xi_1, \xi_2) = c_0 \Delta_{m-1}^\eta + \chi_m \eta_{m-1}, \quad m \geq 1, \quad (16.131)$$

where

$$\Delta_m^\eta = \eta_m - \frac{1}{g} (\sigma_1 \bar{\phi}_m^{1,0} + \sigma_2 \bar{\phi}_m^{0,1} - \Gamma_{m,0}).$$

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